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ON SOME PREVENTIVE MAINTENANCE POLICIES FOR IFR

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1. INTRODUCTION. Several years ago, H. Makabe and the author proposed a preventive maintenance policy and discussed it together with its ramifications [3-5]. Though many preventive maintenance policies are known at the present time, we discussed a series of policies which are named Policy I, II, III and so on, especially Policy III. The policy of type III was proposed in order to improve the policy of type II due to R. Barlow and L. Hunter [1]. We found the optimal policy of type III under the assumption of a Weibull-type failure distribution in the sense of maximizing the limiting efficiency [3]. It was generalized to the sense of minimizing the maintenance cost rate [4]. In comparison with the optimal policy of type II, the optimal policy of type III has a higher efficiency and lower cost rate. Furthermore, it was found numerically that the optimal policy of type III is rather robust [5].

However, in the previous paper [3], the existence and the uniqueness of the optimal policy of type III (in the sense of limiting efficiency) are shown only under a Weibull assumption. We shall generalize the discussion here to the case of a strictly increasing failure rate which is a sufficiently wide class for preventive maintenance problems (Sections 4-5).

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H. Makabe and the author treated the optimality of Policy III from a viewpoint of practical uses [4]. But, in spite of the assumption of a Weibull distribution for failures of a system, the argument in the proof of Theorem 5.1 seems to be incorrect. Thus the author withdraws the theorem and discusses here the optimality of it under weaker conditions (Sections 6-7).

2. **SEVERAL TYPES OF PREVENTIVE MAINTENANCE POLICIES.** In this section, we shall explain various types of preventive maintenance policies and their main properties before proceeding to our discussion in the following sections. But, unfortunately, there seems to be no fixed system for naming preventive maintenance (or replacement) policies. Thus, we shall use here terminologies used in the previous papers.

Preventive maintenance policies of type I, II and III were proposed to be suitable to a simple system (e.g., light bulbs), a complex system (e.g., a computer) and a more complex system (e.g., many machines of same type in a room), respectively. But the third policy may be also used more effectively in the second case than Policy II. These are described as follows.

Note: Hereafter, the word "perform preventive maintenance" means "perform an overhaul" or "replace by a new system". Regardless, we shall have as good a system as new (in the sense of its failure rate) immediately after a preventive maintenance is performed. And we shall reschedule preventive maintenance under the policy.

[Policy I] (age replacement). Perform preventive maintenance after t_0 hours of continuing operation time without failure $(0 < t_0 \leq \infty)$. If the system fails before t_0 hours have elapsed, perform maintenance at the time of failure. Preventive maintenance at the time is rescheduled.

[Policy II] (periodic replacement with minimal repair at time). Perform preventive maintenance after t^* hours operating as a total regardless of the intervening failures $(0 < t^* \leq \infty)$. We assume that after each failure only minimal repairs are made, and that the failure rate of the system is not disturbed after performing minimal repair.

[Policy III]. Perform preventive maintenance at k-th failure; but for first $(k-1)$ times of failures, perform minimal repairs on these occassions. We assume also that the failure rate of the system is not disturbed after performing minimal repair.

Policy I was proposed and discussed by Ph. Morse [6] and R. Barlow and L. Hunter [1]. A general discussion of it is seen in R. Barlow and F. Proschan [2]. As noted above, this policy is suitable for simple systems in which no minimal repair is effective. In this case, it will be sufficient that we discuss the replacement cost per unit time. Usually, it is considered in an infinite time span.

On the other hand, for a complex system, it is too expensive to replace the system by a new system at any failure occassion. Naturally, we have to repair the system and use it. In this case, it is desired that the amount of down time is limited or that the maintenance cost is low. R. Barlow and L. Hunter [1] proposed the maintenance policy of type II and discussed on it to be optimum in the sense of the limiting efficiency which is the expected fraction of operating time in an infinite time span. They also noted that the optimization is equivalent to the optimization in the sense of expected maintenance cost in an infinite time span. However, it will be more natural to consider both concepts simultaneously. Thus, H. Makabe and the author [3] introduced the concept of the maintenance cost rate and defined it as follows.

$$C_\infty = [\text{cost for unit down time}] \times [\text{expected fraction of down time}] + [\text{expected cost of all repairs and replacements during an unit time}],$$

which is a generalized form of the limiting efficiency and the expected maintenance cost per unit time. They proposed Policy III and discussed it in the sense of the limiting efficiency [3, 5] and the maintenance cost rate [4, 5]. They also discussed Policy II based on the maintenance cost rate.

Policy III improved Policy II in the following points:

1) At the time the total operating time reaches t^*, the system is operating generally, so we may continue the use of it without any loss until the next failure occurs. In other words, Policy II wastes a (small) amount of time and/or costs.

2) In the case that there are slight differences between systems of the same kind in their failure rates, Policy III is more robust than Policy II. This fact will be intuitively seen from the observation that the duration of usage of a system is long or short according to its failure rate. This was illustrated numerically in [5] under the assumption that the failure distributions of systems were Weibull-type with a common shape parameter.

3) In many practical cases, an inspection of the system under Policy III is easier and less expensive than under Policy II.

In order to improve Policy II with respect to point 1), we may introduce Policy II' as follows:

[Policy II']. Perform preventive maintenance at the first failure after t^* operating hours as a total regardless of the intervening failures.

H. Makabe and the author also considered in [4] other maintenance policies in order to discuss the effectiveness of Policy III. These are as follows:

[Policy IV]. Perform preventive maintenance when the total operating time t^* without down time, or when the k-th failure occurs. This is a combined policy of type II and III. Of course, this is a generalization of both policies.

[Policy V]. Let $\{a_i\}$ be a preassigned sequence of non-negative numbers (possibly infinite), and Y_i be the total operating time regardless the down time at intervening failures until the i-th failure. Perform preventive maintenance if $Y_i > a_i$ and perform a minimal repair if $Y_i \leq a_i$.

[Policy V']. In Policy V, reverse the inequality.

[Policy IV']. In Policy V, put $a_i = t^*$ for $i < k$, and $a_i = 0$ for $i \geqq k$, where t^* is same as Policy IV.

In this paper, we shall first concentrate upon the following:

[Policy III']. In Policy V, put $a_i = \infty$ for $i < k$, $a_k = \xi$ and $a_i = \infty$ for $i > k$, where ξ is a non-negative real number.

3. **NOTATIONS AND PRELIMINARY LEMMAS.** For convenience we shall establish here some notation and preliminary lemmas. The first lemma is evident and elementary, but since it will play an important role in our arguments we state it here.

LEMMA 1. Let A, B, C, D be positive numbers.

(i) If $\frac{A}{B} > \frac{C}{D}$, then $\frac{A}{B} > \frac{A+C}{B+D} > \frac{C}{D}$.

(ii) If $\frac{A}{B} = \frac{C}{D}$, then $\frac{A}{B} = \frac{A+\lambda C}{B+\lambda D} = \frac{C}{D}$ for every real λ.

Let the failure distribution function of a system be $F(x)$. Assume that $F(o) = 0$ and it is differentiable everywhere on the positive half line. Put for all $x > 0$

$$F'(x) = f(x)$$

$$\bar{F}(x) = 1 - F(x)$$

$$q(x) = \frac{f(x)}{\bar{F}(x)}. \qquad \text{(failure rate)}$$

$$Q(x) = \int_0^x q(t)dt.$$

$$G(t) = \int_t^\infty y\, f(y)dy.$$

It is well known that

(3.1) $$\bar{F}(x) = e^{-Q(x)} .$$

If $q(x)$ is a monotone non-decreasing function, then the failure distribution is called "increasing failure rate" and denoted IFR. We shall say strictly IFR when the monotonity is strict. Quite similarly, we can define DFR (decreasing failure rate) and strictly DFR. However, if the failure distribution is DFR, we must not perform any preventive maintenance. Thus we shall talk about IFR only.

Furthermore, we shall use the following notations.

T_m: mean down time for a minimal repair

T_s: mean down time for a preventive maintenance

C_o: mean cost per unit down time

C_m: mean cost for a minimal repair

C_s: mean cost for a preventive maintenance.

LEMMA 2. Let

(3.2) $$h(t) = e^{Q(t)} \int_t^{\infty} (y-t)f(y)dy.$$

If $f(x)$ is strictly IFR, then $h(t)$ is monotone (strictly) decreasing function and

(3.3) $$h(o) = m$$

(3.4) $$\lim_{t \to \infty} h(t) = 1/q(\infty) ,$$

where m is the expected failure time, i.e., $m = \int_0^{\infty} tdF(t)$.

PROOF. Clearly $h(t)$ is a differentiable function. We have

$$h'(t) = q(t)e^{Q(t)} \int_t^\infty (y-t)f(y)dy - e^{Q(t)} \int_t^\infty f(y)dy \tag{3.5}$$

$$= q(t)h(t) - 1$$

and

$$h(o) = \int_0^\infty y\, f(y)dy = m ,$$

which is (3.3). From this, we can say that $h(+o) < 1/q(+o)$. Because, if it does not hold, since $q(t) > q(+o) \geqq 1/m$, we have $Q(t) > t/m$ and

$$\overline{F}(t) = e^{-Q(t)} < e^{-t/m} \quad \text{for all} \quad t > 0. \tag{3.6}$$

Integrate both sides of (3.6) from zero to infinity. Then we have that $m < m$, which is a contradiction. Thus, the relation $h(+o) < 1/q(+o)$ is true, and $h(t)$ decreases at a neighborhood of origin by (3.5). If there exists a t (= t_0, say) such that $q(t_0)h(t_0) = 1$, then, since we have $h'(t_0) = 0$ by (3.5), it is true that $h(t)$ is decreasing for $0 < t < t_0$. From the assumption of IFR, $\frac{d}{dt}\{1/q(t)\} < 0$ for all t. Thus we have that

$$h(t_0 + \varepsilon) > \frac{1}{q(t_0+\varepsilon)}$$

for arbitrary small $\varepsilon > 0$. Hence, from (3.5) and the monotonity of $1/q(t)$, we can conclude that $h(t)$ increases for $t > t_0$, i.e., $h(t_0) = 1/q(t_0)$ is minimum of $h(t)$. But, on the other hand, we have

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \frac{\int_t^\infty (y-t)f(y)dy}{\bar{F}(t)} = \lim_{t\to\infty} \frac{\int_t^\infty f(y)dy}{f(t)}$$

$$= \lim_{t\to\infty} \frac{1}{q(t)} = \frac{1}{q(\infty)} ,$$

which is (3.4) and implies that $\lim_{t\to\infty} h(t) < h(t_0)$. This is a contradiction. Thus, we can say that t_0 does not exist. So we have $h'(t) < 0$ for all $t > 0$, which completes the proof.

LEMMA 3. If $u_1(x)$ and $u_2(x)$ are both positive functions for all $x > 0$ and $u_1(x)/u_2(x)$ is a (strictly) monotone increasing (decreasing) function, then $w_1(x)/w_2(x)$ is a (strictly) monotone increasing (decreasing) function, where

$$w_i(x) = \int_0^x u_i(t)dS(t) , \quad i = 1, 2$$

and $dS(t)$ is a positive measure.

PROOF. Since for any $t < x < s$

$$\frac{u_1(t)}{u_2(t)} < \frac{u_1(x)}{u_2(x)} < \frac{u_1(s)}{u_2(s)} ,$$

we have

$$\int_0^x u_1(t)dS(t) < \frac{u_1(x)}{u_2(x)} \int_0^x u_2(t)dS(t)$$

and for any $y > x$

$$\frac{u_1(x)}{u_2(x)} \int_x^y u_2(s)dS(s) < \int_x^y u_1(s)dS(s).$$

Hence

$$\frac{w_1(x)}{w_2(x)} < \frac{u_1(x)}{u_2(x)} < \frac{\int_x^y u_1(s)dS(s)}{\int_x^y u_2(s)dS(s)} ,$$

and, using Lemma 1, we have

$$\frac{w_1(x)}{w_2(x)} < \frac{w_1(x) + \int_x^y u_1(s)dS(s)}{w_2(x) + \int_x^y u_2(s)dS(s)} = \frac{w_1(y)}{w_2(y)} \tag{3.7}$$

for $y > x$. Equation (3.7) means that $w_1(x)/w_2(y)$ increases strictly monotonically. Consider the reciprocal of the function $u_1(x)/u_2(x)$; then the assertion about "decreasing" will hold. If the above argument includes the sign of equality at all inequalities, it refers to the same relation but not "strictly".

LEMMA 4. Define $w_i(k,\xi)$ $(i = 1,2)$ as follows:

$$w_1(k,\xi) = \frac{T_m}{(k-1)!} \int_0^\xi \{Q(t)\}^{k-1} e^{-Q(t)} q(t)dt \tag{3.8}$$

$$w_2(k,\xi) = \frac{1}{(k-1)!} \int_0^\xi \{Q(t)\}^{k-1} q(t)[\int_t^\infty (y-t)f(y)dy]dt. \tag{3.9}$$

If the failure distribution is IFR, then

$$\lim_{\xi \to 0} \frac{w_1(k,\xi)}{w_2(k,\xi)} = \frac{T_m}{m} \tag{3.10}$$

and

$$(3.11) \qquad \lim_{\xi \to \infty} \frac{w_1(k,\xi)}{w_2(k,\xi)} = \frac{T_m}{E(X_{k+1})} \quad \text{for} \quad k = 1, 2, \ldots,$$

where $E(X_{k+1})$ is the expected time between the k-th failure and (k+1)-th failure.

PROOF. We have by L'Hospital's rule

$$\lim_{\xi \to o} \frac{w_1(k,\xi)}{w_2(k,\xi)} = \lim_{\xi \to o} \frac{T_m\{Q(\xi)\}^{k-1} e^{-Q(\xi)} q(\xi)}{\{Q(\xi)\}^{k-1} q(\xi) \int_{\xi}^{\infty} (y-\xi) f(y) dy} = \frac{T_m}{m} ,$$

which is (3.10). On the other hand, we have

$$(3.12) \qquad \lim_{\xi \to \infty} w_1(k,\xi) = \frac{T_m}{(k-1)!} \int_0^{\infty} Z^{k-1} e^{-Z} dZ = T_m$$

and

$$\lim_{\xi \to \infty} w_2(k,\xi) = \frac{1}{(k-1)!} \int_0^{\infty} \{Q(t)\}^{k-1} q(t) [\int_t^{\infty} (y-t) f(y) dy] dt$$

$$(3.13) \qquad = \frac{1}{k!} [\{Q(t)\}^k \cdot \int_t^{\infty} (y-t) f(y) dy]_0^{\infty} + \frac{1}{k!} \int_0^{\infty} \{Q(t)\}^k \bar{F}(t) dt$$

$$= \frac{1}{k!} \int_0^{\infty} \{Q(t)\}^k e^{-Q(t)} dt,$$

using (v) of Lemma 5 and the fact that $Q(o) = 0$. Let X_k be the time between (k-1)-th failure and k-th failure of the system. Put $Y_k = \Sigma_{i=1}^{k} X_i$. Y_k means the k-th failure time excluding the down time at the intervening failures. Thus, we have

$$E(Y_k) = \int_0^\infty t \cdot \frac{\{Q(t)\}^{k-1}}{(k-1)!} e^{-Q(t)} q(t)dt$$

(3.14) $$= \left[\frac{\{Q(t)\}^k}{k!} e^{-Q(t)} \cdot t\right]_0^\infty + \int_0^\infty \frac{\{Q(t)\}^k}{k!} e^{-Q(t)} q(t) \cdot tdt$$

$$- \int_0^\infty \frac{\{Q(t)\}^k}{k!} e^{-Q(t)} dt$$

$$= E(Y_{k+1}) - \int_0^\infty \frac{\{Q(t)\}^k}{k!} e^{-Q(t)} dt.$$

By definition and (3.12),

(3.15) $$E(X_{k+1}) = E(Y_{k+1}) - E(Y_k) = \int_0^\infty \frac{\{Q(t)\}^k}{k!} e^{-Q(t)} dt.$$

Equations (3.12), (3.13) and (3.15) imply (3.11). We used, in (3.14), (iii) of Lemma 5

The existence of the integral $E(Y_k)$, the vanishing of $\{Q(t)\}^k e^{-Q(t)}$ as $t \to \infty$, and several similar relations are nearly obvious under the assumption of IFR. But, for convenience, we shall sum up these results in the following.

LEMMA 5. If the failure distribution is IFR, then

(i) $m < \infty$; in general, moments of all orders are finite;

(ii) $\{Q(t)\}^k \bar{F}(t) \to 0$, as $t \to \infty$, for all k;

(iii) $\{Q(t)\}^k t\bar{F}(t) \to 0$, as $t \to \infty$, for all k;

(iv) $\{Q(t)\}^k G(t) \to 0$, as $t \to \infty$, for all k;

(v) $\{Q(t)\}^k \int_t^\infty (y-t)f(y)dy \to 0$, as $t \to \infty$, for all k;

(vi) $E(X_1) > E(X_2) > \cdots > E(X_k) > \cdots;$

(vii) $E(Y_k) < \infty$ for all k.

PROOF. (i) See page 27 of [2].

(ii) Since $Q(t) \to \infty$ as $t \to \infty$, we have

$$\lim_{t \to \infty} \{Q(t)\}^k \, \overline{F}(t) = \lim_{Z \to \infty} Z^k e^{-Z} = 0 \quad \text{for all } k.$$

(iii) Since $Z = Q(t)$ is a monotone increasing convex function, there exist non-negative numbers α and Z_0 such that

$$Q^{-1}(Z) < \alpha Z \quad \text{for all } Z > Z_0.$$

Then we have

$$\lim_{t \to \infty} t\{Q(t)\}^k \, \overline{F}(t) = \lim_{Z \to \infty} Q^{-1}(Z) Z^k e^{-Z}$$

$$\leq \alpha \lim_{Z \to \infty} Z^{k+1} e^{-Z} = 0.$$

(iv) For $|G(t)| < G(o) = m < \infty$ by (i), the integration by parts implies that

$$G(t) = t\overline{F}(t) + \int_t^\infty \overline{F}(y)dy. \tag{3.16}$$

Since

(3.17)

$$\{Q(t)\}^k \int_t^\infty \overline{F}(y)dy \leq \int_t^\infty \{Q(y)\}^k \, \overline{F}(y)dy = \int_{Q^{-1}(t)}^\infty Z^k e^{-Z} \frac{dZ}{q(Q^{-1}(Z))}$$

$$\leq \frac{1}{q(t)} \int_{Q^{-1}(t)}^\infty Z^k e^{-Z} dZ \to 0 \quad \text{as} \quad t \to \infty,$$

(iii) and (3.17) with (3.16) imply (iv).

(v) This is obvious from (iii) and (iv).

(vi) Let $A(x)$ be a set such as

$$A(x) = \{\omega: \; x < Y_{k-1}(\omega) < x + dx\}$$

for arbitrary fixed k. Under the condition that $\omega \in A(x)$, $X_k(\omega)$ is a random variable with failure rate $q(x)$ and $X_{k+1}(\omega)$ is another random variable with failure rate larger than $q(x)$. Then we have

$$E(X_k(\omega)|A(X)) > E(X_{k+1}(\omega)|A(X)),$$

which implies that $E(X_k) > E(X_{k+1})$.

(vii) From (vi), we have

$$E(Y_k) < kE(X_1) = km < \infty.$$

4. Existence and Uniqueness of the Optimal Policy of Type III.

In the previous paper [3], we considered the function

$$f(k) = \frac{(k-1)T_m + T_s}{\sum_{i=1}^{M} p_i \int_0^\infty t\gamma_i^{(k)}(t)dt + (k-1)T_m + T_s} \tag{4.1}$$

in order to find the optimal policy of type III, where

$$\gamma_i^{(k)}(t) = \frac{\{Q_i(t)\}^{k-1}}{(k-1)!} e^{-Q_i(t)} q_i(t) = \frac{\{Q_i(t)\}^{k-1}}{(k-1)!} f_i(t) \tag{4.2}$$

and p_i $(i = 1, \ldots, M)$ is an a priori probability of appearance of a system belonging to i-th category in which every system has the failure density function $f_i(x)$. If $f(k)$ of (4.1) is a convex function, the optimal policy of type III is to perform preventive maintenance at each k_0-th failure, where k_0-1 is a largest integer such that

$$\left(1 - \frac{\overline{U}(k)}{\overline{U}(k+1)}\right)\left(k + \frac{T_s}{T_m}\right) > 1, \tag{4.3}$$

where

$$\overline{U}(k) = \sum_{i=1}^{M} p_i \; E(Y_k^{(i)})$$

and $Y_k^{(i)}$ is the total operating time until the k-th failure of a system belonging to i-th category occurs. Under the assumption of a Weibull-type failure distribution with a common shape parameter, we can easily prove the convexity of $f(x)$. Thus, both existence and uniqueness of the optimal policy of type III are known in this case. But any further discussion on them without the Weibull assumption has not yet been done.

In this section we shall treat the problem under the assumption of IFR. In order to avoid the non-essential discussion, we shall confine ourselves to the case $M = 1$. It is easy to expand the discussion to the case $M > 1$.

First, we have to find the joint distribution of (Y_{i-1}, Y_i) for $i = 2, 3, \dots$. Denoting the joint probability density $h(y_{i-1}, y_i)$, we have

(4.4)

$$h(y_{i-1}, y_i)dy_{i-1}dy_i$$

$$= \Pr\{y_{i-1} < Y_{i-1} < y_{i-1} + dy_{i-1},\ \ y_i < Y_i < y_i + dy_i,\ \ Y_{i-1} < Y_i\}$$

$$= \Pr\{y_{i-1} < Y_{i-1} < y_{i-1} + dy_{i-1}\}$$

$$\cdot \Pr\{y_i < Y_i < y_i + dy_i,\ \ y_{i-1} < Y_i | y_{i-1} < Y_{i-1} < y_{i-1} + dy_{i-1}\}$$

$$= \Pr\{y_{i-1} < Y_{i-1} < y_{i-1} + dy_{i-1}\}$$

$$\cdot \Pr\{y_{i-1} < Y_i | y_{i-1} < Y_{i-1} < y_{i-1} + dy_{i-1}\}$$

$$\cdot \Pr\{y_i < Y_i < y_i + dy_i | y_{i-1} < Y_{i-1} < y_{i-1} + dy_{i-1},\ \ y_{i-1} < Y_i\}$$

$$= \frac{\{Q(y_{i-1})\}^{(i-2)}}{(i-2)!} e^{-Q(y_{i-1})} q(y_{i-1})dy_{i-1} \cdot 1 \cdot \frac{f(y_i)}{\overline{F}(y_{i-1})} dy_i$$

$$= \frac{\{Q(y_{i-1})\}^{(i-2)}}{(i-2)!} q(y_{i-1}) \cdot f(y_i)dy_{i-1}dy_i \ .$$

Of course, if we integrate (4.4), we have marginal density as follows:

$$\gamma^{(i)}(y_i) = \int_0^{y_i} h(y_{i-1}, y_i)dy_{i-1} \tag{4.5}$$

$$= \int_0^{Q(y_i)} \frac{Z^{i-2}}{(i-2)!} dZ \cdot f(y_i)$$

$$= \frac{\{Q(y_i)\}^{(i-1)}}{(i-1)!} \cdot f(y_i),$$

and

$$\gamma^{(i-1)}(y_{i-1}) = \int_{y_{i-1}}^{\infty} h(y_{i-1}, y_i)dy_i \tag{4.6}$$

$$= \frac{\{Q(y_{i-1})\}^{(i-2)}}{(i-2)!} q(y_{i-1})\overline{F}(y_{i-1})$$

$$= \frac{\{Q(y_{i-1})\}^{(i-2)}}{(i-2)!} f(y_{i-1}).$$

We can easily see that (4.2), (4.5) and (4.6) have a common form as was desired.

Now, we shall assume here that $\{a_i\}$ is a sequence of descending order, i.e., $a_1 \geq a_2 \geq \cdots$. In this case, the probability that a preventive maintenance is performed at the i-th failure under Policy V is given by

$$(4.7) \qquad P(1) = \int_{a_1}^{\infty} f(y)dy, \quad \text{for} \quad i = 1;$$

$$(4.8) \qquad P(i) = \left[\int_{a_i}^{a_{i-1}} \int_{y_{i-1}}^{\infty} + \int_{o}^{a_i} \int_{a_i}^{\infty}\right] h(y_{i-1}, y_i)dy_i dy_{i-1}, \quad \text{for} \quad i > 1.$$

These hold because the event that a preventive maintenance is performed at i-th failure will occur in the following two ways:

(A) $a_i < Y_i \leqq a_{i-1}$,

(B) $Y_{i-1} \leqq a_i$ and $a_i < Y_i$.

Hence, we have the expected total operating time $E(Y)$ and the expected number of failures $E(K)$ as follows:

$$(4.9) \qquad E(Y) = \int_{a_1}^{\infty} yf(y)dy + \sum_{i=2}^{\infty} \left[\int_{a_i}^{a_{i-1}} \int_{t}^{\infty} + \int_{o}^{a_i} \int_{a_i}^{\infty}\right] yh(t,y)dydt$$

$$= \sum_{i=1}^{\infty} \left[\frac{\{Q(a_i)\}^{i-1}}{(i-1)!} G(a_i) + \int_{a_{i+1}}^{a_i} \frac{\{Q(t)\}^{i-1}}{(i-1)!} q(t)G(t)dt\right],$$

where

$$G(t) = \int_{t}^{\infty} yf(y)dy;$$

$$(4.13) \qquad E(K) = k \cdot \frac{\{Q(\xi)\}^{k-1}}{(k-1)!} \bar{F}(\xi) + \int_0^{\xi} (k+1) \cdot \frac{\{Q(t)\}^{k-1}}{(k-1)!} q(t)\bar{F}(t)dt$$

$$+ \int_{\xi}^{\infty} k \frac{\{Q(t)\}^{k-2}}{(k-2)!} q(t)\bar{F}(t)dt$$

$$= k + \frac{1}{(k-1)!} \int_0^{\xi} \{Q(t)\}^{k-1} q(t)e^{-Q(t)} dt.$$

Inserting these into (4.11), we get the limiting efficiency of Policy III'. Since maximizing the limiting efficiency is equivalent to minimizing its reciprocal, it is sufficient to minimize the following function for our purpose.

$$(4.14) \qquad v(k,\xi) = \frac{(k-1)T_m + T_s + w_1(k,\xi)}{E(Y_k) + w_2(k,\xi)},$$

where $w_1(k,\xi)$ and $w_2(k,\xi)$ are given by (3.8) and (3.9), respectively.

Now, our first assertion is the following.

THEOREM 1. If the failure distribution of a system is strictly IFR, then there exists an optimal policy of type III. The optimal policy is determined uniquely except in the case in which the succeeding two values of k give the same limiting efficiency.

REMARK 1. The realization of the above exceptional case will be seldom expected, because it will occur when and only when (4.21) below holds. Essentially, the theorem asserts the uniqueness of the optimal policy of type III.

PROOF OF THE THEOREM. $w_1(k,\xi)$ and $w_2(k,\xi)$ are obtained in Lemma 3 by setting

$$(4.15)\qquad u_1(x) = T_m e^{-Q(x)} ,$$

$$u_2(x) = \int_x^\infty (y-x)f(y)dy;$$

$$(4.16)\qquad dS(x) = \frac{1}{(k-1)!}\{Q(x)\}^{k-1}\, q(x)dx.$$

If we define $u_1(x)$ and $u_2(x)$ as above, we can easily see that $h(x)$ in Lemma 2 is equal to $T_m u_2(x)/u_1(x)$. Hence, when $f(x)$ is IFR, the function $w_1(k,\xi)/w_2(k,\xi)$ increases strictly monotonically. This is a direct conclusion by Lemma 2 and Lemma 3.

In order to simplify the notation, we shall define an operation $\oplus$ as follows. Let

$$x(t) = \frac{A(t)}{B(t)} \quad \text{and} \quad \gamma(t) = \frac{C(t)}{D(t)} ,$$

then we denote

$$x(t) \oplus \gamma(t) \equiv \frac{A(t) + C(t)}{B(t) + D(t)} .$$

We can not multiply a factor the numerator and denominator of $x(t)$ and/or $\gamma(t)$ and keep the definition meaningful.

We shall denote

$$Z(k,\xi) \equiv \frac{w_1(k,\xi)}{w_2(k,\xi)} .$$

Now we may write

$$(4.17)\qquad v(k,\xi) = v(k,o) \oplus Z(k,\xi)$$

and

$$(4.18)\qquad v(k+1, o) = v(k,o) \oplus Z(k,\infty).$$

Now, if

$$(4.19)\qquad \frac{T_s + (k-1)T_m}{E(Y_k)} > \frac{T_m}{E(X_{k+1})} ,$$

we have

$$v(k, o) > v(k+1, o)$$

by (4.18) because the LHS of (4.19) equals $v(k,o)$ and the RHS equals $Z(k,\infty)$. Similarly if

$$(4.20)\qquad \frac{(k-1)T_m + T_s}{E(Y_k)} < \frac{T_m}{E(X_{n+1})} ,$$

we have

$$v(k, o) < v(k+1, o).$$

When the equality

$$(4.21)\qquad \frac{(k-1)T_m + T_s}{E(Y_k)} = \frac{T_m}{E(X_{k+1})}$$

holds, we have

$$v(k, o) = v(k+1, o),$$

and, furthermore, since $E(X_{k+1}) > E(X_{k+2})$, we have

$$\frac{kT_m + T_s}{E(Y_{k+1})} = \frac{T_m}{E(X_{k+1})} < \frac{T_m}{E(X_{k+2})} .$$

Thus, if (4.21) holds for a value of k, the inequality (4.20) holds for the next value of k.

We can show that (4.20) holds for a finite value of k except when $\lim_{t\to\infty} q(t) = q(\infty) < \infty$. If it holds for k_0, it is obvious that (4.20) holds for $k > k_0$ by the monotonity of $\{E(X_k)\}$. Thus, if (4.20) holds firstly for k_0, we have

$$v(1,o) > v(2,o) > \cdots > v(k_0,o) < v(k_0+1, o) < \cdots$$

which means k_0 corresponds to the optimal policy of type III and k_0 is finite.

If (4.20) does not hold for every finite k, then (4.25) holds for every k, that is the relation:

$$(4.22) \qquad v(1,o) > \frac{T_m}{E(X_{k+1})} \quad \text{for all } k$$

is true. (4.22) may hold for only a case that

$$(4.23) \qquad \lim_{t \to \infty} q(t) = q(\infty) < \infty.$$

Otherwise, $E(X_k) \to 0$ as $k \to \infty$. If (4.22) holds, the optimal policy of type III is not to perform preventive maintenance at any failure. This corresponds to the case $k_0 = \infty$. Including such a case, the assertion of the theorem is verified.

REMARK 2. The case in which (4.23) holds is essentially identical to the case of an exponential failure distribution for sufficiently large t. In such a case, it is well known that any preventive maintenance has no effect. So, the above conclusion that $k_0 = \infty$ is very natural.

5. **Optimal Policy in the Sense of Maintenance Cost Rate.** In Section 4, we discussed the existence and uniqueness of Policy III in the sense of its limiting efficiency. As was noted in Section 2, the concept of the limiting efficiency are generalized to the concept of the maintenance cost rate. Thus, we have a natural question whether the assertion in Section 4 is true or not in the generalized case. This is answered affirmatively by the following.

Theorem 2. If the failure distribution of a system is strictly IFR, then there exists an optimal policy of type III in the sense of maintenance cost rate. The optimal policy is determined uniquely except in the case when the succeeding two values of k give the same maintenance cost rate.

Proof. We shall consider the following amount

$$C(k) \equiv C_o + \frac{C_m}{T_m} - C_\infty(k) \tag{5.1}$$

$$= \frac{(C_o + C_m/T_m)E(Y_k) + C_mT_s/T_m - C_s}{E(Y_k) + (k-1)T_m + T_s}$$

$$= \left(C_o + \frac{C_m}{T_m}\right)a(k) + \frac{C_mT_s - C_sT_m}{T_m}\, b(k) ,$$

where

$$a(k) = \frac{E(Y_k)}{E(Y_k) + (k-1)T_m + T_s} = \frac{1}{1 + v(k,o)} \tag{5.2}$$

and

$$b(k) = \frac{1}{E(Y_k) + (k-1)T_m + T_s} . \tag{5.3}$$

Hence we have

(5.4)

$$\Delta a(k) \equiv a(k+1) - a(k) = \frac{\{(k-1)T_m + T_s\}E(X_{k+1}) - E(Y_k)T_m}{\{E(Y_k) + (k-1)T_m + T_s\}\{E(Y_{k+1}) + kT_m + T_s\}}$$

and

(5.5) $$\Delta b(k) \equiv b(k+1) - b(k) = -\frac{T_m + E(X_{k+1})}{\text{the denominator of (5.4)}} .$$

Now, we want to maximize $C(k)$ in order to find a k which minimizes the maintenance cost rate $C_\infty(k)$. By Theorem 1, there exists a k (say $k_0 \leqq \infty$) which maximizes $a(k)$. Since $b(k)$ is a monotone decreasing function of k, we can easily see that if $C_m T_s > C_s T_m$, then

(5.6) $$C(k) < C(k_0) \quad \text{for every} \quad k > k_0.$$

This is true even if the equality $v(k_0, 0) = v(k_0+1, 0)$ holds. (This is the exceptional case of Theorem 1.)

On the other hand, the numerator of (5.4) can be rewritten as follows.

(5.7)

$$\{(k-1)T_m + T_s\}E(X_{k+1}) - E(Y_k)T_m = T_s E(X_{k+1}) - T_m[E(Y_{k+1}) - kE(X_{k+1})] .$$

Since

(5.8)

$$\begin{aligned}
&[E(Y_{k+1}) - kE(X_{k+1})] - [E(Y_k) - (k-1)E(X_k)] \\
&\quad = [E(Y_{k+1}) - E(Y_k)] - E(X_{k+1}) + (k-1)[E(X_k) - E(X_{k+1})] \\
&\quad = (k-1)[E(X_k) - E(X_{k+1})] > 0,
\end{aligned}$$

(5.7) is a monotone decreasing function of k. Furthermore, the denominator of (5.4) is a positive and monotone increasing function of k.

These facts mean that $a(k)$ is a convex and monotone increasing function of k for $k \le k_0$.

Let k^* be the k such that $\Delta C(k^*-1) > 0$ and $\Delta C(k) \le 0$ for all $k^* \le k \le k_0$. The existence of k^* is clear for $\Delta C(k_0) < 0$ except the case that $\Delta C(k) \le 0$ for all $k \ge 1$. In the last case, let $k^* = 1$. When $k^* > 1$, we can see that $\Delta C(k) > 0$ for $k \le k^* - 1$ as follows. From the fact that

$$\Delta C(k^*) = C_1 \Delta a(k^*) + C_2 \Delta b(k^*) \le 0$$

and

$$\Delta C(k^*-1) = C_1 \Delta a(k^*-1) + C_2 \Delta b(k^*-1) > 0 ,$$

we have

(5.9)

$$C_1[T_s E(X_{k^*+1}) - T_m\{E(Y_{k^*+1}) - k^* E(X_{k^*+1})\}] \le C_2[T_m + E(X_{k^*+1})]$$

$$C_1[T_s E(X_{k^*}) - T_m\{E(Y_{k^*}) - (k^*-1)E(X_{k^*})\}] > C_2[T_m + E(X_{k^*})] .$$

using (5.4), (5.5) and (5.7), where

$$C_1 = C_0 + \frac{C_m}{T_m} \quad \text{and} \quad C_2 = \frac{C_m T_s - C_s T_m}{T_m} ,$$

(5.9) with (5.8) implies that

$$C_2[E(X_{k^*+1}) - E(X_{k^*})] > C_1(T_s - (k^*-1)T_m)[E(X_{k^*+1}) - E(X_{k^*})] ,$$

which is equivalent to the inequality

(5.10) $$C_1 T_s - C_2 - C_1(k^*-1)T_m > 0$$

by (vi) of Lemma 5. Since $C_1 T_m > 0$, (5.10) implies that

(5.11) $$C_1T_s - C_2 - C_1kT_m > 0 \quad \text{for every} \quad k \le k^* - 1 \ .$$

Now

(5.12)

$$\begin{aligned}
\Delta C(k^*-2) &= C_1\Delta a(k^*-1) + C_2\Delta b(k^*-1) \\
&\quad + C_1[\Delta a(k^*-2) - \Delta a(k^*-1)] + C_2[\Delta b(k^*-2) - \Delta b(k^*-1)] \\
&= C_1\Delta a(k^*-1) + C_2\Delta b(k^*-1) \\
&\quad + b(k^*)b(k^*-1)b(k^*-2)\Big\{C_1[E(Y_{k^*}) \\
&\quad + (k^*-1)T_m + T_s]\Big[T_sE(X_{k^*-1}) - T_m[E(Y_{k^*-1}) \\
&\quad - (k^*-2)E(X_{k^*-1})]\Big] - C_1[E(Y_{k^*-2}) \\
&\quad + (k^*-2)T_m + T_s]\Big[T_sE(X_{k^*}) - T_m[E(Y_{k^*}) \\
&\quad - (k^*-1)E(X_{k^*})]\Big] + C_2[E(Y_{k^*-2}) + (k^*-2)T_m + T_s][T_m + E(X_{k^*})] \\
&\quad - C_2[E(Y_{k^*}) + (k^*-1)T_m + T_s][T_m + E(X_{k^*-1})]\Big\} \\
&= \Delta C(k^*-1) + b(k^*)b(k^*-1)b(k^*-2)[E(Y_{k^*}) + (k^*-1)T_m + T_s] \\
&\quad \times [C_1T_s - C_2 - C_1T_m(k^*-1)][E(X_{k^*-1}) - E(X_{k^*})] \\
&\quad + b(k^*-2) \cdot [E(X_{k^*-1} + X_{k^*}) + 2T_m] \cdot \Delta C(k^*-1) \ .
\end{aligned}$$

By our assumption and (5.10), we can easily see that (5.12) is positive, which means $\Delta C(k^*-2) > 0$. Replacing k^* in (5.12) by k^*-1 and using (5.11) and the above result, we find that $\Delta C(k^*-3) > 0$. Repeating the argument, we can say that $\Delta C(k) > 0$ for $k \le k^*-1$. Thus k^* gives the optimal policy of type III in the sense of the maintenance cost rate.

If $\Delta C(k^*) = 0$ for some integer k^*, then $C(k^*) = C(k^*+1)$. Except in such a case, k^* is unique as was shown in above. Thus the proof of the theorem completes in the case $C_2 > 0$.

For the case $C_2 = 0$ (i.e., $C_mT_s=C_sT_m$), the assertion of the theorem is identical with the one of Theorem 1, already proved.

If $C_2 < 0$, (5.6) does not hold. Moreover, $C(k)$ is a convex function of k for $k < k_o$ because $a(k)$ and $C_2b(k)$ are both convex function in this interval. Hence, there does not exist the optimal k in $[1, k_o)$. Let $\hat{k}$ be a k such that $a(\hat{k}-1) > C(k_o) \geqq a(\hat{k})$ for $k > k_o$. Clearly, $\hat{k}$ exists, since $a(k)$ is a monotone decreasing function for $k > k_o$. Since $a(k)$ dominates $C(k)$ and monotonically decreases for $k > k_o$, there exists a k which maximize $C(k)$ in $[k_o, \hat{k}-1]$. In order to prove the uniqueness, we shall denote by k^* the first value of k which maximize $C(k)$. This means that $\Delta C(k^*) \leq 0$ and $\Delta C(k^*-1) > 0$. We have

$$\begin{aligned}
\Delta C(k^*+1) &= \Delta C(k^*) + [\Delta C(k^*+1) - \Delta C(k^*)] \\
&= \Delta C(k^*) + b(k^*)b(k^*+1)b(k^*+2)\Big\{C_1[E(Y_{k^*}) + (k^*-1)T_m + T_s] \\
&\quad \times \Big[T_sE(X_{k^*+2}) - T_m[E(X_{k^*+2}) - (k^*+1)E(X_{k^*+2})]\Big] \\
&\quad - C_1[E(Y_{k^*+2}) + (k^*+1)T_m + T_s] \\
&\quad \times \Big[T_sE(X_{k^*+1}) - T_m[E(X_{k^*+1}) - k^*E(X_{k^*+1})]\Big] \\
&\quad - C_2[E(Y_{k^*}) + (k^*-1)T_m + T_s] \\
&\quad \times [T_m + E(X_{k^*+2})] + C_2[E(Y_{k^*+2})
\end{aligned}$$

$$+ (k^*+1)T_m + T_s] \cdot [T_m + E(X_{k^*+1})]\Big\}$$

$$= \Delta C(k^*)$$

$$+ b(k^*)b(k^*+1)b(k^*+2)[E(Y_{k^*}) + (k^*-1)T_m + T_s]$$

$$\times [C_1(T_s + k^*T_m) - C_2][E(X_{k^*+2}) - E(X_{k^*+1})]$$

$$+ b(k^*+2)[C_1\Delta a(k^*) + C_2\Delta b(k^*)]$$

$$< 0.$$

A similar argument proves $\Delta C(k) < 0$ for all $k \geq k^*+1$. Thus we have proved the theorem for $C_2 < 0$.

When $k_0 = \infty$, these arguments above are trivially true.

6. **Optimal Policy of Type III'.** Since the class of all policies of type III' includes the class of all policies of type III, the limiting efficiency of the optimal policy of type III' is equal to or higher than the one of the optimal policy of type III. In this section, we shall show that the former is strictly higher than the latter, and we shall discuss how to find the optimal policy of type III'.

Now, if (4.20) holds, from Lemma 3 and Lemma 4 we can say that there exists ξ_0 such that

(6.1) $$v(k,0) = Z(k,\xi_0);$$

hence

$$Z(k,\xi) < v(k,0) \quad \text{for} \quad 0 < \xi < \xi_0$$

and

$$Z(k,\xi) > v(k,0) \quad \text{for} \quad \xi > \xi_0.$$

These inequalities imply that

$$v(k,\xi) < v(k,0) \quad \text{for} \quad 0 < \xi < \xi_0 \tag{6.2}$$

and

$$v(k,\xi) > v(k,0) \quad \text{for} \quad \xi > \xi_0.$$

Thus we have the following.

LEMMA 6. The optimal policy of type III' has strictly higher limiting efficiency than the optimal limiting efficiency of Policy III.

PROOF. Even for k_0, which gives the optimal policy of type III, the inequality (6.2) holds. This means that the assertion of the lemma is true.

Next, we shall show the following.

LEMMA 7. For any k, let

$$v(k, \hat{\xi}(k)) = \min_{\xi} v(k, \xi) .$$

Then $\hat{\xi}(k)$ is unique.

PROOF. It is clear that there exists $\hat{\xi}(k)$, because $v(k,\xi)$ is bounded and continuous. For any positive number ε, assume that there exist ξ_1 and ξ_2 such that

$$0 < \xi_1 < \xi_2 < \xi_0$$

and

$$v(k, 0) - \varepsilon = v(k, \xi_1) = v(k, \xi_2),$$

where ξ_0 was defined in (6.1). From the continuity of $v(k,\xi)$, if we choose ε small enough, we can get such ξ_1 and ξ_2. Without loss of generality, we can assume that the equality

$$v(k,0) - \varepsilon = v(k,\xi) \tag{6.3}$$

does not hold for $\xi < \xi_1$ and $\xi > \xi_2$. Let

$$w_i^*(k, \xi) = \int_{\xi_1}^{\xi} u_i(x)dS(x), \quad (i = 1, 2) ,$$

where $u_i(x)$ and $dS(x)$ are given by (4.15) and (4.16). An analogous argument to the proof of Lemma 3 shows that

$$Z^*(k, \xi) \equiv \frac{w_1^*(k,\xi)}{w_2^*(k,\xi)}$$

is a monotone increasing and continuous function. Hence, since

$$v(k,\xi_2) = v(k,\xi_1) \oplus Z^*(k,\xi_2) = v(k,\xi_1) ,$$

we have

$$Z^*(k,\xi_2) = v(k,\xi_1) .$$

Thus, for any $\xi \in (\xi_1,\xi_2)$, from the fact that

$$Z^*(k,\xi) < v(k,\xi_1),$$

we have

$$v(k,\xi) \;=\; v(k,\xi_1) \oplus Z^*(k,\xi) < v(k,\xi_1) \;=\; v(k,\xi_2) \;.$$

This means that $\hat{\xi}(k) \in (\xi_1,\xi_2)$ and (6.3) does not hold for any $\xi \in (\xi_1,\xi_2)$.

Now, let us fix $\varepsilon > 0$ arbitrary. We examine whether the following equation (6.4) holds or not for $i=0$, $\xi_1^{(i)}=0$, $\varepsilon_i=\varepsilon$, and some $\xi_1^{(i+1)}$, $\xi_2^{(i+1)}$.

$$\text{(6.4)} \qquad v(k,\xi_1^{(i)}) - \varepsilon_i \;=\; v(k,\xi_1^{(i+1)}) \;=\; v(k,\xi_2^{(i+2)}).$$

If it holds, since we can conclude that $\hat{\xi}(k) \in (\xi_1^{(i+1)},\xi_2^{(i+1)})$ using the above argument, proceed with Procedure A in below. If it does not, proceed with Procedure B. Continuing procedure A or B for $i = 1, 2, \ldots,$ we can get $\hat{\xi}(k)$ uniquely.

[Procedure A]. Add 1 to i. Put $\varepsilon_i=\varepsilon_{i-1}$. (i.e., ε's for new and old i are same.) Then examine (6.4) for new i.

[Procedure B]. Add 1 to i. Put $\varepsilon_i=\varepsilon_{i-1/2}$ and $\xi_1^{(i)}=\xi_1^{(i-1)}$. Then examine (6.4) for new i.

Thus, the lemma has been proved.

LEMMA 8. For any positive integer k, $\hat{\xi}(k)$ is the root of the following equation.

$$\text{(6.5)} \qquad v(k,\xi) \;=\; T_m e^{-Q(\xi)} \Big/ \int_\xi^\infty (y-\xi)f(y)dy \;.$$

PROOF. Since $v(k,\xi)$ is a monotone increasing function for $\xi > \xi_0$ and $\hat{\xi}(k)$ is a unique minimum point of $v(k,\xi)$, there exist one and only one point such as $\frac{d}{d\xi} v(k,\xi) = 0$. This is $\hat{\xi}(k)$. Differentiating $v(k,\xi)$, we have

$$\text{(6.6)} \qquad \frac{d}{d\xi}\left\{\frac{(k-1)T_m + T_s + w_1(k,\xi)}{E(Y_k) + w_2(k,\xi)}\right\}$$

$$= \frac{w_1'(k,\xi)\{E(Y_k) + w_2(k,\xi)\} - w_2'(k,\xi)\{(k-1)T_m + T_s + w_1(k,\xi)\}}{\{E(Y_k) + w_2(k,\xi)\}^2} .$$

Since the denominator of the RHS of (6.6) is positive, equating the numerator to zero we get

$$\text{(6.7)} \qquad \frac{w_1'(k,\xi)}{w_2'(k,\xi)} = v(k,\xi) .$$

The LHS of (6.7) is equal to

$$\frac{T_m e^{-Q(\xi)}}{\int_\xi^\infty (y-\xi)f(y)dy} ,$$

which follows from (3.8) and (3.9) and completes the proof of the lemma.

Summarizing Lemma 6-8, we may assert the following.

THEOREM 3. The optimal policy of type III' for a system whose failure distribution is strictly IFR is given by $\hat{\xi} = \min_k \hat{\xi}(k)$, where $\hat{\xi}(k)$ is the unique root of (6.5). The limiting efficiency for the optimal policy of type III' is $h(\hat{\xi})/\{T_m + h(\hat{\xi})\}$ and this is higher than the limiting efficeincy for the optimal policy of type III, where $h(t)$ was defined in Lemma 2.

REMARK 3. This theorem asserts that Policy III is not optimal in the class of Policy V and suggests that Theorem 5.1 of [4] must be false. (An argument in the proof of the latter (p.39, ℓ.11 in [4]) was incorrect.) However, because a tedious computation may be needed for finding $\hat{\xi}$, if the limiting efficiency for the optimal policy of type III' is near to the one for the optimal policy of type III, Policy III may be used from a practical view point. The author hopes to calculate these limiting efficiencies numerically in the near future.

REMARK 4. We shall illustrate the features of several functions $v(k,\xi)$, $Z(k,\xi)$, $Z^*(k,\xi)$ and $T_m u_1(t)/u_2(t)$ in Fig. 1. The author conjectures that such a situation as illustrated in Fig. 2 does not occur. But, unfortunately, the proof has not been discovered. If we prove the fact, we may find $\hat{\xi}$ more easily.

REMARK 5. We can rewrite $w_1(k,\xi)$ as follows:

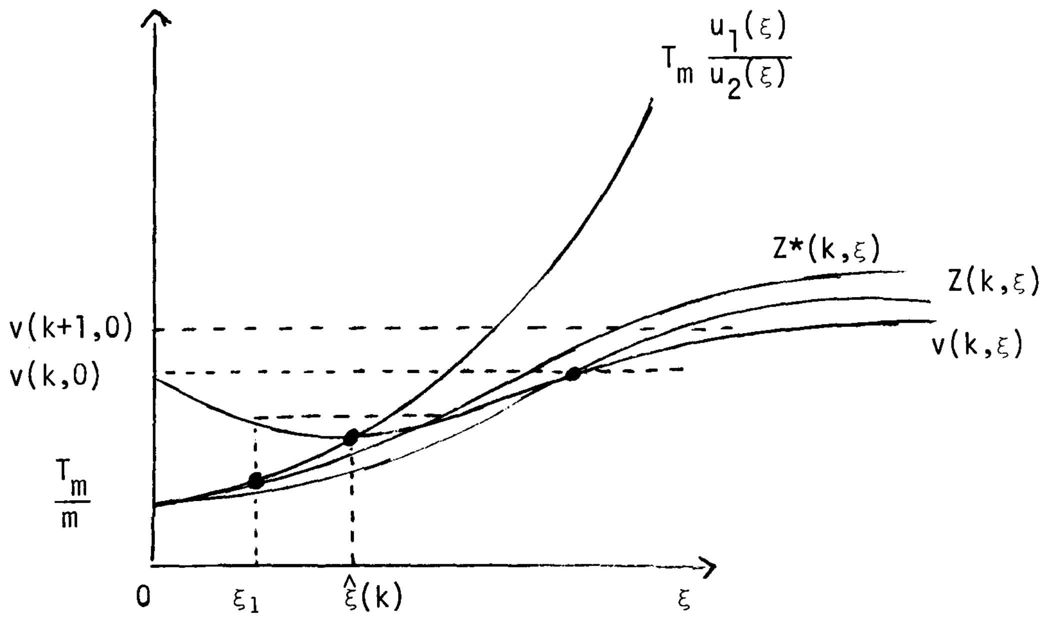
$$w_1(k,\xi) = \frac{T_m}{\Gamma(k)} \int_0^{\xi} e^{-Q(t)} \{Q(t)\}^{k-1} q(t)dt$$

$$= \frac{T_m}{\Gamma(k)} \int_0^{Q(\xi)} e^{-Z} Z^{k-1} dZ$$

$$= \frac{T_m \gamma(k, Q(\xi))}{\Gamma(k)},$$

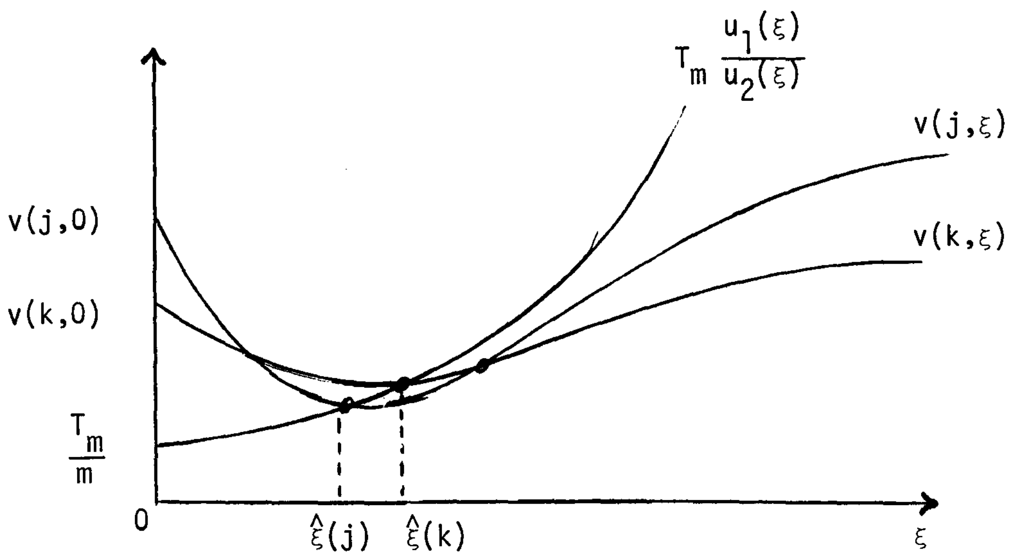
where $\gamma(k,\xi)$ denotes an incomplete Γ function. Thus, we have

$$v(k,\xi) = \frac{\{(k-1)T_m + T_s\}\Gamma(k) + T_m \gamma(k, Q(\xi))}{E(Y_k)\Gamma(k) + \{Q(\xi)\}^k \int_{\xi}^{\infty} (y-\xi)f(y)dy + \gamma(k+1, Q(\xi))}.$$



$T_m \frac{u_1(\xi)}{u_2(\xi)}$
$Z^*(k,\xi)$
$Z(k,\xi)$
$v(k+1,0)$
$v(k,0)$
$v(k,\xi)$
$\frac{T_m}{m}$
0
ξ_1
$\hat{\xi}(k)$
ξ

Fig. 1



$T_m \frac{u_1(\xi)}{u_2(\xi)}$
$v(j,\xi)$
$v(j,0)$
$v(k,\xi)$
$v(k,0)$
$\frac{T_m}{m}$
0
$\hat{\xi}(j)$
$\hat{\xi}(k)$
ξ

Fig. 2

EXAMPLE 1. If the failure distribution of a system is Weibull with a shape parameter β and a scale parameter α, we have

$$\int_\xi^\infty (y-\xi)f(y)dy = \alpha^{-1/\beta}\{\Gamma(\frac{1}{\beta}+1) - \gamma(\frac{1}{\beta}+1, Q(\xi)) - \xi e^{-Q(\xi)}\}$$

and

$$E(Y_k) = \alpha^{-1/\beta}\,\Gamma(k+\frac{1}{\beta})/\Gamma(k) .$$

Then we have

$$v(k,\xi) = \frac{\alpha^{1/\beta}[\{(k-1)T_m + T_s\}\Gamma(k) + T_m\gamma(k,\alpha\xi^\beta)]}{\Gamma(k+\frac{1}{\beta}) + (\alpha\xi^\beta)^k[\Gamma(1+\frac{1}{\beta}) - \gamma(1+\frac{1}{\beta},\alpha\xi^\beta) - \xi e^{-\alpha\xi^\beta}\} + \alpha^{1/\beta}\,\gamma(k+1,\alpha\xi^\beta)} .$$

7. OPTIMAL POLICY OF TYPE N'.

As was shown in the previous section, Policy III could not become the optimal policy in the class of all policies of type III'. This is also true in the class of Policy IV' which includes all the policies of type III. We shall show the fact here.

The limiting efficiency for Policy IV' will be obtained by putting $a_i = \xi$ for $i = 1, \ldots, k-1$ and $a_i = 0$ for $i \geq k$ in (4.9) and (4.10) and then inserting $E(Y)$ and $E(K)$ for (4.9). We shall denote the amount corresponding to $v(k,\xi)$ of (4.14) as $\overline{V}(k,\xi)$, which is given by

(7.1)

$$\overline{V}(k,\xi) = \frac{T_s + (\sum_{i=1}^{k} ie^{-Q(\xi)}\frac{(Q(\xi))^{i-1}}{(i-1)!} + \int_0^\xi kq(t)e^{-Q(t)}\frac{(Q(t))^{k-1}}{(k-1)!}\,dt - 1)T_m}{\sum_{i=1}^{k}\frac{(Q(\xi))^{i-1}}{(i-1)!}\cdot G(\xi) + \int_0^\xi tf(t)\frac{(Q(t))^{k-1}}{(k-1)!}\,dt}$$

$$\equiv \frac{A(k,\xi)}{B(k,\xi)}\ , \quad \text{say.}$$

Since it is clear that this is a differentiable function of ξ, we differentiate it as follows.

(7.2)

$$\frac{d}{d\xi}\overline{V}(k,\xi) = \frac{1}{\{B(k,\xi)\}^2}\left[\left\{\sum_{i=2}^{k} i\,\frac{\{Q(\xi)\}^{i-2}}{(i-2)!}\,e^{-Q(\xi)}\,q(\xi)\right.\right.$$

$$\left. - \sum_{i=1}^{k} iq(\xi)e^{-Q(\xi)}\,\frac{(Q(\xi))^{i-1}}{(i-1)!} + kq(\xi)e^{-Q(\xi)}\,\frac{(Q(\xi))^{k-1}}{(k-1)!}\right\}$$

$$\times\, T_m B(k,\xi) - \left\{\sum_{i=2}^{k}\frac{(Q(\xi))^{i-2}}{(i-2)!}\,q(\xi)G(\xi) - \sum_{i=1}^{k}\frac{(Q(\xi))^{i-1}}{(i-1)!}\,\xi f(\xi)\right.$$

$$\left.\left. + \xi f(\xi)\,\frac{(Q(\xi))^{k-1}}{(k-1)!}\right\} \times A(k,\xi)\right]$$

$$= \frac{1}{(B(k,\xi))^2}\left[T_m\left\{\sum_{i=0}^{k-2}\frac{(Q(\xi))^{i}}{i!}\,e^{-Q(\xi)}\,q(\xi)\right\}\cdot B(k,\xi)\right.$$

$$\left. - \left\{\sum_{i=0}^{k-2}\frac{(Q(\xi))^{i}}{i!}\int_{\xi}^{\infty}(y-\xi)f(y)\,dy\right\}\cdot A(k,\xi)\right]$$

$$= \frac{R(k,\xi)\displaystyle\int_{\xi}^{\infty}(y-\xi)f(y)\,dy}{B(k,\xi)}\,\{\gamma(\xi) - \overline{V}(k,\xi)\}\ ,$$

where

$$\gamma(\xi) = \frac{T_m f(\xi)}{\int_{\xi}^{\infty}(y-\xi)f(y)dy}$$

and

$$R(k,\xi) = \sum_{i=0}^{k-2} \frac{(Q(\xi))^i}{i!} .$$

Here, we can easily show the following.

LEMMA 9.

(i) $$\lim_{\xi \to +0} \gamma(\xi) = \frac{T_m f(+0)}{m} \geq 0.$$

(ii) $$\lim_{\xi \to \infty} \gamma(\xi) = T_m \lim_{\xi \to \infty} \frac{f''(\xi)}{f(\xi)} = T_m\{q(\infty)\}^2 .$$

(iii) If $f(x)$ is strictly IFR, then $\gamma(\xi)$ is a monotone increasing function.

PROOF. (i) is evident. We shall prove (ii) and (iii). The first of these is obvious using L'Hopital's rule. Furthermore, $\gamma(\xi)$ may be rewritten as

$$\gamma(\xi) = \frac{T_m q(\xi)e^{-Q(\xi)}}{\int_{\xi}^{\infty} (y-\xi)f(y)dy} = \frac{T_m q(\xi)}{h(\xi)} ,$$

where $h(\xi)$ was given by (3.2) in Lemma 2, which showed that $h(\xi)$ is a monotone decreasing function if $f(x)$ is strictly IFR and that $\lim_{\xi\to\infty} h(\xi) = 1/q(\infty)$. Thus the assertions of the lemma are true.

Now, we can see that $\overline{V}(k,o) = v(1,o)$ and $\overline{V}(k,\infty) = v(k,o)$, because the former case corresponds to Policy III for $k=1$, where $\overline{V}(k,\infty) = \lim_{\xi\to\infty} \overline{V}(k,\xi)$. Thus $\overline{V}(k,\infty)$ is finite for each k by Theorem 1 and $\overline{V}(k,\xi)$ converges to $\overline{V}(k,\infty)$ smoothly. Because, in the following expression of (7.2),

(7.3)

$$\frac{d\overline{V}(k,\xi)}{d\xi} = \frac{T_m R(k,\xi)f(\xi) - R(k,\xi)\int_\xi^\infty (y-\xi)f(y)dy \cdot \overline{V}(k,\xi)}{R(k,\xi)G(\xi) + \int_0^\xi \frac{(Q(t))^{k-2}}{(k-2)!} \cdot q(t)G(t)dt},$$

the terms $R(k,\xi)f(\xi)$, $R(k,\xi)\int_\xi^\infty(y-\xi)f(y)dy$, and $R(k,\xi)G(\xi)$ will tend to zero as ξ tends to infinity by Lemma 5; while $\overline{V}(k,\xi)$ and the second term of the denominator will tent to constants, respectively. So we have $\lim_{\xi\to\infty} \frac{d\overline{V}(k,\xi)}{d\xi} = 0$. Of course, for any fixed k $\overline{V}(k,\xi)$ becomes its minimum value at $\hat{\xi}(k)$ such that $\gamma(\hat{\xi}) = \overline{V}(k,\hat{\xi}(k))$ if $T_m f(+0) < T_s$. This fact will be shown as follows. If $\gamma(+0) = T_m f(+0)/m < T_s/m = v(1,0) = \overline{V}(k,0)$, then $\overline{V}(k,\xi)$ decreases in a neighbourhood of the origin. In order to have $\frac{d\overline{V}(k,\xi)}{d\xi} = 0$, the relation $\gamma(\xi) = \overline{V}(k,\xi)$ must hold. Hence $\overline{V}(k,\xi)$ will decrease in the interval $(0,\hat{\xi}(k))$, where $\hat{\xi}(k)$ is the first root of the equation

(7.4) $\qquad \gamma(\xi) = \overline{V}(k,\xi).$

Further, at $\hat{\xi}(k)$, since $\gamma(\xi)$ is an increasing function,

$$\gamma(\hat{\xi}(k) + \varepsilon) > \overline{V}(\hat{\xi}(k) + \varepsilon),$$

for sufficiently small $\varepsilon > 0$. This means that $\overline{V}(k,\xi)$ is an increasing function at $\hat{\xi}(k) + \varepsilon$ and that $\hat{\xi}(k)$ is a minimal point. Even if $\hat{\hat{\xi}}(k)$ is another root of the equation (7.4), $\overline{V}(k,\xi)$ will increase at $\hat{\hat{\xi}}(k) + \varepsilon$. Thus, $\hat{\hat{\xi}}(k)$ can not be a maximal point. This fact implies the non-existence of $\hat{\hat{\xi}}(k)$ (i.e., uniqueness of $\hat{\xi}(k)$) and hence the following.

THEOREM 4. If $T_m f(+0) < T_s$, then the optimal policy of type IV' for a system whose failure distribution is strictly IFR is given by $\hat{\xi} = \min_k \hat{\xi}(k)$, where $\hat{\xi}(k)$ is the unique root of the equation $\gamma(\xi) = \overline{V}(k,\xi)$. The limiting efficiency of the optimal policy of type IV' is $\gamma(\hat{\xi})/(1 + \gamma(\hat{\xi}))$. If $T_m f(+0) \geq T_s$, then the optimal policy of type IV' is to replace the system by new one at each failure.

REMARK 6. We shall illustrate the features of $\gamma(\xi)$ and $\overline{V}(k,\xi)$ for several k in Fig. 3. Broken lines indicate the case $T_m f(o) \geq T_s$.

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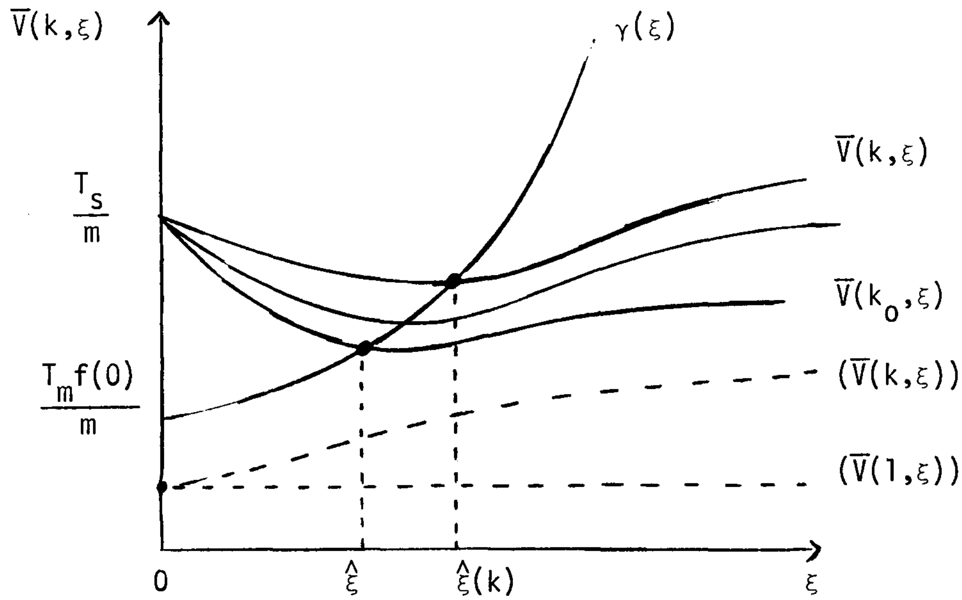

$\overline{V}(k,\xi)$
$\gamma(\xi)$
$\overline{V}(k,\xi)$
$\frac{T_s}{m}$
$\overline{V}(k_o,\xi)$
$\frac{T_m f(0)}{m}$
$(\overline{V}(k,\xi))$
$(\overline{V}(1,\xi))$
0
$\hat{\xi}$
$\hat{\xi}(k)$
ξ

Fig. 3

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