ASYMPTOTIC NON-NULL DISTRIBUTIONS OF THE LIKELIHOOD RATIO CRITERIA FOR COVARIANCE MATRIX UNDER LOCAL ALTERNATIVES

by

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1. Introduction. Asymptotic expansions of the distributions of the likelihood ratio (= LR) criteria based on a random sample from a multivariate normal population under fixed alternative hypothesis have been derived by Sugiura [11], (1) for the equality of covariance matrix to a given matrix, (2) for the equality of mean vector and covariance matrix to a given vector and a given matrix, and also by Sugiura and Fujikoshi [12], (3) for testing the hypothesis of independence between two sets of variates. The limiting non-null distribution of the LR criterion (4) for the equality of several covariance matrices has been obtained by Sugiura [11]. These limiting non-null distributions always degenerate at the null hypothesis so that the asymptotic formulas do not give good approximations when the alternative hypothesis is near to the null hypothesis, as we have experienced in calculating the approximate powers of Bartlett's test for homogeneity of variances in Sugiura and Nagao [14].

In this paper, we shall derive limiting non-null distributions of the LR criteria for the problems (1) and (2) under sequences of alternatives converging to the null hypothesis with the rate of convergence $N^{-\gamma}$, where N means sample size, for arbitrary positive number γ and then asymptotic expansions of the non-null distributions in the case of $\gamma = \frac{1}{2}$ and $\gamma = 1$ in the next two Sections. With the help of the hypergeometric function of matrix argument due to Constantine [2], we shall derive an asymptotic expansion of the distribution for the problem (3) in the case $\gamma = \frac{1}{2}$ in Section 4, and an

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asymptotic expansion of the distribution of the modified LR criterion, in Sugiura and Nagao [13], for the equality of two covariance matrices under the sequence of alternatives with $\gamma = 1$ in Section 5. The formulas in this paper can be applied to compute the approximate power, when the alternative hypothesis is near to the null hypothesis.

2. Asymptotic distributions of the modified LR criterion for $\Sigma = \Sigma_0$.

2.1. Moments of the criterion. Let the $p \times 1$ vectors $X_1, X_2, \ldots, X_N$ be a random sample from a p-variate normal distribution with unknown mean vector and covariance matrix Σ. We shall consider the modified LR criterion λ_1^*, instead of the LR criterion, for testing the hypothesis $H: \Sigma = \Sigma_0$ (a given positive definite matrix) against alternatives $K: \Sigma \neq \Sigma_0$, which is given by

$$(2.1) \qquad \lambda_1^* = \left(\frac{e}{n}\right)^{np/2} |S\Sigma_0^{-1}|^{n/2} \operatorname{etr}\{-\frac{1}{2} \Sigma_0^{-1} S\},$$

where the symbol etr means $\exp \cdot \operatorname{tr}$ and $S = \sum_{j=1}^{N} (X_j - \bar{X})(X_j - \bar{X})'$ with $\bar{X} = \sum_{j=1}^{N} X_j/N$ and $n = N-1$. The unbiasedness of this modified LR criterion was proved by Sugiura and Nagao [13], the monotonicity property of which was established by Nagao [6]. The h-th moment of the statistic λ_1^* under K can be found in Anderson [1, p.266] as

$$(2.2) \qquad E[\lambda_1^{*h}|K] = \left(\frac{2e}{n}\right)^{nph/2} \frac{\Gamma_p(\frac{1}{2}n(1+h))}{\Gamma_p(\frac{1}{2}n)} \frac{|\Sigma\, \Sigma_0^{-1}|^{\frac{1}{2}nh}}{|I + h\Sigma\, \Sigma_0^{-1}|^{\frac{1}{2}n(1+h)}},$$

where the function $\Gamma_p(x)$ is defined by $\pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma(x - \frac{j-1}{2})$.

2.2. Asymptotic distributions when $\gamma < \frac{1}{2}$. First we shall consider the limiting distribution of the statistic $-2\log\lambda_1^*$ under the sequence of alternatives K_γ: $\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2} = I + n^{-\gamma}\,\Theta$ with $\gamma > 0$ and positive definite (= pd) matrix Θ. When $\gamma < \frac{1}{2}$, we can write the characteristic function of $(-\,2\log\lambda_1^*)/n^{1/2-\gamma}$ as

$$(2.3)\qquad \left(\frac{2e}{n}\right)^{-itpn^{\gamma+\frac{1}{2}}} \frac{\Gamma_p(\frac{1}{2}n - itn^{\gamma+\frac{1}{2}})}{\Gamma_p(\frac{1}{2}n)} \left(1 - 2itn^{\gamma-\frac{1}{2}}\right)^{-\frac{1}{2}np+itpn^{\gamma+\frac{1}{2}}}$$

$$\cdot\ |\Sigma\,\Sigma_0^{-1}|^{-itn^{\gamma+\frac{1}{2}}}\ \left|I - \frac{2itn^{-1/2}}{1-2itn^{\gamma-1/2}}\,\Theta\right|^{-\frac{1}{2}n+itn^{\gamma+\frac{1}{2}}}.$$

Applying the asymptotic formula $\log\Gamma(x+h) = \log\sqrt{2\pi} + (x+h-\frac{1}{2})\log x - x + O(|x|^{-1})$ to each gamma function in the first factor of (2.3), we can see that the first factor is equal to $1 + O(n^{\gamma-1/2})$, which tends to one as n tends to infinity. By the formula

$$(2.4)\qquad -\log|I - n^{-1}Z| = \sum_{j=1}^{\ell} n^{-j}\,\mathrm{tr}\,Z^j/j + O(n^{-\ell-1}),$$

which is valid for large n such that all characteristic roots of the symmetric matrix $n^{-1}Z$ have absolute values less than one, the second factor of (2.3) can be evaluated as

$$(2.5)\qquad \exp[it\{\sqrt{n}\,\mathrm{tr}\,\Theta - n^{\gamma+\frac{1}{2}}\log|\Sigma\,\Sigma_0^{-1}|\} - t^2\,\mathrm{tr}\,\Theta^2] + O(n^{\gamma-\frac{1}{2}}).$$

Hence the characteristic function of $n^{\gamma-1/2}[-\,2\log\lambda_1^* - n^{1-\gamma}\,\mathrm{tr}\,\Theta + n\log|\Sigma\,\Sigma_0^{-1}|]$ tends to $\exp[-\,t^2\,\mathrm{tr}\,\Theta^2]$ as n tends to infinity, which implies the following theorem.

Theorem 2.1. Let λ_1^* be the modified LR statistic given by (2.1). Then under the sequence of alternatives K_γ: $\Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2} = I + n^{-\gamma} \Theta$ for $0 < \gamma < \frac{1}{2}$, the limiting distribution of the statistic

$$n^{\gamma-\frac{1}{2}}[- 2\log\lambda_1^* - n\{tr(\Sigma \Sigma_0^{-1} - I) - \log|\Sigma \Sigma_0^{-1}|\}] \tag{2.6}$$

is normal with mean zero and variance $2tr\Theta^2$ as n tends to infinity.

Noting that the asymptotic variance $2tr\Theta^2$ is equal to $2tr(\Sigma \Sigma_0^{-1} - I)^2 \cdot n^{2\gamma}$, we can see that the limiting distribution in Theorem 2.1 is of the same form as under fixed alternative hypothesis given by Sugiura [11]. It may be noted that Theorem 2.1 holds if we replace the modified LR statistic λ_1^* to the LR statistic.

2.3. Asymptotic distributions when $\gamma \geqq \frac{1}{2}$. We shall now consider the sequence of alternative hypothesis K_γ for $\gamma \geqq \frac{1}{2}$. From (2.2), we can get the characteristic function of $- 2\log\lambda_1^*$ under K_γ as

$$C(t) = \left(\frac{n}{2e}\right)^{npit} \frac{\Gamma_p(\frac{1}{2}n(1-2it))}{\Gamma_p(\frac{1}{2}n)(1-2it)^{\frac{1}{2}np(1-2it)}} \cdot \frac{|I + n^{-\gamma} \Theta|^{-nit}}{|I - \frac{2it}{1-2it} n^{-\gamma} \Theta|^{\frac{1}{2}n(1-2it)}} . \tag{2.7}$$

The first factor of $C(t)$ in (2.7) is simply the characteristic function of $- 2\log\lambda_1^*$ under the null hypothesis, which was expanded asymptotically by Sugiura [11] as

(2.8)

$$(1-2it)^{-p(p+1)/4}[1 + \frac{B_2}{n}\{\frac{1}{1-2it} - 1\} + \frac{1}{6n^2}\{\frac{3B_2^2-4B_3}{(1-2it)^2} - \frac{6B_2^2}{1-2it} + 3B_2^2 + 4B_3\} + O(n^{-3})],$$

where

$$B_2 = p(2p^2 + 3p - 1)/24$$

$$B_3 = - p(p-1)(p+1)(p+2)/32 . \tag{2.9}$$

Applying the asymptotic formula (2.4) to the second factor of the characteristic function (2.7), we can get

$$\exp[-\, nit \log|I + n^{-\gamma}\,\Theta| - \frac{n}{2}(1-2it)\log|I - \frac{2it}{1-2it}n^{-\gamma}\,\Theta|] \tag{2.10}$$

$$= \exp\left[\sum_{k=2}^{\ell} \frac{\mathrm{tr}\,\Theta^k}{n^{\gamma\,k-1}}\left\{(-\,1)^k it + \frac{2^{k-1}\,(it)^k}{(1-2it)^{k-1}}\right\} + O(n^{(\ell+1)\gamma-1})\right]$$

$$= \exp[n^{1-2\gamma}\frac{it\ \mathrm{tr}\,\Theta^2}{2(1-2it)} + n^{1-3\gamma}\frac{\mathrm{tr}\,\Theta^3}{6}\left\{\frac{1}{(1-2it)^2} - \frac{3}{1-2it} + 2\right\}$$

$$+\, n^{1-4\gamma}\frac{\mathrm{tr}\,\Theta^4}{8}\left\{\frac{1}{(1-2it)^3} - \frac{4}{(1-2it)^2} + \frac{6}{1-2it} - 3\right\} + O(n^{1-5\gamma})].$$

Multiplying (2.8) and (2.10), we can see that

$$\begin{aligned} C(t) &= (1-2it)^{-p(p+1)/4} + o(1) && \text{when} \quad \gamma > \tfrac{1}{2} \\ &= (1-2it)^{-p(p+1)/4}\exp\left\{\frac{it\ \mathrm{tr}\,\Theta^2}{2(1-2it)}\right\} + o(1) && \text{when} \quad \gamma = \tfrac{1}{2}\,, \end{aligned} \tag{2.11}$$

which implies that the limiting distribution of $-\,2\log\lambda_1^*$ is χ^2 with $f_1 = p(p+1)/2$ degrees of freedom when $\gamma > \frac{1}{2}$ and noncentral χ^2 with f_1 degrees of freedom and noncentrality parameter $\mathrm{tr}\,\Theta^2/4$ when $\gamma = \frac{1}{2}$. Further, we can get the asymptotic expansion of the characteristic function of $-\,2\log\lambda_1^*$ when $\gamma = 1$ as

$$C(t) = (1-2it)^{-f_1/2}\,[1 + \frac{1}{n}\left\{\frac{1}{4}\mathrm{tr}\,\Theta^2 + B_2\right\}\left\{\frac{1}{1-2it} - 1\right\} \tag{2.12}$$

$$+ \frac{1}{6n^2}\left\{\frac{3B_2^2-4B_3}{(1-2it)^2} - \frac{6B_2^2}{1-2it} + 3B_2^2 + 4B_3\right\} + \frac{1}{n^2}\sum_{\alpha=0}^{2} g_{2\alpha}(1-2it)^{-\alpha}] + O(n^{-3})\ ,$$

where

$$
\begin{aligned}
g_0 &= \frac{B_2}{4}\,\mathrm{tr}\,\Theta^2 + \frac{1}{3}\,\mathrm{tr}\,\Theta^3 + \frac{1}{32}\,(\mathrm{tr}\,\Theta^2)^2 \\
g_2 &= -\frac{B_2}{2}\,\mathrm{tr}\,\Theta^2 - \frac{1}{2}\,\mathrm{tr}\,\Theta^3 - \frac{1}{16}\,(\mathrm{tr}\,\Theta^2)^2 \qquad (2.13)\\
g_4 &= \frac{B_2}{4}\,\mathrm{tr}\,\Theta^2 + \frac{1}{6}\,\mathrm{tr}\,\Theta^3 + \frac{1}{32}\,(\mathrm{tr}\,\Theta^2)^2 .
\end{aligned}
$$

When $\gamma = \frac{1}{2}$, we can get another asymptotic formula for $C(t)$ from (2.8) and (2.10) as

$$
(2.14)\qquad C(t) = (1-2it)^{-f_1/2}\exp\left\{\frac{it\,\mathrm{tr}\,\Theta^2}{2(1-2it)}\right\}\cdot\left[1 + \frac{\mathrm{tr}\,\Theta^3}{6\sqrt{n}}\left\{\frac{1}{(1-2it)^2} - \frac{3}{1-2it} + 2\right\}\right.
$$

$$
\left. + \frac{1}{n}\sum_{\alpha=0}^{4} h_{2\alpha}(1-2it)^{-\alpha}\right] + O(n^{-3/2}) ,
$$

where the coefficients $h_{2\alpha}$ $(\alpha = 0, \ldots, 4)$ are given by

$$
\begin{aligned}
h_0 &= -B_2 - \frac{3}{8}\,\mathrm{tr}\,\Theta^4 + \frac{1}{18}\,(\mathrm{tr}\,\Theta^3)^2 \\
h_2 &= B_2 + \frac{3}{4}\,\mathrm{tr}\,\Theta^4 - \frac{1}{6}\,(\mathrm{tr}\,\Theta^3)^2 \\
h_4 &= -\frac{1}{2}\,\mathrm{tr}\,\Theta^4 + \frac{13}{72}\,(\mathrm{tr}\,\Theta^3)^2 \qquad (2.15)\\
h_6 &= \frac{1}{8}\,\mathrm{tr}\,\Theta^4 - \frac{1}{12}\,(\mathrm{tr}\,\Theta^3)^2 \\
h_8 &= \frac{1}{72}\,(\mathrm{tr}\,\Theta^3)^2 .
\end{aligned}
$$

Inverting the characterisitc function (2.8), we have

$$
(2.16)\qquad P_H(-2\log\lambda_1^* < z) = P(\chi^2_{f_1} < z) + \frac{B_2}{n}\{P(\chi^2_{f_1+2} < z) - P(\chi^2_{f_1} < z)\}
$$

$$
+ \frac{1}{6n^2}\{(3B_2^2 - 4B_3)P(\chi^2_{f_1+4} < z) - 6B_2^2P(\chi^2_{f_1+2} < z) + (3B_2^2 + 4B_3)P(\chi^2_{f_1} < z)\}
$$

$$
+ O(n^{-3}),
$$

where $\chi^2_{f_1}$ means a χ^2 variate with f_1 degrees of freedom. It follows from (2.11) and the characteristic functions (2.12), (2.14) that the following theorem holds.

Theorem 2.2. Under the sequence of alternative hypothesis K_γ: $\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2} = I + n^{-\gamma}\,\Theta$ (Θ is pd), the limiting distributions of the modified LR statistic $-2\log\lambda_1^*$ given by (2.1) is χ^2 with $f_1 = p(p+1)/2$ degrees of freedom, when $\gamma > \frac{1}{2}$, and noncentral χ^2 with f_1 degrees of freedom and noncentrality parameter $\mathrm{tr}\,\Theta^2/4$, when $\gamma = \frac{1}{2}$. When $\gamma = 1$,

$$(2.17)\quad P_\gamma(-2\log\lambda_1^* < z) = P_H(-2\log\lambda_1^* < z) + \frac{1}{4n}\,\mathrm{tr}\,\Theta^2\,\{P(\chi^2_{f_1+2} < z) - P(\chi^2_{f_1} < z)\} + \frac{1}{n^2}\sum_{\alpha=0}^{2} g_{2\alpha}\,P(\chi^2_{f_1+2\alpha} < z) + O(n^{-3}),$$

where $P_H(-2\log\lambda_1^* < z)$ is given by (2.16) and the coefficients $g_{2\alpha}$ ($\alpha = 0, 1, 2$) are given by (2.13) with B_2 and B_3 in (2.9). When $\gamma = \frac{1}{2}$, we have

$$(2.18)\quad P_\gamma(-2\log\lambda_1^* < z) = P(\chi^2_{f_1}(\delta^2) < z) + \frac{\mathrm{tr}\,\Theta^3}{6\sqrt{n}}\{P(\chi^2_{f_1+4}(\delta^2) < z) - 3P(\chi^2_{f_1+2}(\delta^2) < z) + 2P(\chi^2_{f_1}(\delta^2) < z)\} + \frac{1}{n}\sum_{\alpha=0}^{4} h_{2\alpha}P(\chi^2_{f_1+2\alpha}(\delta^2) < z) + O(n^{-3/2}),$$

where the symbol $\chi^2_{f_1}(\delta^2)$ means the noncentral χ^2 variate with $f_1 = p(p+1)/2$ degrees of freedom and noncentrality parameter $\delta^2 = \mathrm{tr}\Theta^2/4$, and the $h_{2\alpha}$ are given by (2.15).

2.4. Numerical examples. It may be useful to note that applying the general inverse expansion formula of Hill and Davis [4] to the asymptotic null-distribution of $-2\log\lambda_1^*$ given by Theorem 2.1 in Sugiura [11], we can get an asymptotic formula of the 100α % point of $-2\log\lambda_1^*$ in terms of the 100α % point of the χ^2 distribution with $f_1 = p(p+1)/2$ degrees of freedom as

$$(2.19)\quad u + \frac{2B_2}{nf_1}u + \frac{1}{3n^2}\{\frac{u^2}{f_1^2(f_1+2)}(-6B_2^2 - 4f_1B_3) + \frac{u}{f_1^2}(6B_2^2 - 4f_1B_3)\}$$

$$+ \frac{1}{n^3}(u^3g_3 + u^2g_2 + ug_1) + O(n^{-4}),$$

where u is so chosen that $P(\chi^2_{f_1} > u) = \alpha$ and

$$g_1 = \frac{4}{3f_1^3}(f_1^2B_4 - 2f_1B_2B_3 + B_2^3)$$

$$(2.20)\quad g_2 = \frac{4}{3f_1^3(f_1+2)}(f_1^2B_4 - 2f_1B_2B_3 - 5B_2^3)$$

$$g_3 = \frac{4}{3f_1^3(f_1+2)(f_1+4)}(f_1^2B_4 + 4f_1B_2B_3 + 4B_2^3)$$

with B_2, B_3 in (2.9) and $B_4 = p(6p^4 + 15p^3 - 10p^2 - 30p + 3)/480$.

We shall examine the effectiveness of these formulas in the following simple examples.

Example 2.1. When $p = 1$ and $n = 10$, the exact 5 % point of $-2\log\lambda_1^*$ can be obtained from Table I by Pachares [7] as 3.9682. The asymptotic formula (2.19) gives

5 % points of $-2\log\lambda_1^*$

	$p = 1$ and $n = 10$	$p = 2$ and $n = 100$
first term	3.84146	7.81473
term of order n^{-1}	0.12805	0.05644
term of order n^{-2}	-0.00060	0.00055
term of order n^{-3}	-0.00059	0.00001
approx. value	3.9683	7.87173
exact value	3.9682	

Our asymptotic formula (2.19) gives good approximations for the percentage points.

Example 2.2. When $p = 2$ and $n = 100$, the approximate 5 % point of $-2\log\lambda_1^*$ is given by Example 2.1 as 7.87173. The asymptotic powers should be computed by the different formulas in Theorem 2.2 according to the departure of the alternative hypothesis K from the null hypothesis. The non-centrality parameter $\delta^2 = \text{tr}\,\Theta^2/4$ in Theorem 2.2 may be used as a measure of the "distance" of K from the null hypothesis. Let us specify the alternatives K as $\Sigma_0^{-1/2}\,\Sigma\,\Sigma_0^{-1/2} = I + \Delta I$. Then $\delta^2 = n\Delta^2/2$ and the following case 1 ($\Delta = 0.5$) is computed by the formula (2.16) in Sugiura [11] with the normal distribution function and its derivatives, as well as the cases 2 ($\Delta = 0.1$) and 3 ($\Delta = 0.02$) by the formulas (2.18) and (2.17) respectively.

approximate powers, when $p = 2$ and $n = 100$

	$\Delta = 0.5$	$\Delta = 0.1$	$\Delta = 0.02$
δ^2	12.5	0.5	0.02
first term	0.8651	0.1134	0.05
second term	0.0714	-0.0033	0.00229
third term	0.0052	0.0018	0.00001
approx. power	0.942	0.112	0.0523

3. Asymptotic distributions of the LR criterion for $\Sigma = \Sigma_0$ and $\mu = \mu_0$.

3.1. Moments of the criterion. Let a random sample of size N from a p-variate normal population with mean vector μ $(p\times 1)$ and covariance matrix Σ $(p\times p$, pd$)$ be given. We wish to test the hypothesis H: $\mu = \mu_0$ (a given vector) and $\Sigma = \Sigma_0$ (a given pd matrix) against alternatives K: $\mu \neq \mu_0$ and/or $\Sigma \neq \Sigma_0$. The h-th moment of the LR criterion λ_2 for this problem is given by Sugiura [11] as

$$(3.1)\qquad E[\lambda_2^h|K] = \left(\frac{2e}{N}\right)^{Nph/2} \frac{\Gamma_p((n+Nh)/2)}{\Gamma_p(n/2)} \frac{|\Sigma\, \Sigma_0^{-1}|^{Nh/2}}{|I + h\, \Sigma\, \Sigma_0^{-1}|^{N(1+h)/2}}$$

$$\times \operatorname{etr}[-\frac{Nh}{2} \Sigma_0^{-1}(\mu-\mu_0)(\mu-\mu_0)' \{I - h\, \Sigma_0^{-1}(\Sigma^{-1} + h\, \Sigma_0^{-1})^{-1}\}],$$

where $n = N-1$. The unbiasedness of the LR criterion λ_2 without modification was proved by Sugiura and Nagao [13]. The asymptotic expansions of the distributions of $-2\log\lambda_2$ both under the null hypothesis and a fixed alternative hypothesis have been derived by Sugiura [11].

3.2. Asymptotic distributions when $\gamma < \frac{1}{2}$. Now we shall specify the sequence of alternative hypotheses as K_γ: $\mu - \mu_0 = N^{-\gamma}\, \Sigma_0^{1/2}\, \nu$ and $\Sigma_0^{-1/2}\, \Sigma\, \Sigma_0^{-1/2} = I + N^{-\gamma}\, \Theta$, and consider the case $0 < \gamma < \frac{1}{2}$. From (3.1) we can express the characteristic function of $(-2\log\lambda_2)N^{\gamma-1/2}$ as

$$\left(\frac{2e}{N}\right)^{-pitN^{\gamma+\frac{1}{2}}} \frac{\Gamma_p(\frac{1}{2}N - itN^{\gamma+\frac{1}{2}} - \frac{1}{2})}{\Gamma_p(\frac{1}{2}N - \frac{1}{2})(1 - 2itN^{\gamma-\frac{1}{2}})^{\frac{1}{2}Np-itpN^{\gamma+\frac{1}{2}}}}$$

(3.2)

$$\times\, |\Sigma\, \Sigma_0^{-1}|^{-itN^{\gamma+\frac{1}{2}}} \left|I - \frac{2itN^{-\frac{1}{2}}}{1-2itN^{\gamma-\frac{1}{2}}}\,\Theta\right|^{-\frac{1}{2}N+itN^{\gamma+\frac{1}{2}}}$$

$$\times \operatorname{etr}[itN^{\frac{1}{2}-\gamma}\, \nu'\nu - 2t^2\, \nu'(I - 2itN^{\gamma-\frac{1}{2}}\, \Sigma_0^{-1/2}\, \Sigma\, \Sigma_0^{-1/2})^{-1}(I + N^{-\gamma}\,\Theta)\nu].$$

The first factor and the second factor have the same form as the characteristic function (2.3). Hence we can see that the first factor is equal to $1 + O(N^{\gamma-1/2})$, which tends to one as $N \to \infty$, and the second factor is given by (2.5) after substituting n for N. The third factor is easily evaluated as $\mathrm{etr}[itN^{1/2-\gamma}\nu'\nu - 2t^2\nu'\nu] + o(1)$. It follows that the characteristic function of $(-2\log\lambda_2)N^{\gamma-1/2}$ can be expressed as

$$\text{(3.3)}\quad \exp[it\{\sqrt{N}\mathrm{tr}\Theta - N^{\gamma+\frac{1}{2}}\log|\Sigma\Sigma_0^{-1}| + N^{\frac{1}{2}-\gamma}\nu'\nu\} - t^2(\mathrm{tr}\Theta^2 + 2\nu'\nu)] + o(1),$$

which implies that the characteristic function of the statistic

$$\text{(3.4)}\quad N^{\gamma-\frac{1}{2}}[-2\log\lambda_2 - N\{\mathrm{tr}(\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}| + (\mu-\mu_0)'\Sigma_0^{-1}(\mu-\mu_0)\}]$$

tends to $\exp[-t^2(\mathrm{tr}\,\Theta^2 + 2\nu'\nu)]$ as $N \to \infty$. Thus we can get the following theorem.

Theorem 3.1. Under the sequence of alternatives K_γ: $\mu - \mu_0 = N^{-\gamma}\Sigma_0^{1/2}\nu$ and $\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2} = I + N^{-\gamma}\Theta$, for $0 < \gamma < \frac{1}{2}$, the limiting distribution of the statistic (3.4) is normal with mean zero and variance $2\mathrm{tr}\,\Theta^2 + 4\nu'\nu$ as N tends to infinity.

The limiting distribution in the above theorem is of the same form as under fixed alternative hypothesis given by Sugiura [11].

3.3. Asymptotic distributions when $\gamma \geqq \frac{1}{2}$. We shall now consider the asymptotic distribution of $-2\log\lambda_2$ under K_γ when $\gamma \geqq \frac{1}{2}$. From (3.1), the characteristic function of $-2\log\lambda_2$ can be expressed as

$$\text{(3.5)}\qquad \left(\frac{2e}{N}\right)^{-Npit}\frac{\Gamma_p(\frac{1}{2}N(1-2it)-\frac{1}{2})}{\Gamma_p(\frac{1}{2}N-\frac{1}{2})(1-2it)^{Np(1-2it)/2}}\cdot\frac{|I + N^{-\gamma}\Theta|^{-Nit}}{|I - \frac{2it}{1-2it}N^{-\gamma}\Theta|^{N(1-2it)/2}}$$

$$\cdot \operatorname{etr}\{itN^{1-2\gamma}\nu'\nu - \frac{2t^2}{1-2it}N^{1-2\gamma}\nu'(I - \frac{2it}{1-2it}\frac{\Theta}{N^\gamma})^{-1}(I + \frac{\Theta}{N^\gamma})\nu\} .$$

The first factor is equal to the characteristic function of $-2\log\lambda_2$ under the null hypothesis, which was expanded asymptotically by Sugiura [11] as

(3.6)

$$(1-2it)^{-\frac{1}{2}p-\frac{p}{4}(p+1)}[1 + \frac{B_2'}{N}\{\frac{1}{1-2it} - 1\} + \frac{1}{6N^2}\{\frac{3B_2'^2-4B_3'}{(1-2it)^2} - \frac{6B_2'^2}{1-2it} + 3B_2'^2 + 4B_3'\} + O(N^{-3})],$$

where

$$B_2' = p(2p^2+9p+11)/24$$

$$B_3' = -p(p+1)(p+2)(p+3)/32 . \tag{3.7}$$

The second factor in (3.5) can be expanded by (2.10) after substituting n for N. The third factor can be expanded asymptotically as

(3.8)

$$\operatorname{etr}[\frac{it}{1-2it}N^{1-2\gamma}\nu'\nu + \frac{2(it)^2}{(1-2it)^2}N^{1-3\gamma}\nu'\Theta\nu + \frac{4(it)^3}{(1-2it)^3}N^{1-4\gamma}\nu'\Theta^2\nu + O(N^{1-5\gamma})].$$

When $\gamma > \frac{1}{2}$, both the second and the third factors tend to one as $N \to \infty$. It follows from (3.6) that the characteristic function of $-2\log\lambda_2$ tends to $(1-2it)^{-f_2/2}$ where $f_2 = p + p(p+1)/2$ as $N \to \infty$. Thus the limiting distribution of $-2\log\lambda_2$ is χ^2 with f_2 degrees of freedom. Moreover, we can get the asymptotic formula of the characteristic function when $\gamma = 1$ as

$$(1-2it)^{-f_2/2}[1 + \frac{1}{N}\{\frac{1}{4}\operatorname{tr}\Theta^2 + \frac{\nu'\nu}{2} + B_2'\}\{\frac{1}{1-2it} - 1\}$$
$$+ \frac{1}{6N^2}\{\frac{3B_2'^2-4B_3'}{(1-2it)^2} - \frac{6B_2'^2}{1-2it} + 3B_2'^2 + 4B_3'\} + \frac{1}{N^2}\sum_{\alpha=0}^{2} g_{2\alpha}'(1-2it)^{-\alpha} + O(N^{-3}), \tag{3.9}$$

where

$$g_0' = \frac{B_2'}{4}\,\mathrm{tr}\Theta^2 + \frac{1}{3}\,\mathrm{tr}\Theta^3 + \frac{1}{32}(\mathrm{tr}\Theta^2)^2 + \frac{\nu'\nu}{2}(\frac{1}{4}\mathrm{tr}\Theta^2 + B_2') + \frac{1}{2}\nu'\Theta\nu + \frac{(\nu'\nu)^2}{8}$$

(3.10) $$g_2' = -\frac{B_2'}{2}\,\mathrm{tr}\Theta^2 - \frac{1}{2}\,\mathrm{tr}\Theta^3 - \frac{1}{16}(\mathrm{tr}\Theta^2)^2 - \nu'\nu(\frac{1}{4}\mathrm{tr}\Theta^2 + B_2') - \nu'\Theta\nu - \frac{(\nu'\nu)^2}{4}$$

$$g_4' = \frac{B_2'}{4}\,\mathrm{tr}\Theta^2 + \frac{1}{6}\,\mathrm{tr}\Theta^3 + \frac{1}{32}(\mathrm{tr}\Theta^2)^2 + \frac{\nu'\nu}{2}(\frac{1}{4}\mathrm{tr}\Theta^2 + B_2') + \frac{1}{2}\,\nu'\Theta\nu + \frac{(\nu'\nu)^2}{8}.$$

When $\gamma = \frac{1}{2}$, the second factor in (3.5) tends to $\exp[\frac{1}{2}\,it\,\mathrm{tr}\,\Theta^2/(1-2it)]$ as $N \to \infty$ and the third factor tends to $\exp[it\nu'\nu/(1-2it)]$ as $N \to \infty$. Hence from (3.6) the characteristic function of $-2\log\lambda_2$ tends to $(1-2it)^{-f_2/2}$ $\cdot \exp[it\,\{\frac{1}{2}\,\mathrm{tr}\,\Theta^2 + \nu'\nu\}/(1-2it)]$ as $N \to \infty$, which implies that the limiting distribution is noncentral χ^2 with f_2 degrees of freedom and noncentrality parameter $\delta^2 = \frac{1}{4}\,\mathrm{tr}\,\Theta^2 + \frac{1}{2}\,\nu'\nu$. Further, we can get an asymptotic formula for the characteristic function of $-2\log\lambda_2$ as

(3.11) $$(1-2it)^{-f_2/2}\exp[\frac{it}{1-2it}(\nu'\nu + \frac{1}{2}\,\mathrm{tr}\,\Theta^2)]$$
$$\cdot\,[1 + \frac{1}{6\sqrt{N}}\{\frac{\mathrm{tr}\Theta^3 + 3\nu'\Theta\nu}{(1-2it)^2} - \frac{3\mathrm{tr}\Theta^3 + 6\nu'\Theta\nu}{1-2it} + 2\,\mathrm{tr}\,\Theta^3 + 3\nu'\Theta\nu\}$$
$$+ N^{-1}\sum_{\alpha=0}^{4} h_{2\alpha}'(1-2it)^{-\alpha} + O(N^{-3/2})],$$

where

$$h_0' = -B_2' - \frac{3}{8}\,\mathrm{tr}\,\Theta^4 + \frac{1}{18}(\mathrm{tr}\Theta^3)^2 - \frac{\nu'\Theta^2\nu}{2} + \frac{(\nu'\Theta\nu)^2}{8} + \frac{1}{6}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3$$

$$h_2' = B_2' + \frac{3}{4}\,\mathrm{tr}\,\Theta^4 - \frac{1}{6}(\mathrm{tr}\Theta^3)^2 + \frac{3}{2}\,\nu'\Theta^2\nu - \frac{1}{2}(\nu'\Theta\nu)^2 - \frac{7}{12}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3$$

(3.12) $$h_4' = -\frac{1}{2}\,\mathrm{tr}\,\Theta^4 + \frac{13}{72}(\mathrm{tr}\Theta^3)^2 - \frac{3}{2}\,\nu'\Theta^2\nu + \frac{3}{4}(\nu'\Theta\nu)^2 + \frac{3}{4}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3$$

$$h_6' = \frac{1}{8}\,\mathrm{tr}\,\Theta^4 - \frac{1}{12}(\mathrm{tr}\Theta^3)^2 + \frac{1}{2}\,\nu'\Theta^2\nu - \frac{1}{2}(\nu'\Theta\nu)^2 - \frac{5}{12}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3$$

$$h_8' = \frac{1}{72}(\mathrm{tr}\Theta^3)^2 + \frac{1}{8}(\nu'\Theta\nu)^2 + \frac{1}{12}\,\nu'\Theta\nu\cdot\mathrm{tr}\,\Theta^3\,.$$

Inverting the characteristic function (3.6) under the null hypothesis H, we have

(3.13) $$P_H(-2\log\lambda_2 < z) = P(\chi^2_{f_2} < z) + \frac{B'_2}{N}\{P(\chi^2_{f_2+2} < z) - P(\chi^2_{f_2} < z)\}$$

$$+ \frac{1}{6N^2}\{(3B'^2_2-4B'_3)P(\chi^2_{f_2+4} < z) - 6B'^2_2P(\chi^2_{f_2+2} < z)+(3B'^2_2+4B'_3)P(\chi^2_{f_2} < z)\} + O(N^{-3}).$$

From the asymptotic formulas (3.9) and (3.11) for the characteristic function, we can get the following theorem.

Theorem 3.2. Under the sequence of alternatives K_γ: $\Sigma_0^{-1/2}(\mu - \mu_0) = N^{-\gamma}\nu$ and $\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2} = I + N^{-\gamma}\Theta$, the limiting distribution of the LR statistic $-2\log\lambda_2$ is χ^2 with $f_2 = p + p(p+1)/2$ degrees of freedom, when $\gamma > \frac{1}{2}$ and noncentral χ^2 with $f_2 = p + p(p+1)/2$ degrees of freedom and noncentrality parameter $\delta^2 = \frac{1}{4}\operatorname{tr}\Theta^2 + \frac{1}{2}\nu'\nu$, when $\gamma = \frac{1}{2}$. When $\gamma = 1$, we have

(3.14)

$$P_\gamma(-2\log\lambda_2 < z) = P_H(\chi^2_{f_2} < z) + \frac{1}{N}\{\frac{1}{4}\operatorname{tr}\Theta^2 + \frac{1}{2}\nu'\nu\}\{P(\chi^2_{f_2+2} < z) - P(\chi^2_{f_2} < z)\}$$
$$+ \frac{1}{N^2}\sum_{\alpha=0}^{2} g'_{2\alpha}P(\chi^2_{f_2+2\alpha} < z) + O(N^{-3}),$$

where g_0, g_2 and g_4 are given by (3.10) and $P_H(\chi^2_{f_2} < z)$ is given by (3.13). When $\gamma = \frac{1}{2}$, we have

(3.15) $$P_\gamma(-2\log\lambda_2 < z) = P(\chi^2_{f_2}(\delta^2) < z) + \frac{1}{6\sqrt{N}}\{(\operatorname{tr}\Theta^3+3\nu'\Theta\nu)P(\chi^2_{f_2+4}(\delta^2) < z)$$
$$- (3\operatorname{tr}\Theta^3 + 6\nu'\Theta\nu)P(\chi^2_{f_2+2}(\delta^2) < z) + (2\operatorname{tr}\Theta^3 + 3\nu'\Theta\nu)P(\chi^2_{f_2}(\delta^2) < z)\}$$
$$+ N^{-1}\sum_{\alpha=0}^{4} h'_{2\alpha}P(\chi^2_{f_2+2\alpha}(\delta^2) < z) + O(N^{-3/2}),$$

where $h_{2\alpha}$ $(\alpha = 0, \ldots, 4)$ are given by (3.12).

3.4. <u>More general sequences of alternatives</u>. If we consider the sequences of alternatives $K_{\gamma_1\gamma_2}$: $\Sigma_0^{-1/2}(\mu - \mu_0) = N^{-\gamma_1}\nu$ and $\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2} = I + N^{-\gamma_2}\Theta$, the asymptotic distribution of $-2\log\lambda_2$ can be obtained by the same argument as above.

<u>Theorem 3.3</u>. Under the sequences of alternatives $K_{\gamma_1\gamma_2}(\gamma_1 \neq \gamma_2)$, the limiting distributions of the LR statistic $-2\log\lambda_2$ are given in the following.

(1) When $\gamma_2 < \frac{1}{2}$ and $\gamma_2 < \gamma_1$, the statistic

$$(3.16)\quad \frac{1}{N^{\frac{1}{2}-\gamma_2}}[-2\log\lambda_2 - N\{tr(\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}| + (\mu - \mu_o)'\Sigma_0^{-1}(\mu - \mu_0)\}]$$

is distributed asymptotically according to the normal distribution with mean zero and variance $2tr\,\Theta^2$. If further $\frac{1}{2} \leq \gamma_1$, the term $(\mu - \mu_0)'\Sigma_0^{-1}(\mu - \mu_0)$ in (3.16) may be omitted.

(2) When $\gamma_2 = \frac{1}{2} < \gamma_1$, $-2\log\lambda_2$ has asymptotically the noncentral χ^2 distribution with $f_2 = p + p(p+1)/2$ degrees of freedom and noncentrality parameter $tr\,\Theta^2/4$.

(3) When $\frac{1}{2} < \gamma_2, \gamma_1$, $-2\log\lambda_2$ has asymptotically the χ^2 distribution with $f_2 = p + p(p+1)/2$ degrees of freedom.

(4) When $\gamma_1 < \frac{1}{2}$ and $\gamma_1 < \gamma_2$, the statistic

$$(3.17)\quad \frac{1}{N^{\frac{1}{2}-\gamma_1}}[-2\log\lambda_2 - N\{tr(\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}| + (\mu - \mu_0)'\Sigma_0^{-1}(\mu - \mu_0)\}]$$

has asymptotically the normal distribution with mean zero and variance $4\nu'\nu$. If further $\frac{1}{2} \leq \gamma_2$, the term $tr(\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}|$ in (3.17) may be omitted.

(5) When $\gamma_1 = \frac{1}{2} < \gamma_2$, $-2\log\lambda_2$ has asymptotically the noncentral χ^2 distribution with f_2 degrees of freedom and noncentrality parameter $\nu'\nu/2$.

The above theorem shows that the limiting distributions are normal when $\min(\gamma_1, \gamma_2) < \frac{1}{2}$, noncentral χ^2 when $\min(\gamma_1, \gamma_2) = \frac{1}{2}$ and χ^2 when $\min(\gamma_1, \gamma_2) > \frac{1}{2}$.

3.5. Numerical examples. Since the asymptotic null-distribution of $-2\log\lambda_2$ has the same form as that in the previous section as was shown by Sugiura [11], the asymptotic formula (2.19) for the percentage point can be used, also, in this case after changing n to N and B_α to B'_α $(\alpha = 2, 3, 4)$ in (3.7) with $B'_4 = p(6p^4 + 45p^3 + 110p^2 + 90p + 3)/480$ and putting $f_1 \to f_2 = p(p+1)/2 + p$.

Example 3.1. When $N = 100$ and $p = 2$, we have the following approximations to the 5 % point.

first term	11.0705
term of order N^{-1}	0.1365
term of order N^{-2}	0.0024
term of order N^{-3}	0.0000
approx. value	11.2094

For the alternative hypothesis K: $\Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2} = \mathrm{diag}(1.1, 0.9)$ and $\Sigma_0^{-1/2}(\mu - \mu_0) = (0.05, 0.05)'$, the following approximate powers are computed by the formulas (3.15) and (3.14), based on the above 5 % point.

approximate power	(3.15)	(3.14)
first term	0.1217	0.0500
second term	0	0.0610
third term	0.0036	0.0150
approx. power	0.125	0.127

4. Asymptotic distributions of the LR criterion for independence.

4.1. Preliminaries. Sugiura and Fujikoshi [12] have proved that the moments of the LR statistic λ_3 for testing the independence between p_1 and p_2 sets of variates $(p_1 \leq p_2)$ in a p-variate $(p = p_1 + p_2)$ normal population, based on a random sample of size N, could be expressed, under alternative hypothesis, by using the hypergeometric function with matrix argument due to Constantine [2] as

$$(4.1) \qquad E[\lambda_3^{\frac{2}{N}h}] = \frac{\Gamma_{p_1}(h+\frac{1}{2}(N-p_2-1))\Gamma_{p_1}(\frac{1}{2}(N-1))}{\Gamma_{p_1}(\frac{1}{2}(N-p_2-1))\Gamma_{p_1}(\frac{1}{2}(N-1)+h)} |I - P^2|^{\frac{1}{2}(N-1)}$$

$$\cdot \, {}_2F_1(\tfrac{1}{2}(N-1),\ \tfrac{1}{2}(N-1);\ \ h + \tfrac{1}{2}(N-1);\ \ P^2),$$

where $P^2 = \text{diag}\,(\rho_1^2, \rho_2^2, \ldots, \rho_{p_1}^2)$ and ρ_α $(\alpha = 1, 2, \ldots, p_1)$ are the population canonical correlations. The following lemma for zonal polynomials due to Fujikoshi [3] and Sugiura and Fujikoshi [12] will be used later.

Lemma 4.1. Let $C_\kappa(Z)$ be a zonal polynomial corresponding to the partition $\kappa = \{k_1, k_2, \ldots, k_p\}$ with $k_1 + k_2 + \ldots + k_p = k$ and $k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$. Putting

$$(4.2)\qquad a_1(\kappa) = \sum_{\alpha=1}^{p} k_\alpha(k_\alpha - \alpha)$$
$$a_2(\kappa) = \sum_{\alpha=1}^{p} k_\alpha(4k_\alpha^2 - 6\alpha k_\alpha + 3\alpha^2) ,$$

then the following equalities hold.

$$(4.3)\qquad \sum_{k=\ell}^{\infty} \sum_{(\kappa)} C_\kappa(Z)/(k-\ell)! = (\mathrm{tr}Z)^\ell \,\mathrm{etr}\, Z, \quad \text{for } \ell = 0, 1, 2, \ldots$$

$$(4.4)\qquad \sum_{k=0}^{\infty} \sum_{(\kappa)} a_1(\kappa)\, C_\kappa(Z)/k! = (\mathrm{tr}Z^2)\mathrm{etr}\, Z$$

$$(4.5)\qquad \sum_{k=1}^{\infty} \sum_{(\kappa)} a_1(\kappa)\, C_\kappa(Z)/(k-1)! = (2\mathrm{tr}Z^2 + \mathrm{tr}Z^2\mathrm{tr}Z)\,\mathrm{etr}\, Z$$

$$(4.6)\qquad \sum_{k=0}^{\infty} \sum_{(\kappa)} a_1(\kappa)^2 C_\kappa(Z)/k! = \{(\mathrm{tr}Z^2)^2 + 4\mathrm{tr}Z^3 + \mathrm{tr}Z^2 + (\mathrm{tr}Z)^2\}\mathrm{etr}\, Z$$

$$(4.7)\qquad \sum_{k=0}^{\infty} \sum_{(\kappa)} a_2(\kappa) C_\kappa(Z)/k! = \{4\mathrm{tr}Z^3 + 3\mathrm{tr}Z^2 + 3(\mathrm{tr}Z)^2 + \mathrm{tr}Z\}\,\mathrm{etr}\, Z.$$

It is also convenient to note the following lemma, the proof of which comes by considering $\log(a)_\kappa$, where κ is a partition $\{k_1,k_2,\ldots,k_p\}$ of k and

$$(4.8)\qquad (a)_\kappa = \prod_{\alpha=1}^{p} \prod_{j=1}^{k_\alpha} (a - \tfrac{1}{2}(\alpha+1) + j) .$$

Lemma 4.2.

$$(4.9)\qquad (am + b)_\kappa = (am)^k \Big[1 + \frac{1}{am}\{kb + \frac{1}{2} a_1(\kappa)\}$$
$$+ \frac{1}{24a^2m^2}\{12b^2k(k-1) + 12(k-1)ba_1(\kappa) + 3a_1(\kappa)^2 - a_2(\kappa) + k\} + O(m^{-3})\Big]$$

holds, where $a_1(\kappa)$ and $a_2(\kappa)$ are defined by (4.2).

4.2. Asymptotic distribution when $\gamma = \frac{1}{2}$. We shall now consider the asymptotic distribution of the LR statistic $-2\rho\log\lambda_3$ under the sequence of alternatives K_γ: $P = m^{-1/2}\Theta$, where the correction factor ρ is given by $\rho = 1 - (p_1 + p_2 + 3)/2N$ and $m = \rho N$ (Anderson [1, p.239]). The characteristic function of $-2\rho\log\lambda_3$ under K_γ can be written from (4.1) as

$$\frac{\Gamma_{p_1}(\frac{1}{2}m(1-2it) + \frac{1}{4}(p_1-p_2+1))\Gamma_{p_1}(\frac{1}{2}m + \Delta)}{\Gamma_{p_1}(\frac{1}{2}m + \frac{1}{4}(p_1-p_2+1))\Gamma_{p_1}(\frac{1}{2}m(1-2it) + \Delta)}$$

(4.10)

$$\cdot \left|I - \frac{1}{m}\Theta^2\right|^{\frac{1}{2}m+\Delta} \cdot \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{\{(\frac{1}{2}m + \Delta)_\kappa\}^2}{(\frac{1}{2}m(1-2it) + \Delta)_\kappa m^k} \cdot \frac{C_\kappa(\Theta^2)}{k!},$$

with the abbreviated notation $\Delta = (p_1 + p_2 + 1)/4$. The first factor gives the characteristic function of $-2\rho\log\lambda_3$ under the null hypothesis $(\Theta = 0)$, which can be expanded asymptotically as

$$(4.11) \qquad (1-2it)^{-p_1p_2/2}\left[1 + \frac{p_1p_2}{48m^2}(p_1^2 + p_2^2 - 5)\left\{\frac{1}{(1-2it)^2} - 1\right\} + O(m^{-3})\right].$$

The second factor in (4.10) can be expanded by the formula (2.4) as

$$(4.12) \qquad \left|I - \frac{1}{m}\Theta^2\right|^{\frac{1}{2}m+\Delta} = \operatorname{etr}(-\frac{1}{2}\Theta^2) \cdot [1 - \frac{1}{m}(\Delta\operatorname{tr}\Theta^2 + \frac{1}{4}\operatorname{tr}\Theta^4)$$

$$+ \frac{1}{m^2}\{\frac{1}{2}(\Delta\operatorname{tr}\Theta^2 + \frac{1}{4}\operatorname{tr}\Theta^4)^2 - \frac{1}{2}\Delta\operatorname{tr}\Theta^4 - \frac{1}{6}\operatorname{tr}\Theta^6\} + O(m^{-3})]\,.$$

The third factor in (4.10) can be expressed by Lemma 4.2 as

$$(4.13)\quad \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{C_\kappa(\Theta^2)}{\{2(1-2it)\}^k} \cdot \frac{1}{k!} [1 + \frac{1}{m} \{2k\Delta + a_1(\kappa)\} (2 - \frac{1}{1-2it})$$

$$+ \frac{1}{m^2}\{\frac{1}{2}(2 - \frac{1}{1-2it})^2 (2k\Delta + a_1(\kappa))^2 + \frac{1}{3}(1 - \frac{1}{2(1-2it)^2})(k - 12\Delta^2 k - 12\Delta a_1(\kappa) - a_2(\kappa))\}$$

$$+ 0(m^{-3})],$$

which can be simplified by Lemma 4.1 as

$$(4.14)\quad \exp[\frac{tr\Theta^2}{2(1-2it)}] \cdot [1 + \frac{1}{m} \{\frac{2\Delta tr\Theta^2}{1-2it} + \frac{tr\Theta^4 - 2\Delta tr\Theta^2}{2(1-2it)^2} - \frac{tr\Theta^4}{4(1-2it)^3}\}$$
$$+ \frac{1}{m^2} \sum_{\alpha=1}^{6} g_{2\alpha}(1-2it)^{-\alpha} + 0(m^{-3})],$$

where

$$(4.15)\quad \begin{aligned}
g_2 &= 2\Delta^2 \, tr\, \Theta^2 \\
g_4 &= (3\Delta + \tfrac{1}{4}) tr\, \Theta^4 + (2\Delta^2 + \tfrac{1}{4})\, (tr\Theta^2)^2 - 4\Delta^2 \, tr\, \Theta^2 \\
g_6 &= \tfrac{5}{6} tr\Theta^6 + \Delta tr\Theta^4 \, tr\Theta^2 - (4\Delta + \tfrac{1}{2}) tr\Theta^4 - (2\Delta^2 + \tfrac{1}{2})(tr\Theta^2)^2 + 2\Delta^2 tr\Theta^2 \\
g_8 &= \tfrac{1}{8}(tr\Theta^4)^2 - tr\Theta^6 - \Delta tr\Theta^4 tr\Theta^2 + (\tfrac{3}{2}\Delta + \tfrac{1}{4}) tr\Theta^4 + (\tfrac{1}{2}\Delta^2 + \tfrac{1}{4})(tr\Theta^2)^2 \\
g_{10} &= -\tfrac{1}{8}(tr\Theta^4)^2 + \tfrac{1}{3}\, tr\, \Theta^6 + \tfrac{1}{4}\, \Delta \, tr\, \Theta^4 \, tr\, \Theta^2 \\
g_{12} &= \tfrac{1}{32}\, (tr\Theta^4)^2 .
\end{aligned}$$

It follows from (4.11), (4.12) and (4.14) that the characteristic function (4.10) of $-2\rho\log\lambda_3$ under K_γ can be expanded asymptotically as

$$(4.16)\quad (1-2it)^{-p_1p_2/2}\, etr[\frac{it\ tr\Theta^2}{1-2it}] \cdot [1 + \frac{1}{m} \{-\frac{1}{4}tr\Theta^4 - \Delta tr\Theta^2 + \frac{2\Delta tr\Theta^2}{1-2it}$$

$$+ \frac{tr\Theta^4 - 2\Delta tr\Theta^2}{2(1-2it)^2} - \frac{tr\Theta^4}{4(1-2it)^3}\} + \frac{1}{m^2} \{\frac{p_1p_2}{48}(p_1^2+p_2^2-5)(\frac{1}{(1-2it)^2} - 1) + \sum_{\alpha=0}^{6} h_{2\alpha}(1-2it)^{-\alpha}\}$$

$$+ 0(m^{-3})],$$

where the coefficients $h_{2\alpha}$ $(\alpha = 1, 2, \ldots, 6)$ are given by

(4.17)
$$
\begin{aligned}
h_0 &= \frac{1}{2}\left(\frac{1}{4}\mathrm{tr}\Theta^4 + \Delta\mathrm{tr}\Theta^2\right)^2 - \frac{1}{6}\mathrm{tr}\Theta^6 - \frac{1}{2}\Delta\mathrm{tr}\Theta^4 \\
h_2 &= -\frac{1}{2}\Delta\mathrm{tr}\Theta^4\mathrm{tr}\Theta^2 - 2\Delta^2(\mathrm{tr}\Theta^2)^2 + 2\Delta^2\mathrm{tr}\Theta^2 \\
h_4 &= -\frac{1}{8}(\mathrm{tr}\Theta^4)^2 - \frac{\Delta}{4}\mathrm{tr}\Theta^4\mathrm{tr}\Theta^2 + (3\Delta + \frac{1}{4})\mathrm{tr}\Theta^4 + (3\Delta^2 + \frac{1}{4})(\mathrm{tr}\Theta^2)^2 - 4\Delta^2\mathrm{tr}\Theta^2 \\
h_6 &= \frac{1}{16}(\mathrm{tr}\Theta^4)^2 + \frac{5}{4}\Delta\mathrm{tr}\Theta^4\mathrm{tr}\Theta^2 + \frac{5}{6}\mathrm{tr}\Theta^6 - (4\Delta + \frac{1}{2})\mathrm{tr}\Theta^4 - (2\Delta^2 + \frac{1}{2})(\mathrm{tr}\Theta^2)^2 \\
&\quad + 2\Delta^2\mathrm{tr}\Theta^2 \\
h_\alpha &= g_\alpha \quad (\alpha = 8, 10, 12).
\end{aligned}
$$

Inverting this characterisitc function, we can conclude the following theorem.

<u>Theorem 4.1</u>. Under the sequence of alternatives K_γ: $P^2 = m^{-1}\Theta^2$, the distribution of the LR statistic $-2\rho\log\lambda_3$ for testing the independence between p_1 and p_2 sets of variates $(p_1 \leqq p_2)$ can be expressed asymptotically for large $m = \rho N = N - (p_1+p_2+3)/2N$ as

(4.18)
$$
\begin{aligned}
P(-2\rho\log\lambda_3 < z) &= P(\chi^2_{f_3}(\delta^2) < z) + \frac{1}{m}\{-(\frac{1}{4}\mathrm{tr}\Theta^4 + \Delta\mathrm{tr}\Theta^2)P(\chi^2_{f_3}(\delta^2) < z) \\
&+ 2\Delta\mathrm{tr}\Theta^2 \cdot P(\chi^2_{f_3+2}(\delta^2) < z) + \frac{1}{2}(\mathrm{tr}\Theta^4 - 2\Delta\mathrm{tr}\Theta^2)P(\chi^2_{f_3+4}(\delta^2) < z) \\
&- \frac{1}{4}\mathrm{tr}\Theta^4 P(\chi^2_{f_3+6}(\delta^2) < z)\} + \frac{1}{m^2}[\frac{p_1p_2}{48}(p_1^2+p_2^2-5)\{P(\chi^2_{f_3+4}(\delta^2) < z) \\
&- P(\chi^2_{f_3}(\delta^2) < z)\} + \sum_{\alpha=0}^{6} h_{2\alpha}P(\chi^2_{f_3+2\alpha}(\delta^2) < z)] + O(m^{-3}),
\end{aligned}
$$

where the symbol $\chi^2_{f_3}(\delta^2)$ means the noncentral χ^2 variate with $f_3 = p_1p_2$ degrees of freedom and noncentrality parameter $\delta^2 = \frac{1}{2}\mathrm{tr}\Theta^2$ and the coefficients $h_{2\alpha}$ $(\alpha = 0, 1, \ldots, 6)$ are given by (4.17) with $\Delta = (p_1+p_2+1)/4$.

4.3. <u>Limiting distributions when</u> $\gamma \neq \frac{1}{2}$. We shall now consider the limiting distributions of the LR statistic $-2\rho\log\lambda_3$ under the sequences of alternatives K_γ: $P = m^{-\gamma}\Theta$ $(\gamma < \frac{1}{2})$. Noting that the moments (4.1) can be rewritten by the Kummer transformation formula as in Sugiura and Fujikoshi [12],

$$(4.19)\qquad E[\lambda_3^{\frac{2}{N}h}] = \frac{\Gamma_{p_1}(h + \frac{1}{2}(N-p_2-1))\Gamma_{p_1}(\frac{1}{2}(N-1))}{\Gamma_{p_1}(\frac{1}{2}(N-p_2-1))\Gamma_{p_1}(h + \frac{1}{2}(N-1))} \cdot |I - P^2|^h$$

$$\cdot\ {}_2F_1(h, h;\ h + \frac{1}{2}(N-1);\ P^2)$$

and considering the characteristic function of $(-2\rho\log\lambda_3)/m^{\frac{1}{2}-\gamma}$ based on the above moments, we can easily see that the first four gamma products tend to one and the hypergeometric function ${}_2F_1$ tends to $\exp[-2t^2 \mathrm{tr}\Theta^2]$ as $m \to \infty$. This gives the first part of the following theorem. Also, when $\gamma > \frac{1}{2}$, we can see from (4.1) that the characteristic function of $-2\rho\log\lambda_3$ tends to $(1-2it)^{-p_1p_2/2}$ as $m \to \infty$. Hence, we can imply the following theorem.

Theorem 4.2. Under the sequence of alternatives K_γ: $P = m^{-\gamma}\Theta$ the limiting distributions of the LR statistic $-2\rho\log\lambda_3$ for the hypothesis of independence are given in the following.

(1) When $0 < \gamma < \frac{1}{2}$, the statistic

$$(4.20)\qquad \frac{1}{m^{1/2-\gamma}}[-2\rho\log\lambda_3 - m\log|I - P^2|]$$

has asymptotically the normal distribution with mean zero and variance $4\mathrm{tr}\Theta^2$.

(2) When $\gamma > \frac{1}{2}$, the statistic $-2\rho\log\lambda_3$ has asymptotically the χ^2 distribution with p_1p_2 degrees of freedom.

4.4. Numerical example. Computing the one more term in the asymptotic formula of the characteristic function in the null case given in (4.11), we can get

$$(4.21)\qquad P_H(-2\rho\log\lambda_3 < z) = P(\chi^2_{f_3} < z) + \frac{\omega_2}{m^2}\{P(\chi^2_{f_3+4} < z) - P(\chi^2_{f_3} < z)\}$$

$$+ \frac{1}{m^4}[\omega_4\{P(\chi^2_{f_3+8} < z) - P(\chi^2_{f_3} < z)\} - \omega_2^2\,\{P(\chi^2_{f_3+4} < z) - P(\chi^2_{f_3} < z)\}] + O(m^{-5}),$$

where

$$\omega_2 = \frac{p_1p_2(p_1^2+p_2^2-5)}{48}$$

(4.22)

$$\omega_4 = \frac{1}{2}\ \omega_2^2 + \frac{p_1p_2}{1920}\{3p_1^4 + 3p_2^4 - 50\ p_1^2 - 50\ p_2^2 + 10\ p_1^2p_2^2 + 159\}\ .$$

Since the above formula has the same form as in the case of multivariate linear hypothesis given in Anderson [1, p.208], we can use the asymptotic formula for the percentage point of the LR statistic for multivariate linear hypothesis given by Hill and Davis [4] for our present purpose, that is, the 100α % point of $-2\rho\log\lambda_3$ is given by

$$u + \frac{1}{m^2}\frac{2u\ \omega_2}{f(f+2)}(u + f + 2) + \frac{1}{m^4}\Big[\frac{2u\ \omega_4}{f(f+2)(f+4)(f+6)}\{u^3 + (f+6)u^2$$

$$(4.23)\qquad + (f+6)(f+4)u + (f+6)(f+4)(f+2)\} - \frac{\omega_2^2\ u}{f^2(f+2)^2}\{u^3 + (f-2)u^2$$

$$+ (f-6)(f+2)u + (f-2)(f+2)^2\}\Big] + O(m^{-5}),$$

where $f = p_1p_2$ and u is determined such that $P(\chi^2_f > u) = \alpha$.

<u>Example 4.1</u>. When $N = 87$ and $p_1 = 2$, $p_2 = 3$, the asymptotic formula (4.22) gives the following approximate 5 % point.

first term	12.5916
term of order m^{-2}	0.0016
term of order m^{-4}	0.0000
approx. value	12.5932

Pillai and Jayachandran [8] gave the exact 5 % point as 12.593156 (computed from their Table 8. Lower 5% points of $\{W^{(2)}\}^{1/2}$ $m = 0$ and $n = 40$). For the alternative hypotheses K_1: $\rho_1^2 = 0.001$, $\rho_2^2 = 0.05$ and K_2: $\rho_1^2 = 0.05$, $\rho_2^2 = 0.1$, the following approximate powers are computed by the formula (4.18) in this section and (2.12) in Sugiura and Fujikoshi [12] respectively.

approximate power

	K_1	K_2
first term	0.28505	0.5231
second term	0.00680	0.2871
third term	0.00023	-0.0038
approx. power	0.2921	0.806

Pillai and Jayachandran [8] gave the exact power when the alternative hypothesis is K_1 as 0.2919 in their Table 9. Hence our approximate power 0.2921 shows good approximation to the exact value.

5. Asymptotic distribution of the LR criterion for $\Sigma_1 = \Sigma_2$.

5.1. Moments of the criterion under local alternatives. Let the $p \times 1$ vectors $X_{\alpha 1}, X_{\alpha 2}, \ldots, X_{\alpha N_\alpha}$ be a random sample form a normal population with mean vector μ_α and covariance matrix Σ_α for $\alpha = 1, 2$. The modified LR criterion for testing the hypothesis H: $\Sigma_1 = \Sigma_2$ against alternatives K: $\Sigma_1 \neq \Sigma_2$ with unknown mean vectors, is given by

$$(5.1) \qquad \lambda_4^* = \left(\frac{n^n}{n_1^{n_1} n_2^{n_2}}\right)^{p/2} \frac{|S_1|^{n_1/2} |S_2|^{n_2/2}}{|S_1 + S_2|^{n/2}},$$

where $S_j = \sum_{\alpha=1}^{N_j} (X_{j\alpha} - \bar{X}_j)(X_{j\alpha} - \bar{X}_j)'$ and $\bar{X}_j = N_j^{-1} \sum_{\alpha=1}^{N_j} X_{j\alpha}$ with $n_j = N_j - 1$ and $n = n_1 + n_2$. It is easy to see that the non-null distribution of λ_4^* depends only on p characteristic roots of $\Sigma_1\Sigma_2^{-1}$. Unbiasedness of this modified LR criterion was proved by Sugiura and Nagao [13]. The limiting distribution of $-2\log\lambda_4^*$ under fixed alternative hypothesis K was obtained by Sugiura [11]. We shall now consider the moments of the statistic λ_4^* under the sequence of alternatives K_n: $\Sigma_2^{-1/2}\,\Sigma_1\,\Sigma_2^{-1/2} = I + m^{-1}\,\Theta$, where the $p \times p$ matrix Θ is pd and $m = \rho n$ with the correction factor

$$(5.2)\qquad \rho = 1 - \frac{2p^2+3p-1}{6(p+1)}\left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n}\right) .$$

Without loss of generality, we may assume that the statistic S_1 has the Wishart distribution $W_p(n_1, \Gamma)$ and S_2 has $W_p(n_2, I)$, where $\Gamma = \Sigma_2^{-1/2}\,\Sigma_1\,\Sigma_2^{-1/2}$. Since S_1 and S_2 are independent, we can write the h-th moment of λ_4^* as

$$(5.3)\qquad E[\lambda_4^{*h}\,|K_n] = \left(\frac{n^n}{n_1^{n_1}\,n_2^{n_2}}\right)^{ph/2} \frac{1}{2^{np/2}\Gamma_p(n_1/2)\Gamma_p(n_2/2)} \int \frac{|S_1|^{\frac{n_1(1+h)-p-1}{2}}\,|S_2|^{\frac{n_2(1+h)-p-1}{2}}}{|\Gamma|^{n_1/2}\,|S_1+S_2|^{nh/2}}$$

$$\cdot\ \mathrm{etr}\{-\tfrac{1}{2}(S_1+S_2) + \tfrac{1}{2}(I-\Gamma^{-1})S_1\}\,dS_1\,dS_2,$$

where the range of integration is such that the two $p \times p$ symmetric matrices S_1 and S_2 are pd. We can expand the last part $\mathrm{etr}\{\frac{1}{2}(I - \Gamma^{-1})S_1\}$ in the above integration to infinite series by zonal polynomials as

$$\sum_{k=0}^{\infty}\ \sum_{(\kappa)} C_\kappa(\tfrac{1}{2}(I - \Gamma^{-1})S_1)/k! \ .$$

Transforming the variables (S_1, S_2) to (U_1, U_2) by $U_1 = S_1$ and $U_2 = U_1^{-1/2}\,S_2\,U_1^{-1/2}$ ($U_1^{1/2}$ is chosen pd) with the Jacobian $|\partial(S_1,S_2)/\partial(U_1,U_2)|$

$= |U_1|^{(p+1)/2}$ and integrating out with respect to U_1 by the formula due to Constantine [2]

(5.4) $$\int_{S>0} \{\mathrm{etr}(-RS)\}|S|^{t-(p+1)/2} C_\kappa(ST)dS = \Gamma_p(t)(t)_\kappa |R|^{-t} C_\kappa(TR^{-1}) ,$$

which holds for any $p \times p$ pd matrix R, symmetric matrix T and complex number t satisfying $R(t) > (p-1)/2$, we can rewrite the moment (5.3) as

(5.5) $$\left(\frac{n^n}{n_1^{n_1} n_2^{n_2}}\right)^{ph/2} \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)} \sum_{k=0}^{\infty} \sum_{(\kappa)} \int \frac{|U_2|^{\frac{n_2(1+h)-p-1}{2}} (\frac{1}{2}n)_\kappa}{|\Gamma|^{n_1/2} |I+U_2|^{n(1+h)/2}}$$

$$\cdot\ C_\kappa((I-\Gamma^{-1})(I+U_2)^{-1})/k!\ d\,U_2 .$$

Putting $V = (I + U_2)^{-1}$ with the Jacobian $|\partial U_2/\partial V| = |V|^{-p-1}$ and integrating out with respect to V by the formula due to Constantine [2]

(5.6) $$\int_0^I |S|^{t-(p+1)/2} |I - S|^{u-(p+1)/2} C_\kappa(RS)dS = \frac{\Gamma_p(t)\Gamma_p(u)(t)_\kappa}{\Gamma_p(t+u)(t+u)_\kappa} C_\kappa(R) ,$$

which holds for any $p \times p$ pd matrix R, we can express the h-th moment (5.3) as

(5.7) $$\left(\frac{n^n}{n_1^{n_1} n_2^{n_2}}\right)^{ph/2} \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)} \cdot \frac{\Gamma_p(\frac{1}{2}n_1(1+h))\Gamma_p(\frac{1}{2}n_2(1+h))}{\Gamma_p(\frac{1}{2}n(1+h))\ |\Gamma|^{n_1/2}}$$

$$\cdot\ {}_2F_1(\tfrac{1}{2}n,\ \tfrac{1}{2}n_1(1+h);\ \tfrac{1}{2}n(1+h);\ \ I - \Gamma^{-1}).$$

5.2. Asymptotic distribution under K_n. Applying the Kummer transformation formula ${}_2F_1(a_1, a_2; b; Z) = |I - Z|^{-a_2} \cdot {}_2F_1(b-a_1, a_2; b; -Z(I-Z)^{-1})$ in James [5] to (5.7), we can write the characteristic function of $-2\rho\log\lambda_4^*$ under K_n as

$$\left(\frac{m^m}{m_1^{m_1} m_2^{m_2}}\right)^{-pit} \frac{\Gamma_p(\frac{1}{2}m+\Delta)\Gamma_p(\frac{1}{2}m_1(1-2it)+\Delta_1)\Gamma_p(\frac{1}{2}m_2(1-2it)+\Delta_2)}{\Gamma_p(\frac{1}{2}m_1+\Delta_1)\Gamma_p(\frac{1}{2}m_2+\Delta_2)\Gamma_p(\frac{1}{2}m(1-2it)+\Delta)}$$

(5.8)

$$\cdot \; |I + \frac{1}{m}\Theta|^{-m_1 it} \cdot \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(-mit)_\kappa (\frac{1}{2}m_1(1-2it)+\Delta_1)_\kappa}{(\frac{1}{2}m(1-2it)+\Delta)_\kappa \, m^k} \cdot \frac{C_\kappa(-\Theta)}{k!} ,$$

where $m_\alpha = \rho\, n_\alpha$ and $\Delta_\alpha = \frac{1}{2}(n_\alpha - m_\alpha) = O(1)$ with $m = m_1 + m_2$ and $\Delta = \Delta_1 + \Delta_2$. The first factor in the above expression gives the characteristic function of $-2\rho\log\lambda_4^*$ under the null hypothesis, which can be expanded asymptotically for large m with fixed $\rho_\alpha = m_\alpha/m$ $(\alpha = 1, 2)$ in the usual way as in the previous sections, giving (Anderson [1,p.255])

$$(5.9) \qquad (1-2it)^{-f_4/2} \left[1 + \frac{\omega_2}{m^2}\left\{\frac{1}{(1-2it)^2} - 1\right\} + O\left(\frac{1}{m^3}\right)\right] ,$$

where $f_4 = p(p+1)/2$ and

$$(5.10) \qquad \omega_2 = \frac{1}{48}p(p^2-1)(p+2)\left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} - 1\right) - \frac{1}{2}p(p+1)\Delta^2 .$$

The second factor in (5.8) can be expanded easily by the formula (2.4) as

$$|I + \frac{1}{m}\Theta|^{-m_1 it} = \operatorname{etr}(-\rho_1 it\Theta) \cdot \left[1 + \frac{1}{2m}\rho_1 it \cdot \operatorname{tr}\Theta^2\right.$$

(5.11)

$$\left. + \frac{1}{m^2}\left\{-\frac{1}{3}\rho_1 it\cdot\operatorname{tr}\Theta^3 + \frac{1}{8}(\rho_1 it\cdot\operatorname{tr}\Theta^2)^2\right\} + O\left(\frac{1}{m^3}\right)\right] .$$

The third factor in (5.8) can be written, by Lemma 4.2 , as

$$\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{(\rho_1 it)^k C_\kappa(\Theta)}{k!}\Big[1+\frac{a_1(\kappa)}{m}\Big\{-\frac{1}{2it}+\frac{\rho_2/\rho_1}{1-2it}\Big\}+\frac{1}{m^2}\Big\{-\frac{2\Delta a_1(\kappa)}{(1-2it)^2}\cdot\frac{\rho_2}{\rho_1}$$

(5.12)

$$+(k-a_2(\kappa))\Big(\frac{1}{24(it)^2}+\frac{\rho_1^{-2}-1}{(1-2it)^2}\Big)+\frac{1}{2}a_1(\kappa)^2\Big(\frac{\rho_2/\rho_1}{1-2it}-\frac{1}{2it}\Big)^2\Big\}+O\Big(\frac{1}{m^3}\Big)\Big],$$

which can be simplified, by Lemma 4.1 , as

$$\mathrm{etr}(\rho_1 it\Theta)\cdot\Big[1+\frac{1}{m}\mathrm{tr}\Theta^2\Big\{-\frac{\rho_1^2 it}{2}-\frac{\rho_1\rho_2(it)^2}{1-2it}\Big\}+\frac{1}{m^2}\Big\{-\frac{\rho_1\rho_2}{2}\Delta\mathrm{tr}\Theta^2\Big(1-\frac{1}{1-2it}\Big)^2$$

(5.13)

$$-\frac{1}{24}\Big(1-\frac{2(1-\rho_1^2)}{1-2it}+\frac{1-\rho_1^2}{(1-2it)^2}\Big)(4\rho_1 it\cdot\mathrm{tr}\Theta^3+3\mathrm{tr}\Theta^2+3[\mathrm{tr}\Theta]^2)$$

$$+\frac{1}{8}\Big(-1+\frac{1-\rho_1}{1-2it}\Big)^2([\rho_1 it\cdot\mathrm{tr}\Theta^2]^2+4\rho_1 it\cdot\mathrm{tr}\Theta^3+\mathrm{tr}\Theta^2+[\mathrm{tr}\Theta]^2)\Big\}+O\Big(\frac{1}{m^3}\Big)\Big].$$

Multiplying the three factors (5.9), (5.11), (5.13) together and arranging the terms according to the power of $(1-2it)^{-1}$, we can get the asymptotic formula for the characteristic function of $-2\rho\log\lambda_4^*$ under K_n as

$$(1-2it)^{-f_4/2}\Big[1+\frac{\rho_1\rho_2}{4m}\mathrm{tr}\Theta^2\Big\{\frac{1}{1-2it}-1\Big\}+\frac{1}{m^2}\sum_{\alpha=0}^{2}g_{2\alpha}(1-2it)^{-\alpha}$$

(5.14)

$$+\frac{\omega_2}{m^2}\Big\{\frac{1}{(1-2it)^2}-1\Big\}+O\Big(\frac{1}{m^3}\Big),$$

where

$$g_0=\rho_1\rho_2\Big\{\frac{\rho_1\rho_2}{32}(\mathrm{tr}\Theta^2)^2-\frac{\rho_1^{-2}}{6}\mathrm{tr}\Theta^3-\frac{\Delta}{2}\mathrm{tr}\Theta^2\Big\}$$

(5.15)

$$g_2=\rho_1\rho_2\Big\{-\frac{\rho_1\rho_2}{16}(\mathrm{tr}\Theta^2)^2-\frac{\rho_2}{2}\mathrm{tr}\Theta^3+\Big(\Delta+\frac{1}{4}\Big)\mathrm{tr}\Theta^2+\frac{1}{4}(\mathrm{tr}\Theta)^2\Big\}$$

$$g_4=\rho_1\rho_2\Big\{\frac{\rho_1\rho_2}{32}(\mathrm{tr}\Theta^2)^2+\frac{1-2\rho_1}{6}\mathrm{tr}\Theta^3-\Big(\frac{\Delta}{2}+\frac{1}{4}\Big)\mathrm{tr}\Theta^2-\frac{1}{4}(\mathrm{tr}\Theta)^2\Big\}.$$

Inverting the characteristic function (5.14), we can get the following theorem.

Theorem 5.1. Under the sequence of alternatives $K_n: \Sigma_2^{-1/2}\Sigma_1\Sigma_2^{-1/2} = I + m^{-1}\Theta$, the distribution of the modified LR criterion given by (5.1) can be expanded asymptotically for large $m = \rho n$ with fixed $\rho_\alpha = n_\alpha/n$ $(\alpha = 1, 2)$ as

$$
\begin{aligned}
P(-2\rho\log\lambda_4^* < z) &= P(\chi^2_{f_4} < z) + \frac{\rho_1\rho_2}{4m}\mathrm{tr}\Theta^2\{P(\chi^2_{f_4+2} < z) - P(\chi^2_{f_4} < z)\} \\
&\quad + \frac{1}{m^2}[\omega_2\{P(\chi^2_{f_4+4} < z) - P(\chi^2_{f_4} < z)\} + \sum_{\alpha=0}^{2} g_{2\alpha} P(\chi^2_{f_4+2\alpha} < z)] + O(\frac{1}{m^3}),
\end{aligned} \tag{5.16}
$$

where $f_4 = p(p+1)/2$, and the correction factor ρ is given by (5.9). The coefficients $g_{2\alpha}$ $(\alpha = 0, 1, 2)$ are given by (5.15) with $\Delta = (n-m)/2$ and ω_2 is given by (5.10).

5.3. Numerical examples. Evaluating the asymptotic formula (5.9) for the characteristic function under H more precisely, we can get

$$
\begin{aligned}
P(-2\rho\log\lambda_4^* < z) &= P(\chi^2_{f_4} < z) + \frac{\omega_2}{m^2}\{P(\chi^2_{f_4+4} < z) - P(\chi^2_{f_4} < z)\} \\
&\quad + \frac{\omega_3}{m^3}\{P(\chi^2_{f_4+6} < z) - P(\chi^2_{f_4} < z)\} + \frac{1}{m^4}[\omega_4\{P(\chi^2_{f_4+8} < z) - P(\chi^2_{f_4} < z)\} \\
&\quad - \omega_2^2\{P(\chi^2_{f_4+4} < z) - P(\chi^2_{f_4} < z)\}] + O(m^{-5}),
\end{aligned} \tag{5.17}
$$

where ω_2 is defined by (5.10) and

$$
\begin{aligned}
\omega_3 &= \frac{p}{720}(6p^4 + 15p^3 - 10p^2 - 30p + 3)(\rho_1^{-3} + \rho_2^{-3} - 1) \\
&\quad - \frac{p}{12}(p^2 - 1)(p + 2)\Delta(\rho_1^{-2} + \rho_2^{-2} - 1) + \frac{4}{3}p(p + 1)\Delta^3
\end{aligned}
$$

$$(5.18)\qquad \omega_4 = \frac{1}{2}\omega_2^2 + \frac{p}{480}(p-1)(2p^4 + 8p^3 + 3p^2 - 17p - 14)(\rho_1^{-4} + \rho_2^{-4} - 1)$$
$$- \frac{p}{120}(6p^4 + 15p^3 - 10p^2 - 30p + 3)\Delta(\rho_1^{-3} + \rho_2^{-3} - 1)$$
$$+ \frac{p}{4}(p^2 - 1)(p+2)\Delta^2(\rho_1^{-2} + \rho_2^{-2} - 1) - 3p(p+1)\Delta^4.$$

Applying the inverse expansion formula due to Hill and Davis [4] to the above expression, we get the following asymptotic formula for the 100α% point of $-2\rho\log\lambda_4^*$:

$$u + \frac{2\omega_2 u(u+f+2)}{m^2 f(f+2)} + \frac{2\omega_3 u}{m^3 f(f+2)(f+4)}\{u^2 + (f+4)u + (f+4)(f+2)\}$$
$$(5.19)\qquad + \frac{1}{m^4}\Big[\frac{2\omega_4 u}{f(f+2)(f+4)(f+6)}\{u^3 + (f+6)u^2 + (f+6)(f+4)u + (f+6)(f+4)(f+2)\}$$
$$- \frac{\omega_2^2 u}{f^2(f+2)^2}\{u^3 + (f-2)u^2 + (f+2)(f-6)u + (f+2)^2(f-2)\}\Big] + 0(\frac{1}{m^5}) ,$$

where u is defined such that $P(\chi_f^2 > u) = \alpha$ and $f = p(p+1)/2$.

Example 5.1. The approximate 5% points of the statistic $-2\rho\log\lambda_4^*$ given by (5.1) are computed by the formula (5.19) in the following two cases.

	$p = 1$	$p = 2$
	$n_1 = 4,\ n_2 = 20$	$n_1 = 13,\ n_2 = 63$
first term	3.8415	7.81473
term of order m^{-2}	-0.0389	0.00777
term of order m^{-3}	-0.0045	-0.00010
term of order m^{-4}	0.0030	0.00001
approx. value	3.801	7.82241

The exact 5% point in the first case according to Table 743 in Ramachandran [10] is 3.80. Thus our approximate value is accurate to 2 decimal palces. In the univariate case, an asymptotic expansion of the distribution of $-2\rho\log\lambda_4^*$

under the fixed alternative Hypothesis has been derived by Sugiura and Nagao [14], by which approximate power, when alternative hypothesis is not near to the null hypothesis, can be computed.

Example 5.2. Specifying the alternative hypothesis K as $\Sigma_2^{-1/2} \Sigma_1 \Sigma_2^{-1/2} = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_p)$ and using the 5% points obtained in Example 5.1 , we can get the approximate values of the power of the modified LR test from the formula (5.16).

	$p = 1$	$p = 2$
	$n_1 = 4$, $n_2 = 20$	$n_1 = 13$, $n_2 = 63$
	$\delta_1 = 0.5$	$\delta_1 = \delta_2 = 1.05$
first term	0.0512	0.049828
second term	0.0443	0.001479
third term	0.0083	-0.000008
approx. power	0.104	0.05130

From Table 744a in Ramachandran [10], we can see that the exact power in the first case is 0.113. In the second case, we can see from Table 2 in Pillai and Jayachandran [9] $(m = 5,\ n = 30)$ that the powers of the other tests are given by

Roy's largest root criterion (root of $S_1 S_2^{-1}$)	0.067021
Lawley-Hotelling's trace criterion ($\mathrm{tr}\, S_1 S_2^{-1}$)	0.070087
Pillai's criterion ($\mathrm{tr}\, S_1(S_1 + S_2)^{-1}$)	0.070338
Wilks' criterion ($\lvert I + S_1 S_2^{-1}\rvert$)	0.070273.

However, this does not mean that the modified LR criterion is worse. Because the above four test criteria are to test the null hypothesis $H: \Sigma_1 = \Sigma_2$ against the alternatives $K: \delta_\alpha \geq 1$ for $\alpha = 1, 2, \ldots, p$ and $\sum_{\alpha=1}^{p} \delta_\alpha > 1$, which is an extension of the one-sided test and the modified LR criterion is against all alternatives K: $\Sigma_1 \neq \Sigma_2$, which is an extension of the two-sided test.

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