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THEOREMS OF ERGODIC TYPE FOR STATIONARY SEQUENCES WITH MISSING OBSERVATIONS

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* This work was done while the author was visiting from the University of Sheffield.

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1. INTRODUCTION AND SUMMARY. Let $\{X_n: n = 1,2,\ldots\}$ be a strictly stationary sequence, for which EX_n exists: then according to the Birkhoff ergodic theorem $n^{-1}(X_1 + X_2 + \ldots + X_n)$ converges almost surely (a.s.) to a limit. If, moreover, $E|X_n|^p$ is finite for some p, $1 \leq p < \infty$, then L_p convergence also occurs.

In certain contexts only some of the X_n are available (see e.g. [5]), and it then becomes of importance to know whether the average of those X_n which are in fact available converges, and if so to what limit; such results also have an intrinsic interest, as Blum and Hanson [1] pointed out. Results of this and similar kinds form the subject of this paper. In §§2 and 3 a.s. and L_p properties respectively are discussed.

Analogous questions may be asked in the continuous-time case, when a strictly stationary process $\{X(t): -\infty < t < \infty\}$ is given. Indeed, rather more possibilities suggest themselves: for example, one may have available values at a *sequence* of times $\{t_j\}$, so that the behavior of $n^{-1} \Sigma X(t_j)$ is under scrutiny, or alternatively $X(t)$ may be known on a 'larger' set K, in which case the limiting behavior of

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$T^{-1} \int_0^T X(t)I_K(t)dt$ or something equivalent becomes of interest. (Here I_K is the indicator function of the set K.) Certain applications suggest consideration of a slightly different quantity . Suppose that K is the union of an infinite sequence of (finite) intervals K_n: then it might be appropriate to imagine a new process $\{Y_n\}$ constructed by averaging X over these intervals. If

$$(1) \qquad Y_n = \int X(t)I_{K_n}(t)dt / \int I_{K_n}(t)dt$$

then the ratios $n^{-1} \Sigma^n Y_j$ become candidates for investigation; corresponding definitions and inquiries may be made in the discrete-time case also. Results for the continuous-time case occupy §§4 and 5, again for a.s. and L_p properties respectively.

2. DISCRETE-TIME: A.S. PROPERTIES.

The most direct approach to these problems is, superficially at least, via the 'Basic Ergodic Theorem' on p. 415 of Loève [4], but it is apparently difficult to show that the hypotheses of this theorem follow from any reasonable conditions. Here, in consequence, the method used by Brunel and Keane [2] will be adopted, and indeed the results obtained are little more than elaborations of theirs. It may be useful to emphasize in passing that this slightly changes the problem: the original task was to discuss $n^{-1} \Sigma^n X_{k_i}$, or in operator language $n^{-1} \Sigma T^{k_i} f$ for some specific function f, whereas Brunel and Keane's approach relates to the behavior of this ratio for arbitrary (integrable) f.

The framework is as follows: on some compact metric space X, with Borel sets $\mathcal{X}$, a homeomorphism ϕ is assumed to be given. The system $(X, \mathcal{X}, \phi)$ is assumed to be *strictly L-stable* in the sense that for some $x \in X$ the sequence $\{\phi^n x: \; n = 0,1,2,\ldots\}$ is dense in X, and also the set of mappings $\{\phi^n: \; n \geq 1\}$ is equi-continuous. It then follows that there exists exactly one probability measure μ on $(X, \mathcal{X})$ which is invariant and ergodic under ϕ.

If h is a real-valued function defined on X, let D_h be the set of discontinuity points of h.

In addition to this special framework, there is here, as elsewhere, supposed to exist a probability space $(\Omega, \mathcal{A}, P)$, on which is defined T a measure-preserving transformation. Notation such as $L^p = L^p(\Omega,\mathcal{A},P)$ refers to this space. Although this is phrased in a different language to §1, in which random variables were used, such inconsistency is perhaps not unreasonable. On the one hand, previous work has been expressed in operator language, while on the other, motivation and intuitive content (at least for the probabilist) are clearer in random variable language.

THEOREM 1. If h is a bounded real-valued function on X, such that $\mu(D_h) = 0$, and $f \in L^1$, then for any $y \in X$

$$\frac{1}{n} \sum_{k=1}^{n} h(\phi^k y) f(T^k \omega)$$

converges a.s. as $n \to \infty$ to a limit $f^* \in L^1$.

PROOF. Under the given condition on h, for any $\varepsilon > 0$ there exist continuous functions h_1, h_2 such that for $x \in X$ $h_1(x) \leq h(x) \leq h_2(x)$, and $\int (h_2 - h_1) d\mu < \varepsilon/3$. Now if $h' = h_1 - \varepsilon/3$ and $h'' = h_2 + \varepsilon/3$, it follows that for all $x \in X$

(2) $$h'(x) \le h(x) - \varepsilon/3 < h(x) + \varepsilon/3 \le h''(x)$$

and

(3) $$\int (h'' - h')d\mu < \varepsilon.$$

Given $y \in X$, then, the equi-continuity of $\{\phi^n\}$ and the uniform continuity of h' and h'' imply that there exists an open neighborhood W of y such that, for all n and $x \in W$,

(4) $$h'(\phi^n x) \le h(\phi^n y) \le h''(\phi^n x).$$

From this point on, the only change needed in the proof of Theorem 1 of [2] is the replacement of $I_{Y'}$, I_Y, $I_{Y''}$ by h', h, h'' respectively.

COROLLARY 1. If T is ergodic and if T and ϕ have no eigenvalues (other than 1) in common, then

(5) $$f^* = \int h d\mu \cdot \int f dP \quad \text{a.s.}$$

Thus, in this case, if $\int h d\mu \neq 0$,

(6) $$\frac{\sum_1^n h(\phi^k y) f(T^k \omega)}{\sum_1^n h(\phi^k y)} \to \int f dP \quad \text{a.s.}$$

In particular if T is weakly mixing, (5) and (6) are valid.

It will be recalled that T is *weakly mixing* if

$$\frac{1}{n} \sum_{k=0}^{n-1} |P(T^{-k}E \cap F) - P(E)P(F)| \to 0$$

as $n \to \infty$ for each pair $E, F \in \mathcal{A}$

Brunel and Keane term a sequence $\{k_n\}$ of integers *uniform* if for some strictly L-stable system $(X, \mathcal{X}, \mu, \phi)$ there exists $Y \in \mathcal{X}$ with $\mu(Y) > 0$, $\mu(\partial Y) = 0$, and a point $y \in X$ such that k_n are the successive integers j for which $\phi^j y \in Y$.

COROLLARY 2. (Brunel and Keane) If $f \in L^1$ and $\{k_n\}$ is a uniform sequence then

$$\frac{1}{n}\sum_{1}^{n} f(T^{k_i}\omega)$$

converges a.s. to a limit $\tilde{f}$ in L^1.

The proof requires little more than the choice of I_Y for h in Theorem 1.

Term the set $Y \in \mathcal{X}$ *unsaturated* if whenever $x \in Y-N$, where $N \subset Y$ is a set for which $\mu(N) = 0$, there exist non-negative integers $J(x)$, $K(x)$, such that $\phi^{J(x)+1} x \notin Y$, and $\phi^{-K(x)-1} x \notin Y$; if $\mu(Y) < 1$, Y is unsaturated. Suppose $J(x)$, $K(x)$ are chosen as small as possible and define

$$\begin{aligned} \ell(x) &= 1 + J(x) + K(x) \quad \text{when } x \in Y-N, \\ &= 1 \quad \text{elsewhere.} \end{aligned} \tag{7}$$

Now suppose that $x \in Y$, but that $x \notin N \cup \bigcup_{-\infty}^{\infty} \phi^j(\partial Y)$; the latter is a μ-null set. Then for any given n if z is close enough to x $\phi^n z \in Y$ or Y^c according as $\phi^n x \in Y$ or Y^c. Thus ℓ is continuous at x, and as ℓ is clearly continuous at points in Y^c which are not in the null set just defined it follows that $\mu(D_\ell) = 0$.

If Y is an unsaturated set, with $\mu(\partial Y) = 0$ and $\mu(Y) > 0$, and if $y \notin \bigcup_{-\infty}^{\infty} \phi^j(N)$, call the uniform sequence generated by y and Y *regular*. Then a regular uniform sequence decomposes uniquely into an infinite union of disjoint, finite, maximal subsets A_i of consecutive integers, and each k_j belongs to precisely one A_i; the A_i are supposed labelled in the obvious way, so that A_{i+1} lies to the right of A_i.

Now define the *smoothed* f *sequence* in such a situation as $\{g_n\}$, where

$$(8) \qquad g_n(\omega) = \sum_{j\in A_n} f(T^j\omega) / \sum_{j\in A_n} 1:$$

then the following result holds.

THEOREM 2. If $f \in L^1$ and $\{g_n\}$ is the smoothed f sequence associated with a regular uniform sequence for which $\int \ell d\mu < \infty$,

$$\frac{1}{n}\sum_{1}^{n} g_j$$

converges a.s. to a limit $\tilde{g}$ in L^1, and if the conditions of Corollary 1 of Theorem 1 are satisfied, $\tilde{g} = \int f dP$ a.s.

PROOF. Let

$$(9) \qquad h(x) = I_Y(x)/\ell(x),$$

where ℓ is defined by (7). Then $1 \geq h \geq 0$, and $\mu(D_h) = 0$, so that Theorem 1 may be applied, and it follows that $(N_n^{-1})\Sigma_1^n g_j$ converges, where N_n is the maximum element of A_n; provided n/N_n converges to a non-zero limit the proof will be complete. Now it follows immediately that

$$(10) \qquad n/N_n \to \int h d\mu = \int_Y \frac{d\mu}{\ell(x)}$$

which is non-zero if $\int \ell d\mu < \infty$. It seems plausible that $\mu(Y) < 1$ implies $\int \ell d\mu < \infty$, but no proof has been found.

An alternative approach is occasionally possible: if L^p-convergence can be proved, and is rapid enough, then a.s. convergence will follow. See for example Theorem 6.2 in Chapter X, [3], and, for slightly different conditions, Problem 13 on p.265 of Loève.

3. DISCRETE-TIME: L^p PROPERTIES.

Since the sequence of functions involved is uniformly integrable, the next results follow directly from Theorem 1, together with its Corollary 2, and Theorem 2.

THEOREM 3. If h is a bounded real-valued function on X, such that $\mu(D_h) = 0$, and $f \in L^p$, then for any $y \in X$

$$\frac{1}{n} \sum_{k=1}^{n} h(\phi^k y) f(T^k \omega)$$

converges to f^* in the L^p norm.

COROLLARY 1. (Brunel and Keane) If $f \in L^p$ and $\{k_n\}$ is a uniform sequence, then

$$\frac{1}{n} \sum_{1}^{n} f(T^{k_i} \omega)$$

converges to $\tilde{f}$ in the L^p norm.

THEOREM 4. If $f \in L^p$ and $\{g_n\}$ in the smoothed f-sequence associated with a uniform sequence derived from an unsaturated set for which $\int \ell d\mu < \infty$, then $n^{-1} \Sigma^n g_j$ converges in the L^p norm to $\tilde{g}$.

The following results are based on those of Blum and Hanson [1].

THEOREM 5. Let $\{h_n\}$ be a sequence of real numbers satisfying

(i) $\sup_{1 \le i \le n} |h_i| \cdot \sum_1^n |h_i| = o(n^2)$,

(ii) $\sum_1^n |h_i| = O(n)$, and

(iii) $n^{-1} \sum_1^n h_i \to \ell$.

Then if T is strongly mixing and $f \in L^p$

$$\frac{1}{n} \sum_1^n h_i \; f(T^i\omega) \to \ell \int f dP \tag{11}$$

in the L^p-norm, and hence if $\ell \neq 0$

$$\frac{\sum_1^n h_i \; f(T^i\omega)}{\sum_1^n h_i} \to \int f dP. \tag{12}$$

It will be recalled that T is *strongly mixing* if $P[T^n A \cap B] \to P[A]P[B]$ as $n \to \infty$ for every pair $A, B \in \mathcal{A}$

PROOF. Because of (iii) one may as well suppose $\int f dP = 0$, and then in L^2 one has

$$\left|\left|\frac{1}{n} \sum_1^n h_i \; f(T^i\omega)\right|\right|_2^2 = \frac{1}{n^2} \sum h_i h_j \; V(i,j) \tag{13}$$

where $V(i,j) = \int f(T^i\omega)f(T^j\omega)dP$. On separating the sum in (13) into two parts, that for which $|i-j| \le M$ and that for which $|i-j| > M$, and using (i) and (ii) it follows that the result is true in L^2, and it is extended to other L^p by approximation.

THEOREM 6. Let $\{h_n\}$ be a sequence of real numbers satisfying

(i) $\sup_{1 \le i \le n} |h_i| = o(\sum_1^n h_i)$ and

(ii) $\sum_1^n |h_i| = O(\sum_1^n h_i)$.

Then if T is strongly mixing and $f \in L^p$

$$\frac{\sum_1^n h_i \, f(T^i\omega)}{\sum_1^n h_i} \to \int f dP \tag{14}$$

in the L^p-norm.

The proof uses the same technique as before. The only point which is not completely trivial is in showing that $(\Sigma_1^n h_i)^{-2} \Sigma_{|i-j| \le M} h_i h_j V(i,j)$ tends to 0. This, however, is bounded by a constant multiple of $(2M + 1) \cdot \sup_{1 \le i \le n} h_i \cdot \Sigma^n |h_i| \cdot (\Sigma h_i)^{-2}$, and the use of (i) and (ii) completes this part of the proof.

COROLLARY. (Blum and Hanson) If T is strongly mixing and $\{k_n\}$ is any strictly increasing sequence of integers, and if $f \in L^p$, then $n^{-1} \Sigma_1^n f(T^{k_i}\omega)$ converges to $\int f dP$ in the L^p-norm.

If A is an infinite subset of the positive integers, term it *decomposable* if it is the union of disjoint, finite, maximal subsets A_i

of consecutive integers; in §2, an unsaturated set gave rise to a decomposable sequence. Write $n(A_i) = n_i$, and again, for $f \in L^p$, define g_n by (8).

THEOREM 7. If $f \in L^p$ and $\{g_n\}$ is the smoothed f-sequence associated with a decomposable subset of the integers, and if T is strongly mixing, then

$$\frac{1}{n}\sum_{1}^{n} g_j \to \int f dP$$

in the L^p-norm.

PROOF. Define
$$h_k = 0 \quad \text{if} \quad k \notin A,$$
$$\quad = n_j^{-1} \text{ if } k \in A_j,$$

and replace n in Theorem 6 by N_n. Then $1 \geq h_i \geq 0$ for all i, while $\Sigma_1^{N_n} h_i = n$, and conditions (i) and (ii) are trivially satisfied.

It will be observed that, although Theorem 7 is stated in this form so as to show the parallel with Theorem 2, there is in the present case no need for the subsets A_i to be maximal: any decomposition for which A_{i+1} lies to the right of A_i for all i will suffice.

4. **CONTINUOUS-TIME: A.S. PROPERTIES.** As far as one of the two formulations possible in continuous time is concerned, that in which a sequence $\{t_j\}$ of times is involved, no progress of any significance has been made; results are easily obtained, of course, if the gaps between

successive members of the sequence $\{t_j\}$ form a periodic sequence, or if the t_j are all integral multiples of some unit of time, or finally if L^p-convergence occurs sufficiently rapidly, as was observed for the discrete-time case at the end of §2.

Results for the other kind of situation are straightforwardly found, only obvious changes being necessary in the arguments of §2. First, a measurable semi-group of measure-preserving transformations $\{T_t: t > 0\}$ is supposed given on the probability space (Ω, A, P). Next, it is supposed that there exists a semi-group $\{\phi_t: t > 0\}$ of homeomorphisms on X, with dense orbit $\{\phi_t x: t > 0\}$ for at least one x, and this semi-group is assumed to be equi-continuous and measurable; the latter condition of measurability is satisfied if $\phi_t x$ is continuous on *both* sides at $t = 0$ for every x. (ϕ_t is defined for negative t as the inverse of ϕ_{-t}.)

THEOREM 8. If h is a bounded real-valued function on X, and that $\mu(D_h) = 0$, and $f \in L^1$, then for any $y \in X$

$$\frac{1}{t}\int_0^t h(\phi_s y) f(T_s \omega)\, ds$$

converges a.s. as $n \to \infty$ to a limit $f^* \in L^1$.

COROLLARY 1. If $\{T_t\}$ is ergodic and if $\{T_t\}$ and $\{\phi_t\}$ have no eigenvalues (other than 1) in common, then

$$(15) \qquad f^* = \int h\,d\mu \cdot \int f\,dP \quad \text{a.s.}$$

Thus in this case if $\int h\,d\mu \neq 0$,

$$(16) \qquad \frac{\int_0^t h(\phi_s y) f(T_s \omega) ds}{\int_0^t h(\phi_s y)} \to \int f dP \quad \text{a.s.}$$

In particular, if $\{T_t\}$ is weakly mixing, (15) and (16) are valid.

The definition of weakly mixing in the present context is obtained from the discrete-time version by making the obvious change of replacing summation over $k=0$ to n by integration over $s=0$ to t.

Let a subset K of the non-negative real axis be called *uniform* if for some $Y \in X$, with $\mu(Y) > 0 = \mu(\partial Y)$, and some $y \in X$, K is the set of those s for which $\phi_s y \in Y$.

COROLLARY 2. If $f \in L^1$ and K is a uniform set, then

$$\frac{\int_0^t I_K(s) f(T_s \omega) ds}{\int_0^t I_K(s) ds}$$

converges a.s. to a limit $\tilde{f}$ in L^1.

It does not seem possible to find reasonable general conditions which would imply an analogue of Theorem 2, and it appears easier to investigate particular cases when the need arises.

5. **CONTINUOUS-TIME: L^p PROPERTIES.** For this mode of convergence, it is very simple to obtain results for a sequence $\{t_j\}$.

THEOREM 9. Let $\{t_j\}$ be a sequence of real numbers such that $t_i - t_j \to \infty$ as $|i-j| \to \infty$, and let $\{h_n\}$ be a sequence of real numbers satisfying

(i) $$\sup_{1 \le i \le n} |h_i| \sum_1^n |h_i| = o(n^2),$$

(ii) $$\sum_1^n |h_i| = O(n),$$

and

(iii) $$n^{-1} \sum_1^n h_i \to \ell.$$

Then if $\{T_t\}$ is strongly mixing and $f \in L^p$

$$\frac{1}{n} \sum_1^n h_i \; f(T_{t_i} \omega) \to \ell \int f dP \tag{17}$$

in the L^p-norm, and hence if $\ell \neq 0$

$$\frac{\sum_1^n h_i \; f(T_{t_i} \omega)}{\sum_1^n h_i} \to \int f dP. \tag{18}$$

THEOREM 10. Let $\{t_j\}$ be a sequence of real numbers such that $t_i - t_j \to \infty$ as $|i-j| \to \infty$, and let $\{h_n\}$ be a sequence of real numbers satisfying

(i) $$\sup_{1 \le i \le n} |h_i| = o(\sum^n h_i)$$

and

(ii) $\sum_1^n |h_i| = O(\sum_1^n h_i)$.

Then if $\{T_t\}$ is strongly mixing and $f \in L^p$

$$\text{(19)} \qquad \frac{\sum_1^n h_i \, f(T_{t_i}\omega)}{\sum_1^n h_i} \to \int f dP$$

in the L^p-norm.

THEOREM 11. If $\{T_t\}$ is strongly mixing and $\{t_j\}$ is a sequence of real numbers such that $t_i - t_j \to \infty$ as $|i-j| \to \infty$, and if $f \in L^p$, then $n^{-1} \Sigma_1^n f(T_{t_i}\omega)$ converges to $\int f dP$ in the L^p norm.

For an analogue to Theorem 7, it is necessary to take up the last remark of Section 3. Define a *decomposition* of the set Z of positive integers as an expression of Z in the form $\cup A_i$, where each A_i is finite, and A_{i+1} lies to the right of A_i. Then corresponding to (8), define the smoothed f-sequence associated with $\{t_j\}$ and $\{A_i\}$ as $\{g_n\}$, where

$$\text{(20)} \qquad g_n = \sum_{j \in A_n} f(T_{t_j}\omega) / \sum_{j \in A_n} 1.$$

THEOREM 12. If $f \in L^p$ and $\{g_n\}$ is the smoothed f-sequence associated with a decomposition of the integers and a sequence $\{t_j\}$ of real numbers such that $t_i - t_j \to \infty$ as $|i-j| \to \infty$, and if $\{T_t\}$ is strongly

$$\frac{1}{n}\sum_{1}^{n} g_j \rightarrow \int f dP$$

in the L^p norm.

The proofs of these results are exactly similar to those of the corresponding ones in Section 3.

Next, results for the case in which a set K of positive measure appears. The following are immediate consequences of Theorem 8 and its Corollary 2.

THEOREM 13. If h is a bounded real-valued function on X, such that $\mu(D_h) = o$, and $f \in L^p$, then for any $y \in X$

$$\frac{1}{t}\int_0^t h(\phi_s y) f(T_s \omega)\, ds$$

converges to f^* in the L^p norm.

COROLLARY 1. If $f \in L^p$ and K is a uniform set then

$$\frac{\int_0^t I_K(s) f(T_s \omega)\, ds}{\int_0^t I_K(s)\, ds}$$

converges to $\tilde{f}$ in the L^p norm.

Finally, results which parallel Theorems 5, 6, 7, 9, 10, 11 and 12.

THEOREM 14. Let h be a measurable real-valued function satisfying

(i) $$\sup_{0 \le s \le t.} |h(s)| \cdot \int_0^t |h(s)|ds = o(t^2),$$

(ii) $$\int_0^t |h(s)|ds = O(t),$$

and

(iii) $$t^{-1} \int_0^t h(s)ds \to \ell.$$

Then if $\{T_t\}$ is strongly mixing and $f \in L^p$

$$\frac{1}{t} \int_0^t h(s)f(T_s\omega)ds \to \ell \int fdP \tag{21}$$

in the L^p norm, and hence if $\ell \neq 0$

$$\frac{\int_0^t h(s)f(T_s\omega)ds}{\int_0^t h(s)ds} \to \int fdP . \tag{22}$$

THEOREM 15. Let h be a measurable real-valued function satisfying

(i) $$\sup_{0 \le s \le t} |h(s)| = o(\int_0^t h(s)ds)$$

and

(ii) $$\int_0^t |h(s)|ds = O(\int_0^t h(s)ds).$$

Then if $\{T_t\}$ is strongly mixing and $f \in L^p$

$$\frac{\int_0^t h(s)f(T_s\omega)ds}{\int_0^t h(s)ds} \to \int fdP \tag{22}$$

in the L^p norm.

COROLLARY 1. If $\{T_t\}$ is strongly mixing, and K is a set with Lebesgue measure ∞, then for any $f \in L^p$

$$(23) \qquad \frac{\int_0^t I_K(s) f(T_s \omega)\, ds}{\int_0^t I_K(s)\, ds} \to \int f dP$$

in the L^p norm.

If $\{A_n\}$ is a sequence of measurable sub-sets of the real line, having positive Lebesgue measures $\{\ell_n\}$, and if for each i A_{i+1} is to the right of A_i, term $\{A_n\}$ *admissible* and define the smoothed f-sequence corresponding to $\{A_n\}$ by analogy with (8), viz.,

$$(24) \qquad g_n = \ell_n^{-1} \int I_{A_n}(s) f(T_s \omega)\, ds.$$

THEOREM 16. If $\{g_n\}$ is the smoothed f-sequence corresponding to an admissible sequence of subsets $\{A_n\}$, for which $n\ell_n \to \infty$, and if $\{T_t\}$ is strongly mixing, then in L^p

$$n^{-1} \sum_1^n g_j \to \int f dP.$$

The proofs of Theorems 14 and 15 are straightforward and Theorem 16 follows from Theorem 15 on writing

$$h(s) = \sum_1^\infty \ell_i^{-1} I_{A_i}(s).$$

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13. ABSTRACT

According to the Birkhoff and von Neumann ergodic theorems, the arithmetic mean of the first n variables in a strictly stationary stochastic process converges as n tends to ∞, almost surely and in L^2. If not all variables are available, it is natural to consider the average of the first n available variables, and closely related questions. Such topics, including the analogous situations in continuous time, form the subject of the present paper.

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