

Research partially supported by the U.S. Department of Navy, Office of Naval Research, Contract Nonr 855(09).

STOPPING TIMES ON BROWNIAN MOTION:
SOME PROPERTIES OF ROOT'S CONSTRUCTION

by

R. M. LOYNES¹
Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 637

JULY 1969

This work was done while the author was visiting from the University of Cambridge.

Stopping Times on Brownian Motion: Some Properties of Root's Construction¹

by

R. M. Loynes²

University of Cambridge

1. Introduction and Summary. The problem of finding a stopping-time for Brownian motion in such a way that the stopped value will have a given distribution has received considerable attention since the original work of Skorokhod [6]: see Breiman [1], Dubins [2], and Root [5]. (A further method has also been given by W. J. Hall, but has not been published.) Although all approaches coincide when the given distribution is concentrated on just two points, they otherwise give different results, and have different characteristics. Skorokhod's method requires external randomization, for example, as does Breiman's modification; Dubin's method involves, in an essential way, a limiting process. As these features could be disadvantages in certain circumstances, it seems worthwhile to consider Root's method carefully, since it is at least in principle very straightforward, defining the stopping-time as the hitting-time of a certain barrier. (It is not, however, easy to find the barrier corresponding to any particular distribution of the stopped process.)

Here we shall investigate the relationship between the barrier, the stopping-time and the distribution of the stopped process, obtaining uniqueness and continuity properties. This work was begun in an attempt to solve the embedding problem for reverse martingales in the following way. If $\{X_n : n \geq 1\}$ is the martingale, then for every

¹ Research partially supported by the Office of Naval Research, Contract Nonr 855(09).

² This work was done while the author was at the Department of Statistics, University of North Carolina at Chapel Hill

the set $\{X_j: n \geq j \geq 1\}$ may be embedded as a forward martingale, with stopping times $\{T_j^n: n \geq j \geq 1\}$ (Strassen [7], Theorem 4.3): there seems a possibility that, for fixed j , T_j^n may converge to some T_j and solve the problem. The attempt was unsuccessful, but the results, nevertheless, seemed worth presenting.

Certain general results are given in Section 2. The main results, which require certain restrictions, form Section 3.

2. General Results. The notation and terminology will be as in [5]. In particular, the Brownian motion sample path will be denoted by ω , and a barrier will be defined as follows.

Definition. A barrier is a subset B of $[0, +\infty] \times [-\infty, -\infty]$ satisfying

- (1) B is closed;
- (2) $(+\infty, x) \in B$ for all x ;
- (3) $(t, \pm\infty) \in B$ for all $t \in [0, +\infty]$;
- (4) if $(t, x) \in B$ then $(s, x) \in B$ whenever $s > t$.

The set of all barriers is denoted by \mathcal{B} , and the stopping time corresponding to $B \in \mathcal{B}$ will be denoted by

$$\tau_B(\omega) = \inf\{t: (t, \omega(t)) \in B\}.$$

Here and elsewhere we shall omit labels, such as B , when no confusion can be caused.

Proposition 1: If $B \in \mathcal{B}$, then either

- (i) B contains at least one point (t, x) with both t and x finite and $P[\tau_B < \infty] = 1$
- or (ii) B is the barrier consisting of all points at ∞ only, and $P[\tau_B = \infty] = 1$.

The proof is trivial since in case (i) B contains a horizontal line $\{(s, x): s \geq t\}$.

It will be recalled that Root defined a metric r on the space of barriers, by

$$r(C, D) = \max\{\sup_{x \in C} r(x, D), \sup_{y \in D} r(y, C)\},$$

where the metric r on the points of the closed half plane H is that induced by the homeomorphism $(t, x) \rightarrow (t/(1+t), x/(1+|x|))$ of H onto the rectangle $[0, 1] \times [-1, +1]$ endowed with the Euclidean metric ρ . The space \mathcal{B} is then compact and separable.

The following lemma is a strengthened version of Lemma 2.4 of [5].

Lemma 1: Let B be a barrier with stopping time τ . Then if $P[\tau < \infty] = 1$, for any $\varepsilon > 0$ there exists $\delta > 0$ and that $r(B, \bar{B}) < \delta$ implies $P[|\tau - \bar{\tau}| > \varepsilon] < \varepsilon$, where $\bar{\tau}$ is the stopping time corresponding to $\bar{B} \in \mathcal{B}$. If $P[\tau = \infty] = 1$, then given A there exists δ and that $r(B, \bar{B}) < \delta$ implies $P[\bar{\tau} < A] < \varepsilon$.

Clearly both parts of the lemma can be combined into one statement if desired.

Proof: As this is quite similar to Root's proof, we omit some of the details. Since, moreover, the second part is trivial, we confine our attention to the first part. Choose η as in [5], and T so that

$T > \varepsilon$ and $P[\tau \geq T] < \varepsilon/3$. Next choose M so that

$P[0 \leq \sup_{t \leq 2T} |\omega(t)| \geq M] < \varepsilon/3$, and after that choose δ so that if

$(t, x) \in B$ with $t \leq 2T$, $|x| \leq M$, then $r(B, \bar{B}) < \delta$ implies

$\rho((t, x), \bar{B}) < \eta$. If the set A is defined as in [5], it follows, as there, that $P[\bar{\tau} - \tau > \varepsilon] < \varepsilon$.

Now $P[\bar{\tau} \geq 2T] \leq P[\tau \geq T] + P[\bar{\tau} > \tau + \varepsilon] \leq 4\varepsilon/3$, and if $(t, x) \in \bar{B}$ with $t \leq 2T$, $|x| \leq M$ then $r(B, \bar{B}) < \delta$ implies $\rho((t, x), B) < \eta$. If we define $\bar{A} = \{\omega: \omega \text{ satisfies (i)' , (ii)' , (iii)' below}\}$, we find $\omega \in \bar{A}$ implies $\tau < \bar{\tau} + \varepsilon$, while $P[\bar{A}] > 1 - 2\varepsilon$. The first statement of the lemma then follows except that 3ε appears instead of ε , while the proof of the second part is trivial.

- (i)' Same as (i) of [5], with $\bar{\tau}$ replacing τ ;
- (ii)' $\bar{\tau}(\omega) < 2T$;
- (iii)' Same as (iii) of [5], with $\bar{\tau}$ replacing τ .

Let us call a probability law $L(X)$ achievable if there exists a barrier B with stopping time τ such that $L(\omega(\tau)) = L(X)$: then the question naturally arises as to how large the class of achievable laws is. Certainly, every law with finite variance and zero mean is achievable, for this is Theorem 2.1 of [5]. Possibly every law is achievable, though we have not succeeded in proving this; the following is a partial result.

Proposition 2: If $L(X)$ is concentrated on a half line, it is achievable.

Proof: First consider the case in which $L(X)$ is concentrated on a finite interval $[a, b]$, where $a < b$, and suppose $EX > 0$. For every n construct a law L_n , with zero mean and finite variance, by writing

$P_n = (1 - \frac{1}{n})P + \frac{1}{n}P_{1,n}$, where P_n, P , are the measures corresponding to $L_n, L(X)$, respectively, and $P_{1,n}$ places probability 1 at $-(n-1)EX$, and let B_n, τ_n , be barriers and stopping times corresponding to L_n , for which $E(\tau_n) < \infty$. Then, as in [5], there exists a limit point B of the B_n , with stopping time τ , and provided B is not the barrier at ∞ , B generates $L(X)$. (It is this provision that makes it difficult to apply the argument more widely.) If we suppose, as we may, that b is in the support of $L(X)$, then we may also suppose that $(0, x) \in B_n$ for $x \geq b$ for all n because of the finiteness of $E(\tau_n)$, and this implies that B contains finite points.

If now we return to the general case, in which $L(X)$ is concentrated on $(-\infty, b]$ or $[a, \infty)$ for some a or b , then truncation to a finite interval together with an argument of the above type will complete the proof of the proposition.

It will be convenient to define, for a given barrier B , the barrier function f_B as follows:

$$(1) \quad f_B(x) = \inf\{t: (t, x) \in B\} \quad (-\infty \leq x \leq \infty).$$

Proposition 3: The barrier function has the following properties:

- (i) $0 \leq f_B(x) \leq \infty$ for all x ;
- (ii) $(f_B(x), x) \in B$ for all x ;
- (iii) f_B is lower semi-continuous;
- (iv) $B = \{(t, x): t \geq f_B(x)\}$;
- (v) $f_B(\pm\infty) = 0$.

Moreover any lower semi-continuous function f taking values in $[0, \infty]$ and vanishing at $\pm\infty$ is the barrier function of the barrier

$$(2) \quad B_f = \{(t, x) : t \geq f(x)\}.$$

Properties (i), (iv) and (v) are obvious, and properties (ii) and (iii) follow from the fact that barriers are closed sets; (iii) was observed by Root [4]. The usual definition of lower semi-continuity, that given $\varepsilon > 0$ and x_0 , there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $f(x) > f(x_0) - \varepsilon$, has of course to be modified in the obvious way if $|x_0|$ or $f(x_0) = \infty$.

As above, we shall refer to the law or distribution of $\omega(\tau_B)$ as being generated by B (or τ_B).

Proposition 4: If B_1 and B_2 , with stopping times τ_1 and τ_2 , generate the same law, then $B_1 \cup B_2$ and $B_1 \cap B_2$ also generate this same law and have stopping times $\tau_1 \wedge \tau_2 = \min(\tau_1, \tau_2)$, and $\tau_1 \vee \tau_2 = \max(\tau_1, \tau_2)$ respectively.

Corollary: If B_i , with stopping times τ_i , ($i = 1, 2, \dots$), generate the same law, then this is also generated by $\overline{\cup B_i}$ and $\cap B_i$, with respective stopping times $\inf \tau_i$, $\sup \tau_i$, provided in the latter case that $\cap B_i$ contains finite points, or equivalently $\sup \tau_i > \infty$ with probability 1.

Proof: It will be convenient to denote that part of a barrier B for which x lies in a set K by $B(K)$.

Let us write $K = \{x : f_1(x) < f_2(x)\}$. Now suppose that for a given sample function ω , $\omega(\tau_2) \in K$. This means that ω hits $B_2(K)$ before $B_2(K^c)$, which implies that it hits $B_1(K)$ before $B_2(K^c)$, and this in turn implies that it hits $B_1(K)$ before $B_1(K^c)$: thus $P[\omega(\tau_1) \in K^c, \omega(\tau_2) \in K] = 0$. From this and the fact that

$$\begin{aligned}
& P[\omega(\tau_1) \in K, \omega(\tau_2) \in K] + P[\omega(\tau_1) \in K^c, \omega(\tau_2) \in K] \\
& = P[\omega(\tau_1) \in K, \omega(\tau_2) \in K] + P[\omega(\tau_1) \in K, \omega(\tau_2) \in K^c]
\end{aligned}$$

it follows that if we write

$$\begin{aligned}
(3) \quad A_1 &= \{\omega: \omega(\tau_1) \in K, \omega(\tau_2) \in K\} \\
A_2 &= \{\omega: \omega(\tau_1) \in K^c, \omega(\tau_2) \in K^c\}
\end{aligned}$$

then

$$(4) \quad P[A_1 \cup A_2] = 1.$$

But it is now clear that if $\omega \in A_1$ we must have $\tau_B = \tau_1 \leq \tau_2$, where $B = B_1 \cup B_2$, while if $\omega \in A_2$ on the other hand $\tau_B = \tau_2 \leq \tau_1$; thus $\tau_B = \tau_1 \wedge \tau_2$. That $\omega(\tau_B)$ has the same distribution as $\omega(\tau_1)$ and $\omega(\tau_2)$ is also easily seen, and completes the proof of one half of the proposition; the other half follows similarly.

In the part of the corollary dealing with unions, we may suppose B_i increasing, since if necessary, we may replace the original B_i by $\bigcup_{j=i} B_j$. Now a subsequence of $\{B_i\}$ converges, to B say, and by Lemma 1 $\tau_B = \lim \tau_i$: thus it is only necessary to show that $B = \overline{\bigcup B_i}$, since $L(\omega(\tau_B)) = \lim L(\omega(\tau_i))$. This is straightforward, and as the other part may be dealt with similarly, the proof is complete.

Whether it is possible to have two regular barriers (see Section 3 for the definition) generating the same distribution, neither of which includes the other, is unknown, though it seems a little unlikely.

3. Uniqueness and Continuity Results. A given barrier B gives rise to a well defined stopping time τ_B , and consequently (provided it has some finite points) generates exactly one distribution, $L(\omega(\tau_B))$. It would be useful to be able to reverse this statement, and conclude that each achievable law arises from exactly one barrier, but unfortunately this is not true without certain restrictions. In the first place, if for a barrier B $f_B(x_0) = 0$ for some x_0 , then the barrier beyond x_0 (i.e., with $x > x_0$ if $x_0 \geq 0$, and with $x < x_0$ if $x_0 \leq 0$) is inaccessible, and consequently f_B can be changed arbitrarily without changing $L(\omega(\tau_B))$; and in the second place essentially different barriers can generate the same law, as the example of the barriers $B_c = \{(t,0): t \geq c\}$ for arbitrary $c \geq 0$ shows. The first difficulty is easily overcome, but the second is more troublesome. A possibility of resolving it arises from the corollary to Proposition 4, since we might agree to use only barriers with minimal stopping times, but apart from the fact that uncountable collections might occur, there is also the drawback that it would be difficult to recognise such barriers. As it happens, however, in the practically important case of zero mean and finite variance laws, these difficulties disappear, and allow an attractive solution.

Definition: The first positive zero of a non-negative lower semi continuous function f is at x_0 if $x_0 \geq 0$, $f(x_0) = 0$ and $f(x) > 0$ for $0 \leq x < x_0$. The first negative zero is similarly defined.

Note that although barrier functions have first zeroes, they may be at $\pm \infty$.

Definition: Two barriers B and C or barrier functions f_B and f_C are equivalent if the latter agree on the interval $[x_-, x_+]$ where x_- , x_+ are the first negative and positive zeroes of either.

Definition: A barrier B or barrier function f_B is regular if f_B vanishes outside the interval $[x_-, x_+]$.

Clearly any barrier is equivalent to exactly one regular barrier.

Lemma 2: If B and C are two barriers, $\tau_B = \tau_C$ with probability 1 if and only if B and C are equivalent.

Corollary: If B and C are two regular barriers, $\tau_B = \tau_C$ with probability 1 if and only if B and C coincide.

The qualification 'with probability 1' may of course be omitted. That the corollary follows from the lemma is obvious, and the 'if' part of the lemma needs no proof. Suppose, therefore, that B and C are two regular barriers which are not identical. Then without loss of generality, we may suppose that for some finite $x_0 > 0$ $f_B(x_0) < f_C(x_0)$. Thus $f_C(x_0) \neq 0$, and because C is regular and a lower semi-continuous function achieves its minimum on a compact set, there exists an $\varepsilon > 0$ such that $f_C(x) > \varepsilon$ for $0 \leq x \leq x_0$. Moreover, ε may be chosen so that $2\varepsilon < f_C(x_0) - f_B(x_0)$, and then there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $f_C(x) > f_C(x_0) - \varepsilon > f_B(x_0) + \varepsilon$. Clearly there is a non-zero probability that ω will hit the line $x = x_0$ in the time-interval $[f_B(x_0), f_B(x_0) + \varepsilon]$ while not hitting C until after time $f_B(x_0) + \varepsilon$, and thus $P[\tau_B \neq \tau_C] \neq 0$.

Theorem 1: If $L(X)$ has zero mean and finite variance, it is generated by exactly one regular barrier B with finite $E(\tau_B)$.

That there is at least one such barrier is Root's Theorem 2.1, so we need only prove uniqueness. Suppose however that there are two regular barriers, B and C , generating $L(X)$: then by Proposition 4 $B \cup C$ also generates $L(X)$, and has stopping time $\tau_B \wedge \tau_C$, also with finite mean. Thus $E(\tau_B) = E(X^2) = E(\tau_B \wedge \tau_C)$, which shows that $\tau_B = \tau_B \wedge \tau_C$ with probability 1: by the corollary to Lemma 2 B and C must coincide.

The essentials of the proof are contained in the following result.

Proposition 5: If $L(X)$, with zero mean and finite variance, is generated by two barriers B and C , and $E(\tau_B) < \infty$, then $\tau_B \leq \tau_C$.

Under these conditions, there is therefore a minimum stopping time and a maximum boundary.

In the first paragraph of this section, it was shown by example that non-equivalent barriers can generate the same law. Whether this is true for every law is not known, though it seems quite likely. One can discover some properties of the situation without much difficulty. Consider, for example, the uniform distribution on $[-1, +1]$: this is generated by a regular barrier with finite stopping time, and for this barrier $f_B(\pm 1) = 0$, and with this latter condition the barrier is unique. If we relax this, at $+1$ say, we must have $f_B(+1) = \infty$, since otherwise there would be positive probability attached to $[+1, \infty)$.

Our final results concern continuity properties. To state them, the following notation will be convenient.

$$\begin{aligned}
\mathcal{B}_{\text{reg}} &= \{B: B \text{ is a regular barrier}\}; \\
T &= \{\tau: \tau = \tau_B \text{ for some barrier } B\}; \\
T^0 &= \{\tau: \tau \in T, \tau < \infty \text{ with probability } 1\}; \\
T^f &= \{\tau: \tau \in T, E(\tau) < \infty\}; \\
T^K &= \{\tau: \tau \in T, E(\tau) \leq K\}, \text{ where } 0 \leq K < \infty; \\
\mathcal{D}^a &= \{F: F \text{ is an achievable probability distribution}\}; \\
\mathcal{D}^f &= \{F: F \text{ is a probability distribution with mean } \mu(F) = 0 \\
&\quad \text{and variance } v(F) < \infty\}; \\
\mathcal{D}^K &= \{F: F \in \mathcal{D}^f, v(F) \leq K\}.
\end{aligned}$$

On these spaces, we shall put a topology: on \mathcal{B}_{reg} the metric topology corresponding to r ; on T and its subsets, the topology of convergence in probability, denoted by cp ; and on \mathcal{D}^a and its subsets, the topology of weak convergence, denoted by w . The natural maps from barrier to stopping time and from stopping time to distribution are the obvious ones.

Theorem 2: The following spaces are compact: (T, cp) , and, for all $K \geq 0$, (T^K, cp) and (\mathcal{D}^K, w) .

The following natural maps are onto, and also have the properties indicated:

- (a) $(\mathcal{B}_{\text{reg}}, r) \rightarrow (T, cp)$ is one-one and continuous;
- (b) $(T^0, cp) \rightarrow (\mathcal{D}^a, w)$ is continuous;
- (c) $(T^f, cp) \rightarrow (\mathcal{D}^f, w)$ is one-one and continuous;
- (d) $(T^K, cp) \rightarrow (\mathcal{D}^K, w)$ is a homeomorphism.

Proof: That the maps are onto is a consequence of definition, and Root's main theorem. Moreover, (a) follows from Lemmas 1 and 2, and since

$(\mathcal{B}, r) \rightarrow (T, cp)$ is also continuous (though not one-one) and \mathcal{B} is compact, it follows that T is compact: from Fatou's lemma, it follows that T^K is closed, and thus compact. For (b), it is necessary to show that if $\tau_n \rightarrow \tau$, then $F_n \Rightarrow F$, where F, F_n correspond to τ, τ_n . For this, it is sufficient to show that every subsequence of F_n contains a subsubsequence converging to F : if $F_{n'}$ is the subsequence, then $\tau_{n'} \rightarrow \tau$ in probability, so that there exists a subsubsequence $\tau_{n''} \rightarrow \tau$ with probability 1, and this implies $F_{n''} \Rightarrow F$. From Theorem 1, (c) now follows, and (d) is then obvious in view of the compactness of T^K ; the compactness of \mathcal{D}^K is a consequence of (d) and the proof is complete.

None of these results except possibly (c) can be improved in any direct and obvious way. Clearly \mathcal{B}_{reg} is not closed in \mathcal{B} , and therefore not compact (and none of the other spaces is compact): thus (a) can not be a homeomorphism. The map in (b) is not one-one. Whether (c) can be improved to state that the map is a homeomorphism has not been settled: an affirmative answer is equivalent to proving the impossibility of $\tau_n \rightarrow \infty$ with probability 1 while $F_n \rightarrow F$, where τ_n corresponds to F_n and $E(\tau_n), v(F_n), v(F) < \infty$.

The impossibility of improving (a) is merely another way of saying that r is not the appropriate metric on \mathcal{B}_{reg} for the present purposes. It appears to be possible to modify the metric in a natural way to r^* in such a way that $(\mathcal{B}_{reg}, r^*) \rightarrow (T, cp)$ is a homeomorphism; it then follows that \mathcal{B}_{reg} (and \mathcal{B}_{reg}^K , the inverse image of T^K) is compact under τ^* . As, however, the details are rather tedious, and for many purposes it is the stopping times rather than the barriers which are important, they will not be given here.

A probably more useful result is obtained by strengthening the topology on T^f and \mathcal{D}^f . On T^f , we introduce the usual L_1 metric, $d(\tau_1, \tau_2) = E|\tau_1 - \tau_2|$. On \mathcal{D}^f , we introduce the topology defined by $F_n \rightarrow F$ if $F_n \Rightarrow F$ and $v(F_n) \rightarrow v(F)$. This we shall denote by q .

Proposition 6: The following are equivalent for \mathcal{D}^f :

- (i) $F_n \Rightarrow F$ and $v(F_n) \rightarrow v(F)$
- (ii) $Q_n \Rightarrow Q$, where $Q_n(A) = \int_A x^2 dF_n(x)$, $Q(A) = \int_A x^2 dF(x)$.

It will not usually be the case that Q_n, Q are probability measures; however, we define $Q_n \Rightarrow Q$ in the usual way by requiring $\int h(x) dQ_n(x) \rightarrow \int h(x) dQ(x)$ for every bounded continuous h .

Proof: Assume (i): then from Theorem A of [3], p. 183, it follows that x^2 is uniformly integrable in F_n , and since h is bounded it follows that $h(x)x^2$ is uniformly integrable in F_n . Now since $\int h(x) dQ_n(x) = \int h(x)x^2 dF_n(x)$ the truth of (ii) is clear. Conversely, suppose (ii) holds: then if $\pm\epsilon$ are continuity points of Q , we can choose in effect $h(x) = x^{-2}$ if $|x| > \epsilon$, $= 0$ if $|x| \leq \epsilon$, and we find $F_n(|x| > \epsilon) \rightarrow F(|x| > \epsilon)$, so that $F_n(|x| \leq \epsilon) \rightarrow F(|x| \leq \epsilon)$. Because of the continuity of the admissible h it is therefore sufficient to show that $\int_{|x| > \epsilon} h(x) dF_n(x) = \int_{|x| > \epsilon} h(x)x^{-2} dQ_n(x) \rightarrow \int_{|x| > \epsilon} h(x)x^{-2} dQ(x) = \int_{|x| > \epsilon} h(x) dF(x)$ for every ϵ such that $\pm\epsilon$ are continuity points, and this is straightforward.

Theorem 3: The natural map $(T^f, L_1) \rightarrow (\mathcal{D}^f, q)$ is a homeomorphism.

Proof: The map is known to be one-one. Suppose that $\tau_n \rightarrow \tau$ in L_1 : then $E(\tau_n) \rightarrow E(\tau)$, so that $v(F_n) \rightarrow v(F)$, and $\tau_n \rightarrow \tau$ is probability, so that $F_n \Rightarrow F$ from Theorem 2. Suppose on the other hand that $F_n \rightarrow F$ in q -topology: then $F_n, F \in \mathcal{D}^K$ for some K , and from Theorem

$\tau_n \rightarrow \tau$ in probability, and in addition $E(\tau_n) \rightarrow E(\tau)$, and from the result at the top of p. 163 of [3], $E|\tau_n - \tau| \rightarrow 0$.

References:

- [1] Breiman, L. Probability Theory. Reading: Addison-Wesley, 1968.
- [2] Dubins, L. On a theorem of Skorokhod. Ann. Math. Statist. 39 (1968), 2094-2097.
- [3] Loève, Michel Probability Theory (3rd edition). Princeton: Van Nostrand, 1963.
- [4] Root, D.H. The existence of certain stopping times on Brownian motion. Technical report no. 15, Dept. of Mathematics, University of Washington.
- [5] Root, D.H. The existence of certain stopping times on Brownian motion. Ann. Math. Statist. 40 (1969), 715-718.
- [6] Skorokhod, A. Studies in the theory of random processes. Reading: Addison-Wesley, 1965.
- [7] Strassen, Volker. Almost sure behavior of sums of independent random variables and martingales. Proc. 5th Berkeley Symposium on Mathematical Statistics and Probability 2 (1), 315-343.

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Department of Statistics University of North Carolina Chapel Hill, North Carolina 27514		2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	
		2b. GROUP	
3. REPORT TITLE STOPPING TIMES ON BROWNIAN MOTION: SOME PROPERTIES OF ROOT'S CONSTRUCTION			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) TECHNICAL REPORT			
5. AUTHOR(S) (First name, middle initial, last name) Robert M. Loynes			
6. REPORT DATE July 1969	7a. TOTAL NO. OF PAGES -15-	7b. NO. OF REFS -07-	
8a. CONTRACT OR GRANT NO. Nonr 855(09)	9a. ORIGINATOR'S REPORT NUMBER(S) INSTITUTE OF STATISTICS MIMEO SERIES NUMBER 637		
b. PROJECT NO. RR-003-05-01	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)		
c.			
d.			
10. DISTRIBUTION STATEMENT THE DISTRIBUTION OF THIS REPROT IS UNLIMITED			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch Office of Naval Research Washington, D.C. 20360	
13. ABSTRACT Root recently [Ann. Math. Statist. 40] described a method of constructing an absorbing barrier for Brownian motion with the property that the value of the process at the moment it stops has any desired distribution with zero mean and finite variance. Further properties of the construction are investigated here, concerning uniqueness and continuity.			