The work described in this paper was supported by the Office of Naval Research under Contract N0014-67-A-0002.

ON CERTAIN RESULTS FOR STATIONARY POINT PROCESSES AND THEIR APPLICATION

by

M. R. LEADBETTER
Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 641

MAY 1969

On Certain Results for Stationary Point Processes and Their Application[1]

by

M. R. Leadbetter

SUMMARY

The purpose of this report is to describe some viewpoints and certain recent results of point process theory, and their application.

Section 2 contains series expressions for the distribution of the times between events both when measurement is made from a fixed time, and from an "arbitrary event" of the process. Relations between these series are also discussed. (These results were obtained in collaboration with R. J. Serfling.)

Section 3 concerns applications, particularly to zero-crossings problems. In Section 4, the method of defining a stationary point process by means of a stationary sequence of non negative random variables (due to Kaplan) is discussed and related to the previously described framework.

Section 5 deals with the approximation of stationary point processes consisting of rare events by Poisson processes.

[1] The work described in this report was supported by the Office of Naval Research under Contract N0014-67-A-0002.

1. INTRODUCTION

The theory of stationary point processes has application in many areas in which the simpler Poisson and renewal models do not apply because of a possibly complex dependence structure. An important example is provided by the zeros of a real valued stationary process. These zeros form a stationary point process of very real interest in certain electronic applications such as the "hard limiting" of noise waveforms.

Our purpose in this report is to describe some viewpoints and certain recent results of point process theory which we feel are relevant for applications in various such areas.

A great deal of research effort has been directed towards the discussion of the statistical properties of the zeros of a stationary process (usually assumed also normal) since the pioneering work of S. O. Rice ([19]). In particular, a great deal of attention has been given to the problem of obtaining good approximations to the distribution of the time between consecutive zeros (the "zero crossing problem"). A summary and comparison of much of this literature is given in a thesis by Levenbach [14].

From our point of view, perhaps the most interesting discussion of this problem is that given by Longuet-Higgins [15]. In that paper, series expressions are obtained for the distribution of such quantities as the time between a zero and the n-th subsequent zero, an upcrossing of zero and the n-th subsequent zero, and so on. In [11], we have obtained results similar to those of [15], for a general stationary point process (and "mixtures" of such processes). Specifically, the distribution function $F_n(t)$ of the time between an "arbitrary" event of the process and the n-th subsequent event, was expressed in terms of a series involving (factorial) moments of the number $N(0,t)$ of events in the interval $(0,t)$.

Recently, R. J. Serfling and the present author ([13]) have obtained further results along these lines for general stationary point processes. Specifically, two types of results are possible depending on whether measurements are made from a fixed time point, or relative to an "arbitrary" event of the process. This gives what we feel to be an illuminating point of view and in particular yields some of the results of [11]. We shall indicate some of these ideas in Section 2. Applications of these results have also been given by Serfling ([19]) in the field of traffic theory. In Section 3, we comment on the applicability of the results of Section 2.

As noted above, one may take measurements from a fixed time point or relative to an "arbitrary event" of the process (See e.g. [4, Chap.4]). The notion of an "arbitrary event" is an intuitive one which may be made precise by definitions such as those given for $F_n(t)$ in Section 2. Another point of view is to regard the process as being defined from a stationary sequence of non negative random variables (to which the times between events are closely related). In Section 4, we make this notion precise along lines given by Kaplan [9] and relate it to our Section 2 viewpoint. The description of a stationary point process described in Section 4 is, we feel, likely to be helpful in traffic theory and other applications where it is desired to relate the "fixed time" and "arbitrary event" approaches.

The series given in Section 2, though exact, are not universally applicable because of the increasing complexity of the terms involved. In the case of rare events in a stationary point process, it is sometimes possible to obtain a Poisson approximation, by means of arguments along the lines of those used by Cramér ([5][6]) for upcrossings of a high level by a stationary normal process. From a practical standpoint, such an approximation naturally yields a considerable simplification. This topic is discussed in Section 5.

2. THE TIMES BETWEEN EVENTS CONDITIONAL AND UNCONDITIONAL DISTRIBUTIONS

We consider a stationary point process and write, as above, $N(s,t)$ for the number of events in the interval (s,t). We shall suppose that the process has (with probability one) no multiple events, and that its intensity $\lambda = EN(0,1)$ is finite and non zero. Define

(1) $$G_n(t) = \Pr\{N(0,t) \geq n\} \qquad n = 0, 1, 2, \ldots .$$

$G_n(t)$ is thus the distribution function for the time to the n-th event after $t = 0$. We now modify $G_n(t)$ to define a corresponding probability but now conditioned by the occurrence of an event "at" $t=0$ in the following precise sense

(2) $$F_n(t) = \lim_{\delta \downarrow 0} \Pr\{N(0,t) \geq n | N(-\delta,0) \geq 1\}$$

and we shall write $F_n(t) = \Pr\{N(0,t) \geq n | \text{Event at } 0\}$, it being understood that the condition "event at 0" is (here and throughout) to be interpreted precisely as indicating a limit such as that in (2). That is, the zero probability condition is to be replaced by the positive probability condition $N(-\delta,0) \geq 1$, and the limit taken as $\delta \to 0$ (cf. the "horizontal window" conditions of [8]).

It is shown in [10] that the limit in (2) exists and that $F_n(t)$ is a distribution function. Clearly $F_n(t)$ has the intuitive interpretation of being the distribution of the time to the n-th event after t=0 given an event "at" t=0. Alternatively (as discussed in [10]) we may interpret $F_n(t)$ as the distribution function for the time from an "arbitrary event" to the n-th subsequent event.

It will also be convenient to introduce factorial moments of $N(0,t)$- both unconditional and conditional. Specifically (writing N for $N(0,t)$) we define

(3) $$\beta_k(t) = EN(N-1) \dots (N-k+1)$$

and $\alpha_k(t) = E\{N(N-1) \dots (N-k+1) | \text{Event at } 0\}$, i.e., strictly

(4) $$\alpha_k(t) = \lim_{\delta \downarrow 0} E\{N(N-1) \dots (N-k+1) | N(-\delta,0) \geq 1\} .$$

Now corresponding to (1) and (2), let

(5) $$v_n(t) = \Pr\{N(0,t) = n\} = G_n(t) - G_{n+1}(t)$$

(6) $$u_n(t) = \Pr\{N(0,t) = n | \text{Event at } 0\} = F_n(t) - F_{n+1}(t) .$$

Then it follows at once that

(7) $$\beta_k(t)/k! = \sum_{n=k}^{\infty} \binom{n}{k} v_n(t) .$$

Further, it is shown in [13] that the corresponding result

(8) $$\alpha_k(t)/k! = \sum_{n=k}^{\infty} \binom{n}{k} v_n(t) .$$

also holds, as intuitively expected, provided $EN^{k+1}(0,t) < \infty$. The proof of (8) makes use of a result (shown in [13] and stated here for its possible independent interest) to the effect that when $\mathcal{E}N^{k+1}(0,t)$ is finite, then

$$\mathcal{E}\left\{\binom{N(0,t)}{k} \middle| \text{Event at } 0\right\} = \lambda^{-1}\, \mathcal{E}\left\{\sum_{t_i \in (0,1)} \binom{N(t_i,\, t_i + t)}{k}\right\}$$

where t_i denote the positions of the events.

It is easy to show from (7) and (8) that

$$(9) \qquad \beta_k(t)/k! = \sum_{n=k}^{\infty} \binom{n-1}{k-1} G_n(t)$$

$$(10) \qquad \alpha_k(t)/k! = \sum_{n=k}^{\infty} \binom{n-1}{k-1} F_n(t) \; .$$

Equation (9) expresses the factorial moments $\beta_k(t)$ of $N(0,t)$ in terms of the distributions $G_n(t)$ of the time to the n-th event after t=0. Equation (10) does the same except that the factorial moments and distributions are all conditioned by the occurrence of an event "at" t=0.

Inversion of (9) and (10) yields the following important results (see [13])

$$(11) \qquad G_n(t) = \sum_{k=n}^{\infty} (-)^{k-n} \binom{k-1}{n-1} \beta_k(t)/k!$$

$$(12) \qquad F_n(t) = \sum_{k=n}^{\infty} (-)^{k-n} \binom{k-1}{n-1} \alpha_k(t)/k!$$

in the sense that whenever either of the series in (11) or (12) converges absolutely, the corresponding equality holds.

Equation (12) relates the conditional distribution function $F_n(t)$ to the conditional factorial moments, whereas (11) is the corresponding result for unconditioned distribution function and moments. In applications to zero crossing problems, it is of interest to obtain an expression for the conditional distribution functions $F_n(t)$ in terms of the unconditional factorial moments $\beta_k(t)$ of $N(0,t)$. This may be accomplished by means of the following result of [13].

$$(13) \qquad \beta_{k+1}(t)/(k+1)! = \lambda \int_0^t \alpha_k(u)/k!\, du \le \infty \quad k = 1, 2, \ldots \; .$$

Thus if $EN^{k+1}(0,t) < \infty$ for some k, then $\beta_{k+1}(t)$ is absolutely continuous with density $\lambda(k+1)\alpha_k(t)$. Further, this latter quantity is also easily seen to be the right hand derivative $D^+\beta_{k+1}(t)$ of $\beta_{k+1}(t)$.

Combining (12) and (13), we have the result given previously in [11], viz.,

$$(14) \qquad F_n(t) = \lambda^{-1} \sum_{k=n+1}^{\infty} (-)^{k-n} \binom{k-2}{n-1} D^+\beta_k(t)/k!$$

which holds in particular whenever this series converges absolutely.

A sufficient condition for the absolute convergence of the series in (14) is that the probability generating function $P_t(z) = \Sigma\, v_k z^k$ of $N(0,t)$ should be regular in a circle of radius $\rho_t > 2$. On the other hand, if $\rho_t < 2$ the series does not converge. However, it can be shown that, as long as $\rho_t > 1$,

$$(15) \qquad F_n(t) = \lambda^{-1} \sum_{k=n+1}^{\infty} \frac{1}{(1+q)^{k+1}} \sum_{s=n+1}^{k} (-)^{s-k-1} \binom{k}{s}\binom{s-2}{n-1} q^{k-s} D^+\mu_s(t)/s!$$

for any $q > (\rho_t-1)^{-2} -1$. This result (which we have shown in cooperation with P. Imrey) is proved by the use of Euler's "q-transform" by similar methods to those used in [20].

3. REMARKS CONCERNING APPLICATION OF THE GENERAL RESULTS

One of the main applications of the results of the previous section is to problems of zero crossing type. For example, one may consider the distribution of the time between an upcrossing of zero and the n-th subsequent upcrossing by a stationary normal process. The k-th factorial moment $\beta_k(t)$ of the number of upcrossings of zero in $(0,t)$ are given in [6, p.204] by

$$(16) \qquad \beta_k(t) = k! \int_{0 < t_1 \cdots < t_k < t} \cdots\cdots \int W_k(t_1 \ldots t_k)dt_1 \ldots dt_k$$

where

$$(17) \qquad W_k(t_1 \ldots t_k) = \int \overset{\infty}{\underset{0}{\cdots}} \int |y_1 \ldots y_k| P_{t_1 \ldots t_k}(0,\ldots 0, y_1, \ldots y_k) dy, \ldots dy_k$$

in which $P_{t_1 \ldots t_k}(x_1 \ldots x_k, y_1 \ldots y_k)$ denotes the joint density for the process values $x(t_1) \ldots x(t_k)$ and its derivatives $x'(t_1) \ldots x'(t_k)$ at $t_1 \ldots t_k$. Thus for the distribution function $F_n(t)$ of the time from an arbitrary upcrossing of zero to the n-th subsequent upcrossing, we have from (14)

$$(18) \qquad F_n(t) = \lambda^{-1} \sum_{k=n+1}^{\infty} (-)^{k-n} \binom{k-2}{n-1} \int_{0<t_1 \ldots <t_{k-1}<t} \cdots \int W(t_1 \ldots t_{k-1}, t) dt_1 \ldots dt_{k-1}$$

which may be shown to agree with the corresponding result of [15, 1.2.10] for the density of F_{n+1}. (Note $\lambda = \beta_1(1) = W_1(0)$.)

To consider downcrossings instead of upcrossings, it is only necessary to change the ranges of integration in (17) to $(-\infty, 0)$ instead of $(0,\infty)$. For crossings of _whatever_ type, the ranges should be $(-\infty,\infty)$.

Longuet-Higgins [15] gives a thorough discussion and comparison between approximations based on series such as (17) and other proposed approximations. We shall content ourselves with some general observations concerning applicability.

In the first place, equalities such as (18) are valid typically when the series converge absolutely. As far as we are aware, no simple sufficient conditions for such convergence are known - the approximations given in [15] appear to presuppose this convergence. Belayev [2] gives sufficient conditions for finiteness of the $\beta_k(t)$ but to our knowledge the question of absolute convergence of (18) or (14) is still an open one.

Second, there are a great many problems related to the zero crossing problem and which might be called "of zero crossing type". The occurrence of local extrema, fades below a low level or exceedances of a high level provide typical examples. In many cases, it is possible to write down formulae of the form (16) for the corresponding $\alpha_k(t)$ and then (18) again holds. For example, (16) and (17) both hold in a wide variety of cases (see [12] for the appropriate conditions). In such a case, $W_k(t_1 \ldots t_k)dt_1 \ldots dt_k$ represents intuitively the "probability of an event in each $(t_i, t_i + dt_i)$, $i = 1 \ldots k$. This point of view is used in [15] and also described in [11]. (See also [1, 3.42] and references therein for an interesting historical discussion of the use of such formulae.)

Thus while computational difficulties will be formidable in many cases, there is a wide variety of situations in which equations such as (18) will provide useful approximations, to whatever degree the calculation of the successive terms of the series is feasible.

Finally we refer to [19] for applications of the results of Section 2 to traffic theory. One example given there concerns the modelling of traffic flow by means of a "Pólya process" for which the probabilities $v_k(t)$ have negative binomial form

$$v_k(t) = \binom{m-1+k}{k}(t\Delta)^k\ (1 + t\Delta)^{-m-k} \quad k = 0, 1 \ldots ,$$

where m is a non negative integer. This yields (unconditioned) factorial moments given by

$$\beta_k(t) = k!\binom{m-1+k}{k}(t\Delta)^k .$$

In [19], relationships between the "unconditional" and "conditional" (i.e., measurement with respect to "fixed time" or relative to an "arbitrary event") are discussed. Specifically Equation (13) is used to relate the conditional factorial moments $\alpha_k(t)$ to the β_k, yielding

$$\alpha_k(t) = k!\binom{m+k}{k}(t\Delta)^k .$$

That is $\alpha_k(t)$ is just $\beta_k(t)$ with $m + 1$ replacing m. Hence the (conditional) probabilities $u_k(t)$ are obtained from the v_k's by this replacement. For a full discussion of this process and its motivation for traffic theory use, we refer to [19].

4. ALTERNATIVE DESCRIPTION OF A STATIONARY POINT PROCESS, AND ITS USE IN APPLICATIONS

$G_n(t)$ defined by (1) is the distribution function of a random variable, namely the time to the n-th event after the fixed zero time point. $F_n(t)$ is also a distribution function but was defined by (2) in a conditional fashion and not as the distribution function of a random variable. However, as noted, $F_n(t)$ may be regarded intuitively as the distribution function for the time from an "arbitrary event" to the n-th subsequent event. (We should perhaps note here also that (for example) the distribution function $H_n(t)$, say, of the time between the first and (n+1)st events after $t=0$ refers to a fixed time point and is not in general identical with $F_n(t)$.)

G_n (or H_n) and F_n thus correspond to the two well known points of view (cf. [4], Chap 4]) which may be adopted concerning a point process, namely whether one is making measurements from a fixed time point, or relative to an arbitrary event of the process. Usually, perhaps, observations will, in practice, begin at a fixed time point and quantities such as $G_n(t)$ are appropriately considered. Even then, however, it is important to remember that if one measures the lengths of intervals between events for a long time period and counts the proportion not exceeding t, one is estimating the distribution function $F_1(t)$. Further, in some experimental situations it may be reasonably supposed that observation starts from an "arbitrary" event.

In applications to traffic theory, measurement relative to a fixed time point (e.g., measurement of $N(0,t)$) has been termed "asynchronous counting" whereas "synchronous counting" refers to measurement beginning from an arbitrary event (the time a vehicle passes an observation point). For a discussion of such terminology of applications, we refer to [19], and references therein. The point of view of "synchronous counting" is really that one wishes to think intuitively of the times between events as a stationary sequence of random variables, even though this is not true when the events are labelled relative to a fixed time origin. However, one can view a stationary point process in this way by, in essence, "choosing a random time origin". We shall now indicate specifically how this may be done, along the lines given by

Kaplan [9] (who generalized the treatment given by Doob [7] in the renewal theory case).

Specifically, let $\{\tau_i : i = 0, \pm 1 \ldots\}$ denote a stationary sequence of non negative random variables. We shall require that the distribution function $F(t)$ of each τ_i satisfies $F(0+) = 0$ (to avoid multiple events). Define the (consistent) finite dimensional distributions of a new sequence of non negative random variables $\{t_o, \tau_i^*, i = 0, \pm 1 \ldots\}$ by

(19) $$\Pr(t_o \leq \alpha, \tau_i^* \leq \beta_i, i \in I\} = \mu^{-1} \int_0^\alpha \Pr\{\tau_0 > x, \tau_i \leq \beta_i, i \in I\} dx$$

for any finite index set I, $\alpha \geq 0$ $\beta_i \geq 0$, $\mu = \mathcal{E} u_i$, assumed finite. It may be shown that t_o may be taken as the time of the first event after $t=0$ and τ_i^* as the times between consecutive events in a stationary point process. Specifically define

(20) $$t_m = t_o + \sum_{i=1}^{m} \tau_i^* \qquad \text{or} \qquad t_m = t_o - \sum_{i=m+1}^{o} \tau_i^*$$

according as $m > 0$ or $m < 0$. Then $\{t_m : m = 0, \pm 1 \ldots\}$ represent the events of a stationary point process, labelled so that t_o is the first event after $t=0$. (Note that it follows from (19) that $t_o < \tau_o$ with probability one and hence $t_{-1} < 0 \leq t_o$).

Thus a stationary point process may be specified by starting with a stationary sequence of non negative random variables τ_i and defining closely related random variables τ_i^* to represent the times between events. (A converse result to this is incidentally also shown in [9].)

To relate this approach to the discussion of the previous sections, we write $N(s,t)$ for the number of $t_i \in (s,t]$. Then it follows, for example, that for $\delta > 0$

$$\Pr\{N(0,t) \geq 1 | N(-\delta,0) \geq 1\} = \Pr(t_o \leq t,\ t_{-1} > -\delta)/\Pr(t_{-1} > -\delta).$$

Now, from (19)

$$\Pr\{t_o \leq t,\ t_{-1} > -\delta\} = \mu^{-1} \int_0^t \Pr(x < \tau_o \leq x + \delta)dx$$

and this expression is readily seen to be $\frac{\delta}{\mu} F(t) + o(\delta)$. Similarly $\Pr(t_1 > -\delta) \sim \delta/\mu$ and hence

$$\Pr\{N(0,t) \geq 1 | N(-\delta,0) \geq 1\} \to F(t) \quad \text{as} \quad \delta \to 0\ .$$

That is $F_1(t)$, defined by (2) is identical with the distribution function $F(t)$ of each τ_i, as one expects intuitively. Similarly, it may be shown that $F_n(t)$ is also the distribution function of $\sum_1^n \tau_i$ (or the sum of any other n consecutive τ_i).

Thus $F_n(t)$ is the distribution function of a random variable very closely and naturally related to the point process. We may further take the point of view that the random variables τ_m(or $\sum_1^m \tau_i$ correspond to measurement "from the occurrence of an arbitrary event".

This discussion also provides a criterion for deciding whether a given sequence of distribution functions $F_n(t)$ can be taken to correspond (by Equation (2)) to a stationary point process. Specifically, this can be done provided $F_n(t)$ is the distribution function of $\sum_1^n \tau_i$ for some stationary sequence $\{\tau_i\}$ of non negative random variables. This may be easy to check and would be useful in postulating models for particular physical point processes.

In particular, of course, one may specify a stationary renewal process by means of one "lifetime" distribution. The so called "Erlang" process used in traffic theory applications ([19]) is of this type with a gamma distribution for the times between events. In traffic theory applications, it appears (as already noted) to be of importance to be able to move easily between and to relate the "synchronous" and "asynchronous" counting procedures. We feel that the relations described above provide a useful viewpoint for this.

5. POISSON APPROXIMATION

For a stationary normal process, the series expressions given in [15] provide exact solutions to certain zero crossing problems. As already noted, these series will be very useful in some situations and not so useful in others where the calculations to obtain good approximations will be prohibitive. This comment applies also to the generalized versions of Section 2. Hence, from a practical standpoint, it is desirable to investigate other possible methods of approximation. A number of such approximations have been used (and some of these are compared in [15] and [14]).

When the events of the process are rare (i.e., widely separated), there is the possibility that a Poisson model may provide a good approximation. For example, it is known ([21], [5], [3], [17]) that the upcrossings of a very high level by a stationary normal process, tend to behave in a Poisson fashion. Intuitively, this is so because the upcrossings occur at intervals which are sufficiently long so that the dependence of the process has "fallen off sufficiently" between successive upcrossings.

We shall now briefly indicate how results of this type may be obtained in a general setting along the lines of the arguments used by Cramér (see [5] or [6]) for upcrossings of a high level. These results are also related to some quite detailed corresponding results for discrete time obtained by R. Meyer ([16]). Further it has been pointed out to me by Professor Harald Cramér that the result stated under a mixing condition is very close to a theorem of Volkonski and Rozanov [22].

Specifically, we consider a sequence $\{P_r\}$ of stationary point processes without multiple events and having finite intensities $\mu_r \to 0$ as $r \to \infty$. Write $N_r(s,t)$ for the numbers of P_r-events in $(s,t]$ Corresponding to each P_r, we define a "standardized" point process P_r^* having events at the points $\mu_r t_i$, where t_i denote the times of occurrence of the events of P_r. That is if $N_r^*(s,t)$ corresponds to P_r^*, $N_r^*(s,t) = N_r(s/\mu_r, t/\mu_r)$.

The processes P_r^* are stationary with unit intensities, and we have further

$$(21) \qquad P\{N_r^*(0,t) \geq 1\}/t \to 1 \quad \text{as} \quad t \downarrow 0$$

$$(22) \qquad P\{N_r^*(0,t) > 1\}/t \to 0 \quad \text{as} \quad t \downarrow 0 \ .$$

If the convergence properties indicated in (21) and (22) are each uniform in $r = 1, 2 \ldots$, we shall refer to $\{P_r\}$ as a uniform family of point processes. (We note that it may be shown that a sufficient condition for this uniformity is that $EN_r^*(0,t)[N_r^*(0,t) - 1]$ should be $o(t)$ uniformly in r, as $t \to 0$.)

It follows that if $\{P_r\}$ is a uniform family, then

$$(23) \qquad P\{N_r(0, \gamma_r) \geq 1\} \sim \mu_r \gamma_r \quad \text{as} \quad r \to \infty$$

(24) $P\{N_r(0, \gamma_r) > 1\} = o(\mu_r\gamma_r)$ as $r \to \infty$

whenever $\{\gamma_r\}$ is a sequence of non negative constants such that $\mu_r\gamma_r \to 0$.

For such a uniform family, it may be shown under various dependence restrictions that

(25) $P\{N_r(0, \tau/\mu_r) = k\} \to e^{-\tau} \tau^k/k!$, $k = 0, 1, 2 \ldots$,

for $\tau > 0$, as $r \to \infty$, and that corresponding multidimensional limits hold. Here we shall restrict attention to (25) and indicate the lines of its derivation (using arguments of [5], as noted).

Specifically, for each $r = 1, 2 \ldots$ consider $n_r(\to\infty)$ pairs of consecutive intervals alternately of length γ_r, δ_r where

(26) $\delta_r/\gamma_r \to 0$, $T_r = n_r(\gamma_r + \delta_r) \sim \tau/\mu_r$ as $r \to \infty$.

These intervals, which we refer to as "γ_r-intervals", "δ_r-intervals", together make up the interval $(0, T_r)$. Writing $p_r = P\{N(0,\gamma_r) \geq 1\}$ if follows from (23) and (26) that

(27) $p_r \sim \mu_r\gamma_r \sim \tau/n_r$ as $r \to \infty$.

Let, now, $D_{k,r}$ denote the occurrence of at least one P_r-event in exactly k of the n_r γ_r-intervals and no P_r-event in the remaining n_r-k such intervals. We require that the dependence structure of the process be such that it can be shown that

(28) $P(D_{k,r}) - \binom{n_r}{k} p_r^k(1 - p_r)^{n_r-k} \to 0$ as $r \to \infty$.

A sufficient condition for the truth of (28) (with γ_r, δ_r, n_r satisfying (26)) is that each P_r-process be strongly mixing with the same mixing funtion $\psi(h) = 0(h^{-\alpha})$ for some $\alpha > 0$ as $h \to \infty$. (i.e., for each r $|Pr(A \cap B) - Pr(A)Pr(B)| \le \psi(h)$ whenever $A \in S_{-\infty,t-h}$, $B \in S_{t,\infty}$ for any t, where $S_{-\infty,t}$ is the σ-field generated by the random variables $N_r(u,v)$ for all $u \le v \le t$, and similarly for $S_{t,\infty}$.) However, in other examples, (28) may be shown in different ways.

It then follows from (27) and (28) that

$$(29) \qquad P(D_{k,r}) \to e^{-\tau} \tau^k/k! \quad .$$

Now the probability of at least one event in any of the n_r δ_r-intervals does not exceed $n_r EN(0,\delta_r) = n_r\mu_r\delta_r \to 0$ by (26). Similarly, the probability of more than one event in any one of the n_r γ_r-intervals does not exceed $n_r P\{N(0,\gamma_r) > 1\} = o(n_r\mu_r\gamma_r)$ by (24), and this tends to zero by (26).

Combining these statements with (29), it can be shown that the probability of exactly k P_r-events in $(0,T_r)$ is given by

$$(30) \qquad P\{N_r(0,T_r) = k\} \to \tau^k e^{-\tau}/k! \quad \text{as} \quad r \to \infty .$$

Finally, it may be shown that $P|\{N_r(0,T_r) - N_r(\tau/\mu_r)| > 0\} \to 0$ as $r \to \infty$, from which it follows that

$$P\{N_r(0,T_r) = k\} - P\{N(\tau/\mu_r) = k\} \to 0$$

and by combining this with (30), we see that Equation (25) holds. Sufficient conditions, then, for the Poisson limit (25) are that the family $\{P_r\}$ be uniform, and each strongly mixing with mixing function $\psi(h) = O(h^{-\alpha})$ for some $\alpha > 0$ as $h \to \infty$. Of course, mixing conditions are not readily verifiable in general. However, we feel that the existence of the Poisson limit given, such a mixing condition, is good justification for the expectation that the Poisson limit will hold in many practical cases, and encouragement to test it with data. Also, as noted, the mixing assumption is used through Equation (28). In some cases, it is possible to verify (28) directly from other considerations. In particular, we refer to the derivation for the upcrossings of a high level by a stationary normal process given in [5], where the assumed properties of the covariance function yield (28).

REFERENCES

[1] Bartlett, M.S., (1956), An introduction to stochastic processes, Cambridge University Press.

[2] Belayev, Yu.K., (1966), On the number of intersections of a level by a Gaussian stochastic process, Teor. Ver. i Prim. 11, pp. 120-128.

[3] Belayev, Yu.K., (1967), On the number of intersections of a level by a Gaussian stochastic process II, Teor. Ver. i Prim. 12, pp. 444-457.

[4] Cox, D.R., and Lewis, P.A.W., (1960), The statistical analysis of series of events, Methuen and Company, Ltd., London.

[5] Cramér, Harald, (1966), On the intersections between the trajectories of a normal stochastic process and a high level, Arkiv. Mat. 6, pp. 337-349.

[6] Cramér, Harald, and Leadbetter, M.R., (1967), Stationary and related stochastic processes, John Wiley and Sons, New York.

[7] Doob, J.L., (1948), Renewal theory from the point of view of probability, Trans. Amer. Math. Soc. 63, pp. 422-438.

[8] Kac, M., and Slepian, D., (1959), Large excursions of Gaussian processes, Ann. Math. Statist. 30, pp. 1215-1228.

[9] Kaplan, E.L., (1955), Transformation of stationary random sequences, Math Scand 3, pp. 127-149.

[10] Leadbetter, M.R., (1966), On streams of events and mixtures of streams, J. Roy. Stat. Soc. (B) 28, pp. 218-227.

[11] Leadbetter, M.R., (1969), On the distributions of the times between events in a stationary stream of events, To appear in J. Roy. Stat. Soc. (B).

[12] Leadbetter, M.R., (1967), A note on the number of axis crossings by a stochastic process, Bull. Amer. Math. Soc. 73, pp. 129-132.

[13] Leadbetter, M.R., and Serfling, R.J., (1969), On distribution function - moment relationships in a stationary point process, (Unpublished), Inst. of Stat. Mimeo Series No. 611, University of North Carolina at Chapel Hill.

[14] Levenbach, Hans, (1963), The zero-crossing problem, Queen's University Research Report No. 63-4.

[15] Longuet-Higgins, M.S., (1962), The distribution of the intervals between zeros of a stationary random function, Phil. Trans. Roy. Soc. 254, pp. 557-599.

[16] Meyer, R.M., (1967), Some Poisson type theorems for sequences of dependent rare events, Ph.D. Thesis, University of North Carolina at Chapel Hill.

[17] Qualls, Clifford, (1968), On a limit distribution of high level crossings of a stationary Gaussian process, Ann. Math. Statist. 39, pp. 2108-2113.

[18] Rice, S.O., (1945), Mathematical analysis of random noise, Bell Syst. Tech. J. 24, pp. 46-156.

[19] Serfling, R.J., (1969), Fundamental relationships for traffic flow models, (Unpublished), Florida State University Statistics Report M154.

[20] Takacs, Lajos, (1967), On the method of inclusion and exclusion, J. Amer. Stat. Assn. 62, pp. 102-113.

[21] Volkonski, V.A., and Rozanov, Yu.A., (1959), Some limit theorems for random functions I, Teor. Ver. i Prim. 4, pp. 186-207.

[22] Volkonski, V.A., and Rozanov, Yu.A., (1961), Some limit theorems for random functions II , Teor. Ver. i Prim. 6, pp. 202-215.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Department of Statistics The University of North Carolina Chapel Hill, North Carolina 27514	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

On Certain Results for Stationary Point Processes and their Application

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

Malcolm R. Leadbetter

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1969	-21-	-22-

8a. CONTRACT OR GRANT NO. N00014-67-A-0321-0002 b. PROJECT NO. NR 042-214/1-6-69 (436) c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Institute of Statistics Mimeo Series Number 641 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Distribution unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch Office of Naval Research Washington, D.C. 20360

13. ABSTRACT

The purpose of this report is to describe some viewpoints and certain recent results of point process theory, and their application.

Section 2 contains series expressions for the distribution of the times between events both when measurement is made from a fixed time, and from an "arbitrary event" of the process. Relations between these series are also discussed. (These results were obtained in collaboration with R.J. Serfling.)

Section 3 concerns applications, particularly to zero-crossings problems. In Section 4, the method of defining a stationary point process by means of a stationary sequence of non negative random variables (due to Kaplan) is discussed and related to the previously described framework.

Section 5 deals with the approximation of stationary point processes consisting of rare events by Poisson processes.

DD FORM 1 NOV 65 1473

Security Classification