NOTE ON SOME FORMULAS FOR WEIGHTED SUMS OF ZONAL POLYNOMIALS[1]

by

NARIAKI SUGIURA[2]

Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mineo Series No. 610

JANUARY 1969

1 The research was partially sponsored by the National Science Foundation Grant No. GU-2059, U. S. Air Force Grant AFOSR 68-1415, and the Sakko-kai Foundation.
2 University of North Carolina and Hiroshima University.

# Note on Some Formulas for Weighted Sums of Zonal Polynomials[†]

by

Nariaki Sugiura

1. Introduction. The first purpose of this paper is to prove a stronger result than the previous lemma for the sum of zonal polynomials given in Sugiura and Fujikoshi [5], which played an important role in deriving the asymptotic expansions of the non-null distributions of the likelihood ratio criteria in multivariate analysis by these authors, Sugiura [6] and of the generalized variance by Fujikoshi [1].

THEOREM 1. Let $C_\kappa(Z)$ be the zonal polynomial of degree $k$ corresponding to a partition $\kappa = \{k_1, k_2, \ldots, k_p\}$ of $k$ $(k_1 \geq k_2 \geq \cdots \geq k_p \geq 0)$ for a $p \times p$ positive definite matrix $Z$. Put

$$a_1(\kappa) = \sum_{\alpha=1}^{p} k_\alpha (k_\alpha - \alpha)$$

$$a_2(\kappa) = \sum_{\alpha=1}^{p} k_\alpha (4k_\alpha - 6\alpha k_\alpha + 3\alpha^2). \tag{1.1}$$

Then the following equalities hold:

$$\sum_{(\kappa)} a_1(\kappa) C_\kappa(Z) = k(k-1)\, \mathrm{tr}Z^2\, (\mathrm{tr}Z)^{k-2} \tag{1.2}$$

$$\sum_{(\kappa)} a_1(\kappa)^2 C_\kappa(Z) = k(k-1)\, [\{\mathrm{tr}Z^2 + (\mathrm{tr}Z)^2\}\, (\mathrm{tr}Z)^{k-2} + 4(k-2)\, \mathrm{tr}Z^3\, (\mathrm{tr}Z)^{k-3} + (k-2)(k-3)(\mathrm{tr}Z^2)^2\, (\mathrm{tr}Z)^{k-4}] \tag{1.3}$$

$$\sum_{(\kappa)} a_2(\kappa) C_\kappa(Z) = k[(\mathrm{tr}Z)^k + 3(k-1)\, \{\mathrm{tr}Z^2 + (\mathrm{tr}Z)^2\}\, (\mathrm{tr}Z)^{k-2} + 4(k-1)(k-2)(k-3)\, \mathrm{tr}Z^3\, (\mathrm{tr}Z)^{k-3}], \tag{1.4}$$

where the symbol $\Sigma_{(\kappa)}$ means the sum of all possible partition $\kappa$ of $k$.

[†] The research was partially sponsored by the National Science Foundation Grant No. GU-2059, United States Air Force Grant AFOSR-68-1415, and the Sakko-kai Foundation.

Dividing by k! on both sides of each of the equations (1.2), (1.3) and (1.4) and summing with respect to k from zero to infinity, we obtain the lemma given by Sugiura and Fujikoshi [5].

The second purpose of this paper is to give an alternative proof of the following theorem due to Fujikoshi [2], by using a differential equation for zonal polynomials obtained recently by James [4].

THEOREM 2. (Fujikoshi). With the same notation as in Theorem 1, put

$$(1.5)\qquad (b)_\kappa = \prod_{\alpha=1}^{p} (b - \tfrac{\alpha-1}{2})(b + 1 - \tfrac{\alpha-1}{2}) \dots (b + k_\alpha - 1 - \tfrac{\alpha-1}{2}).$$

Then the following equalities hold:

$$(1.6)\qquad \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_\kappa a_1(\kappa) C_\kappa(Z)}{k!} = \frac{b}{2} |I - Z|^{-b} \{(2b + 1)\ trW^2 + (trW)^2\}$$

$$(1.7)\qquad \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_\kappa a_1(\kappa)^2 C_\kappa(Z)}{k!} = \frac{b}{4} |I - Z|^{-b} \{(2b + 1)(2b^2 + b + 2)(trW^2)^2$$

$$+ 2(2b^2 + b + 2)trW^2(trW)^2 + 2(8b^2 + 10b + 5)trW^4 + 8(2b + 1)trW^3\ trW$$

$$+ b(trW)^4 + 8(2b^2 + 3b + 2)\ trW^3 + 12(2b + 1)trW^2\ trW + 4(trW)^3$$

$$+ 2(2b + 1)(trW)^2 + 2(2b + 3)\ trW^2\},$$

where the positive definite matrix $Z$ is assumed to have characteristic roots less than one, and $W = Z(I - Z)^{-1}$.

2. Proof of Theorem 1. Since the zonal polynomial $C_\kappa(Z)$ is a homogeneous symmetric polynomial of degree $k$ with respect to the $p$ characteristic roots of $Z$, it is sufficient to prove the equalities in Theorem 1 and Theorem 2, when $Z$ is a diagonal matrix $Y = \mathrm{diag}(y_1, y_2, \dots, y_p)$. Fujikoshi [2] has shown, in the proof of Theorem 2, that the following differential relations hold:

$$(2.1)\qquad C_\kappa(Y)a_1(\kappa) = tr(Y\partial)^2\ C_\kappa(\Sigma)|_{\Sigma=Y}$$

(2.2) $$C_\kappa(Y)\,\{3a_1(\kappa)^2 - a_2(\kappa) + k\} = [3\{\mathrm{tr}(Y\partial)^2\}^2 + 8\mathrm{tr}(Y\partial)^3]C_\kappa(\Sigma)|_{\Sigma=Y},$$

where the symbol $\partial$ means a matrix of differential operators given by $\partial = (\frac{1}{2}(1+\delta_{ij})\partial/\partial\sigma_{ij})$, operating on a positive definite matrix $\Sigma = (\sigma_{ij})$, ($\delta_{ij}$ is a Kronecker delta.) The proof of this formula is based on the asymptotic expression of the equality (1.22) in Sugiura and Fujikoshi [5]. By the formula $\Sigma_{(\kappa)} C_\kappa(\Sigma) = (\mathrm{tr}\Sigma)^k$ in James [3], we have from (2.1)

(2.3) $$\sum_{(\kappa)} a_1(\kappa)C_\kappa(Y) = \sum_{\alpha=1}^{p} y_\alpha^2(\partial^2/\partial\sigma_{\alpha\alpha}^2)(\mathrm{tr}\Sigma)^k|_{\Sigma=Y} = k(k-1)\mathrm{tr}Y^2(\mathrm{tr}Y)^{k-2}$$

(2.4) $$\sum_{(\kappa)} a_1(\kappa)^2C_\kappa(Y) = \sum_{(\kappa)} \mathrm{tr}(Y\partial)^2a_1(\kappa)C_\kappa(\Sigma)|_{\Sigma=Y} = \mathrm{tr}(Y\partial)^2k(k-1)(\mathrm{tr}\Sigma^2)(\mathrm{tr}\Sigma)^{k-2}|_{\Sigma=Y}$$

$$= k(k-1)\,\{\sum_{\alpha=1}^{p} y_\alpha^2(\partial^2/\partial\sigma_{\alpha\alpha}^2) + \tfrac{1}{2}\sum_{\alpha<\beta} y_\alpha y_\beta(\partial^2/\partial\sigma_{\alpha\beta}^2)\}\,\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^{k-2}|_{\Sigma=Y}$$

$$= k(k-1)\,\{2\mathrm{tr}Y^2(\mathrm{tr}\Sigma)^{k-2} + 4(k-2)\mathrm{tr}Y^3(\mathrm{tr}\Sigma)^{k-3} + (k-2)(k-3)(\mathrm{tr}Y^2)^2(\mathrm{tr}\Sigma)^{k-4}$$

$$+ 2\sum_{\alpha<\beta} y_\alpha y_\beta(\mathrm{tr}\Sigma)^{k-2}\}|_{\Sigma=Y},$$

which imply equalities (1.2) and (1.3) respectively. From (2.2) we have

(2.5) $$\sum_{(\kappa)} a_2(\kappa)C_\kappa(Y) = 3\sum_{(\kappa)} a_1(\kappa)^2C_\kappa(Y) + k(\mathrm{tr}Y)^k - [3\{\sum_{\alpha=1}^{p} y_\alpha^4(\partial^4/\partial\sigma_{\alpha\alpha}^4)$$

$$+ 2\sum_{\alpha<\beta} y_\alpha^2y_\beta^2(\partial^4/\partial\sigma_{\alpha\alpha}^2\partial\sigma_{\beta\beta}^2)\} + 8\sum_{\alpha=1}^{p} y_\alpha^3(\partial^3/\partial\sigma_{\alpha\alpha}^3)](\mathrm{tr}\Sigma)^k|_{\Sigma=Y},$$

which yields (1.4) of Theorem 1.

It is interesting to note that the first two equations (1.2) and (1.3) in Theorem 1 can also be obtained from the following linear partial differential equation of second degree derived by James [4]:

(2.6)

$$\sum_{\alpha=1}^{p} y_\alpha^2(\partial^2/\partial y_\alpha^2)C_\kappa(Y) + \sum_{\alpha\neq\beta} y_\alpha^2(y_\alpha - y_\beta)^{-1}(\partial/\partial y_\alpha)C_\kappa(Y) = \{a_1(\kappa) + (p-1)k\}C_\kappa(Y).$$

Summing both sides of the above formula with respect to $\kappa$ for fixed $k$, yields equation (1.2). Operating $\Sigma_{(\kappa)}a_1(\kappa)$ on both sides of (2.6) yields equation (1.3).

3. Alternative proof of Theorem 2. Noting that

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_\kappa C_\kappa(Z)}{k!} = |I - Z|^{-b}, \tag{3.1}$$

when all characteristic roots of positive definite matrix $Z$ are less than one (James [3]), we can get from (2.6)

(3.2)

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_\kappa a_1(\kappa) C_\kappa(Y)}{k!} = \{\sum_{\alpha=1}^{p} y_\alpha^2(\partial^2/\partial y_\alpha^2) + \sum_{\alpha \neq \beta} y_\alpha^2 (y_\alpha - y_\beta)^{-1}(\partial/\partial y_\alpha)\}|I-Y|^{-b}$$
$$- (p-1)(d/dt)\ |I - tY|^{-b}|_{t=1}$$
$$= b|I - Y|^{-b}\ \{(b+1)\mathrm{tr}W^2 + \sum_{\alpha \neq \beta} y_\alpha^2 (y_\alpha - y_\beta)^{-1}(1 - y_\alpha)^{-1} - (p-1)\mathrm{tr}W\}$$

where $W = Y(I-Y)^{-1}$. The second term in the above equation can be simplified by noting $(I-Y)^{-1} = I + W$ as

(3.3)

$$\sum_{\alpha \neq \beta} y_\alpha^2 (y_\alpha - y_\beta)^{-1}(1 - y_\alpha)^{-1} = \sum_{\alpha < \beta} (y_\alpha + y_\beta - y_\alpha y_\beta)(1 - y_\alpha)^{-1}(1 - y_\beta)^{-1}$$
$$= \tfrac{1}{2}\{2\mathrm{tr}W\ \mathrm{tr}(I + W) - (\mathrm{tr}W)^2 - 2\mathrm{tr}W(I + W) + \mathrm{tr}W^2\}$$
$$= \tfrac{1}{2}\{2(p-1)\mathrm{tr}W + (\mathrm{tr}W)^2 - \mathrm{tr}\ W^2\},$$

which implies the first equation (1.6) in Theorem 2. From the differential equation (2.6), we have

(3.4)

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(b)_\kappa a_1(\kappa)^2 C_\kappa(Y)}{k!} = \{\sum_{\alpha=1}^{p} y_\alpha^2 \frac{\partial^2}{\partial y_\alpha^2} + \sum_{\alpha \neq \beta} \frac{y_\alpha^2}{y_\alpha - y_\beta} \frac{\partial}{\partial y_\alpha}\}\ f(Y)|I - Y|^{-b}$$
$$- (p-1)(d/dt)\ |I - tY|^{-b}\ f(tY)|_{t=1},$$

where $f(Y) = (b/2)\ \{(2b + 1)\mathrm{tr}\ W^2 + (\mathrm{tr}W)^2\}$ with $W = Y(I-Y)^{-1}$. The first term in the right hand side of (3.4) can be verified after some computation as

(3.5)

$$\frac{b}{2}\ |I - Y|^{-b}\ \{b(b+1)(2b+1)(trW^2)^2 + b(b+1)trW^2(trW)^2 + 8(b+1)^2trW^4$$

$$+ 4(b+1)trW\ trW^3 + 4(2b^2+5b+3)trW^3 + 4(b+1)trW\ trW^2 + 4(b+1)trW^2\}\ .$$

The second term in (3.4) can be written as

(3.6)

$$\frac{b}{2}\ |I - Y|^{-b}\ [2(2b+1)\sum_{\alpha\neq\beta} y_\alpha^3(y_\alpha-y_\beta)^{-1}(1-y_\alpha)^{-3} + 2trW\sum_{\alpha\neq\beta} y_\alpha^2(y_\alpha-y_\beta)^{-1}(1-y_\alpha)^{-2}$$

$$+ b\{(2b+1)trW^2 + (trW)^2\}\sum_{\alpha\neq\beta} y_\alpha^2(y_\alpha-y_\beta)^{-1}(1-y_\alpha)^{-1}].$$

Noting that

(3.7)

$$\sum_{\alpha\neq\beta}\frac{y_\alpha^3}{(y_\alpha-y_\beta)(1-y_\alpha)^3} = \sum_{\alpha<\beta}\ \frac{y_\alpha^2}{(1-y_\alpha)^3(1-y_\beta)} + \frac{y_\alpha y_\beta}{(1-y_\alpha)^2(1-y_\beta)^2} + \frac{y_\beta^2}{(1-y_\alpha)(1-y_\beta)^3}\ \}$$

$$= \tfrac{1}{2}\{- 3trW^4 + 2trW^3trW + (trW^2)^2 + 2(p-3)trW^3 + 4trW^2trW + (2p-3)trW^2 + (trW)^2\}$$

(3.8)

$$\sum_{\alpha\neq\beta} y_\alpha^2(y_\alpha-y_\beta)^{-1}(1-y_\alpha)^{-2} = \sum_{\alpha<\beta}\ \{y_\alpha(1-y_\alpha)^{-2}(1-y_\beta)^{-1} + y_\beta(1-y_\alpha)^{-1}(1-y_\beta)^{-2}\}$$

$$= - trW^3 + trW^2trW + (trW)^2 + (p-2)trW^2 + (p-1)trW,$$

we can simplify the second term in (3.4) as

(3.9)

$$\frac{b}{2}\ |I - Y|^{-b}\ [(b^2+2)trW^2(trW)^2 + \tfrac{1}{2}b(trW)^4 + (1-\tfrac{1}{2}b)(2b+1)(trW^2)^2$$

$$- 3(2b+1)trW^4 + 4btrW^3\ trW + \{8b + 2 + (2b^2 + b + 2)(p-1)\}\ trW^2\ trW$$

$$+ \{b(p-1) + 2\}(trW)^3 + 2(p-3)(2b+1)trW^3 + (2b+1)(2p-3)trW^2 + \{2b + 1 + 2(p-1)\}(trW)^2].$$

The third term in (3.4) can be written as

(3.10)

$$(d/dt)\ |I - tY|^{-b}\ f(tY)\Big|_{t=1} = \tfrac{1}{2}b\,|I - Y|^{-b}\ \{2(2b+1)\mathrm{tr}W^3 + (2b^2+b+2)\mathrm{tr}W^2\ \mathrm{tr}W$$

$$+ b(\mathrm{tr}W)^3 + 2(\mathrm{tr}W)^2 + 2(2b+1)\mathrm{tr}W^2\}.$$

Substituting (3.5), (3.9), and (3.10) for the right hand side of (3.4), we can derive the second equality (1.7) in Theorem 2.

Fujikoshi [2] obtained a futher formula concerning

$$\Sigma_{k=0}^{\infty}\ \Sigma_{(\kappa)}\ a_2(\kappa)\ C_{\kappa}(Z)\ /\ k!\ ,$$

based on the equality (2.2). It seems difficult, however, to give an alternative proof from the formula (2.6).

## REFERENCES

[1] Fujikoshi, Y. (1968). Asymptotic expansion of the distribution of the generalized variance in the non-central case. J. Sci. Hiroshima University. 32. In press.

[2] Fujikoshi, Y. (1968). Asymptotic expansions of the distributions of some statistics in multivariate analysis. (Reported in the symposium on multivariate analysis at the Research Institute for Mathematical Sciences. (Kyoto) August, 1968)

[3] James, A.T. (1964). The distribution of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 35 475-501.

[4] James, A.T. (1968). Calculation of zonal polynomial coefficients by use of the Laplace-Beltrami operator. Ann. Math. Statist. 39 1711-1718.

[5] Sugiura, N. and Fujikoshi, Y. (1969). Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence. Ann. Math. Statist. 40 No. 3. To appear.

[6] Sugiura, N. (1969) Asymptotic distributions of the likelihood ratio criteria for covariance matrix under local alternatives. To appear.