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A MARTINGALE DECOMPOSITION THEOREM

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Let Z be a random variable with $E|Z| < \infty$ and define recursively

(1) $$Z_0 = EZ, \quad Z_n = E^{F_n} Z,$$

where

(2) $$F_n = \mathcal{B}(Z_{n-1}, I(Z \geq Z_{n-1})) \quad \text{for} \quad n = 1, 2, \ldots \quad .$$[1]

The Z_n sequence constitutes a martingale decomposition of Z in the sense of the following

<u>THEOREM</u>.

(i) $Z_0, Z_1, \ldots, Z_n, \ldots, Z$ is a martingale.

(ii) The conditional distribution of Z_n given Z_{n-1} is a one or two point distribution a.s. for $n = 1, 2, \ldots$.

(iii) $Z_n \to Z$ a.s. as $n \to \infty$.

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[1] We shall assume that everything is defined on a basic probability space (Ω, F, P). For an arbitrary event $A \in F$ and arbitrary random vector W, we denote $I(A)$ and $\mathcal{B}(W)$ as the indicator function (taking the value 1 <u>on</u> A and 0 off A) and the σ-field generated by W respectively. $\bar{\mathcal{B}}(W)$ will refer to the smallest σ-field containing $\mathcal{B}(W)$ and the null sets of F.

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Proof. It is useful to define a closely related sequence by

(3) $$Y_0 = EZ, \quad Y_n = E^{\mathcal{G}_n} Z,$$

where

(4) $$\mathcal{G}_n = \mathcal{B}(Y_i, I(Z \geq Y_i);\ i = 0, \ldots, n-1) \quad \text{for} \quad n = 1,2,\ldots\ .$$

We shall show that

(5) $$\bar{F}_n = \bar{\mathcal{G}}_n$$

from which we may conclude (i) (cf., [1] pg 293) and

(6) $$Y_n = Z_n \quad \text{a.s. for} \quad n = 0, 1, \ldots\ .$$

To show (5), it suffices to show for $0 \leqq j < k$ that

(7) $$Z \geqq Y_j \quad \text{if, and only if,} \quad Y_k \geqq Y_j \quad \text{a.s.}$$

and

(8) $$Y_j \text{ is measurable with respect to } \bar{\mathcal{B}}(Y_k).$$

For then

$$\begin{aligned}
\bar{\mathcal{G}}_n &= \bar{\mathcal{B}}(Y_i, I(Z \geqq Y_i);\ i = 0, \ldots, n-1) \\
&= \bar{\mathcal{B}}(Y_{n-1}, I(Z \geqq Y_{n-1}), Y_i, I(Y_{n-1} \geq Y_i);\ i = 0, \ldots, n-2) \quad \text{(cf., (7))} \\
&= \bar{\mathcal{B}}(Y_{n-1}, I(Z \geqq Y_{n-1})) \quad \text{(cf., (8))} \\
&= \bar{\mathcal{B}}(Z_{n-1}, I(Z \geqq Z_{n-1})) \quad \text{(cf., (1), (2), (3), (4))} \\
&= \bar{F}_n
\end{aligned}$$

(7) follows from

(9) $$I(Z \geqq Y_j)(Y_k - Y_j) = E^{\mathcal{G}_k} I(Z \geqq Z_j)(Z - Y_j) \geqq 0 \quad \text{a.s.}$$

and

(10) $$I(Z < Y_j)(Y_k - Y_j) = E^{\mathcal{G}_k} I(Z < Y_j)(Z - Y_j) < 0 \quad \text{a.s. on } [Z < Y_j] .$$

(8) is true for $j = 0$ and if true for $j = 0, \ldots, \alpha-1 < k-1$, then

$$\bar{\mathcal{G}}_\alpha = \bar{\mathcal{B}}(Y_i, I(Y_k \geq Y_i);\ i = 0, \ldots, \alpha-1) \subset \bar{\mathcal{B}}(Y_k)$$

and, hence, (8) is true for $j = \alpha$.

(ii) is immediate from (1) and (2). Preliminary to showing (iii), we observe that for $0 \leqq j < k$,

$$E|Z - Y_j| = EE^{\mathcal{G}_k}(I(Z \geqq Y_j) - I(Z < Y_j))(Z - Y_j)$$

(11) $$= E(I(Z \geqq Y_j) - I(Z < Y_j))(Y_k - Y_j)$$

$$= E(I(Y_k \geqq Y_j) - I(Y_k - Y_j))(Y_k - Y_j) = E|Y_k - Y_j| .$$

It is easily seen that $\sup E|Y_n| \leq E|Z| < \infty$ and that the Y_n are uniformly integrable. Hence, by the martingale convergence theorem, there is a random variable Y_∞ with $Y_n \to Y_\infty$ a.s. and $E|Y_\infty - Y_n| \to 0$ as $n \to \infty$. In view of (6) and (11), (iii) clearly follows.

REMARKS:

(A) It is easy to see that if one of the Z_k is decomposed as we have Z into a sequence Z_{kn} $(n = 0, 1, \ldots)$, say, then $Z_{kn} = Z_{k \wedge n}$ a.s. where $k \wedge n$ is the smaller of k and n.

(B) This decomposition leads to an obvious procedure for embedding Z into Brownian motion when $EZ = 0$ and more generally for embedding zero mean martingales. (cf., Skorokhod [3] page 163, Strassen [4] page 318) The procedure, which does not require external randomization, becomes identical to the one suggested by Lester Dubins [2]. His paper, which hints at such a decomposition, has provided some of the motivation for this note. Primarily, we have found this type of decomposibility of interest in itself.

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