

The research was partially sponsored by NSF Grant No. GU-2059 and the U.S. Air Force Grant No. AFOSR 68-1415.

This work was completed while the author was at the University of North Carolina at Chapel Hill, Department of Statistics.

ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF

$$|S_1 S_2^{-1}| \quad \text{and} \quad |S_2 (S_1 + S_2)^{-1}|$$

FOR TESTING THE EQUALITY OF TWO COVARIANCE MATRICES

by

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DECEMBER 1969

Asymptotic Expansions of the Distributions of $|S_1 S_2^{-1}|$ and $|S_2(S_1+S_2)^{-1}|$ for Testing the Equality of Two Covariance Matrices*

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1. INTRODUCTION. Let $p \times 1$ vectors $X_{i1}, X_{i2}, \dots, X_{iN_i}$ be a random sample from p -variate normal distribution with mean μ_i and covariance matrix Σ_i for $i = 1, 2$. Put $S_i = \sum_{\alpha=1}^{N_i} (X_{i\alpha} - \bar{X}_i)(X_{i\alpha} - \bar{X}_i)'$ and $\bar{X}_i = N_i^{-1} \sum_{\alpha=1}^{N_i} X_{i\alpha}$. Then the statistic $|S_1 S_2^{-1}|$ or $|S_2(S_1+S_2)^{-1}|$ may be used to test the null hypothesis $H: \Sigma_1 = \Sigma_2$ for unknown μ_i against one-sided alternatives $K: \gamma_i \geq 1$ and $\sum_{i=1}^p \gamma_i > p$, where γ_i means the i -th largest characteristic root of $\Sigma_1 \Sigma_2^{-1}$. We can reject H , when the observed value of $|S_1 S_2^{-1}|$ (or $|S_2(S_1+S_2)^{-1}|^{-1}$) is larger than a pre-assigned constant. The monotonicity of the power functions of these tests with respect to γ_j among other tests, were proved by Anderson and Das Gupta [2] from a more general point of view.

Putting $n_i = N_i - 1$ and $n_1 + n_2 = n$, we shall derive, in this paper, asymptotic expansion of the non-null distribution of $\lambda_1 = \sqrt{n} \log |(n_2/n_1) S_1 S_2^{-1}|$, as well as of the null distribution, according to Box [3] and then asymptotic expansions of the distributions of $\lambda_2 = -\sqrt{n} \log |(n/n_2) S_2(S_1+S_2)^{-1}|$ both under the null hypothesis and local alternatives, based on the hypergeometric function of matrix argument due to Constantine [4], James [7] and used by Fujikoshi [5], Sugiura and Fujikoshi [13], Sugiura [12]. The limiting distribution of λ_2 under

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K is also derived. The asymptotic expansions are given in terms of the normal distribution function and its derivatives, which are different from the expansions of the distributions of the likelihood ratio criteria for similar problems in Sugiura [11], [12]. In fact, the null distributions of the likelihood ratio criteria were expanded in terms of the χ^2 distributions with different degrees of freedom. Asymptotic formulas for the upper percentage points of λ_1 and λ_2 will also be given by applying the general inverse expansion formula of Hill and Davis [6], by which some numerical values of the power of these tests are computed and compared with the exact value obtained by Pillai and Jayachandran [9] for the test λ_2 with $p=2$.

In the following discussion, we shall always assume without loss of generality that, under K , S_1 has the Wishart distribution $W_p(n_1, \Gamma)$ and S_2 has $W_p(n_2, I)$, where $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_p)$. Then the null hypothesis H can be expressed by $\Gamma=I$. Put $n_i = \rho_i n$ with $\rho_1 + \rho_2 = 1$ and let n tend to infinity for fixed $\rho_i > 0$.

2. ASYMPTOTIC DISTRIBUTIONS OF $|S_1 S_2^{-1}|$. It is easy to see that the characteristic function of $\lambda_1 = \sqrt{n} \log |(\rho_2/\rho_1) S_1 S_2^{-1}|$ under K , is given by

(2.1)

$$(\rho_2/\rho_1)^{\sqrt{np}it} |\Gamma|^{\sqrt{nit}} \Gamma_p(\frac{1}{2}n_1 + \sqrt{nit}) \Gamma_p(\frac{1}{2}n_2 - \sqrt{nit}) / [\Gamma_p(\frac{1}{2}n_1) \Gamma_p(\frac{1}{2}n_2)],$$

where $\Gamma_p(x) = \pi^{p(p-1)/4} \prod_{j=1}^p \Gamma(x-(j-1)/2)$. Applying the asymptotic formula of $\log \Gamma(x+h)$ for large $|x|$ with fixed h , to the above expression as in Box [3], Anderson ([1], p.204), Sugiura [12] and etc.,

we can rewrite (2.1) as

(2.2)

$$\begin{aligned}
 |\Gamma|^{\sqrt{n}it} \exp \left[-(\rho_1^{-1} + \rho_2^{-1})pt^2 + \sum_{r=1}^{\infty} (2it)^r p n^{-r/2} \right. \\
 \left. \left\{ 2(it)^2(r+1)^{-1}(r+2)^{-1} [(-1)^r \rho_1^{-r-1} + \rho_2^{-r-1}] \right. \right. \\
 \left. \left. + (p+1)[(-1)^r \rho_1^{-r} + \rho_2^{-r}]/(4r) \right\} \right. \\
 \left. - \sum_{r=1}^{\infty} (-2/n)^r B_{r+1} \{r(r+1)\}^{-1} \{(\rho_1 + 2itn^{-\frac{1}{2}})^{-r} \right. \\
 \left. + (\rho_2 - 2itn^{-\frac{1}{2}})^{-r} - \rho_1^{-r} - \rho_2^{-r} \} \right],
 \end{aligned}$$

where $B_{r+1} = \sum_{j=1}^p B_{r+1}(-j-1)/2$ with the Bernoulli polynomial $B_r(h)$

of degree r . Specifying the sequence of alternatives K as

$K_\delta: \Gamma = I + n^{-\delta}\Theta$ ($\delta \geq 0$) for a fixed diagonal matrix Θ having non-

negative diagonal elements and considering the first term of (2.2),

we easily get

THEOREM 2.1. Under K , the statistics λ_1 , $\lambda_1 - \text{tr}\Theta$ and $\lambda_1 - \sqrt{n} \log|\Gamma|$, have asymptotically the same normal distribution with mean zero and variance $\tau^2 = 2p(\rho_1^{-1} + \rho_2^{-1})$, according as $\delta > \frac{1}{2}$, $\delta = \frac{1}{2}$ and $\delta < \frac{1}{2}$ respectively.

There is no singularity of the limiting distribution at the null hypothesis as was the case with the likelihood ratio tests for covariance

matrix in Sugiura [11], [12]. We can write the first four terms of (2.2) as

$$(2.3) \quad \exp\left[-\frac{1}{2}\tau^2 t^2 + it\sqrt{n} \log|\Gamma|\right] \cdot \left[1 + n^{-\frac{1}{2}} \{a_3(it)^3 + a_1(it)\} + n^{-1} \sum_{j=1}^3 b_{2j}(it)^{2j} + n^{-3/2} \sum_{j=1}^5 c_{2j-1}(it)^{2j-1}\right],$$

where $\tau^2 = 2p(\rho_1^{-1} + \rho_2^{-1})$ and

$$(2.4) \quad \begin{aligned} a_1 &= \frac{1}{2}p(p+1)(-\rho_1^{-1} + \rho_2^{-1}), & a_3 &= \frac{2}{3}p(-\rho_1^{-2} + \rho_2^{-2}), \\ b_2 &= \frac{1}{2}a_1^2 + \frac{1}{2}p(p+1)(\rho_1^{-2} + \rho_2^{-2}), & b_4 &= a_1 a_3 + \frac{2}{3}p(\rho_1^{-3} + \rho_2^{-3}), \\ b_6 &= \frac{1}{2}a_3^2, \end{aligned}$$

$$(2.4) \quad \begin{aligned} c_1 &= \frac{1}{12}p(2p^2 + 3p - 1)(-\rho_1^{-2} + \rho_2^{-2}), \\ c_3 &= p(p+1)\left\{\frac{2}{3}(-\rho_1^{-3} + \rho_2^{-3}) + \frac{1}{2}(\rho_1^{-2} + \rho_2^{-2})a_1\right\} + \frac{1}{6}a_1^3, \\ c_5 &= \frac{4}{5}p(-\rho_1^{-4} + \rho_2^{-4}) + \frac{1}{2}p(p+1)(\rho_1^{-2} + \rho_2^{-2})a_3 + \frac{2}{3}p(\rho_1^{-3} + \rho_2^{-3})a_1 + \frac{1}{2}a_3 a_1^2, \\ c_7 &= \frac{1}{2}a_1 a_3^2 + \frac{2}{3}p(\rho_1^{-3} + \rho_2^{-3})a_3, & c_9 &= \frac{1}{6}a_3^3. \end{aligned}$$

Inverting the characteristic function (2.3) yields

THEOREM 2.2. Put $\lambda_1 = \sqrt{n} \log|(n_2/n_1)S_1 S_2^{-1}|$ and $\tau = \{2p(\rho_1^{-1} + \rho_2^{-1})\}^{\frac{1}{2}}$.

Then under alternatives K , we have

$$(2.5) \quad \begin{aligned} P(\{\lambda_1 - \sqrt{n} \log|\Gamma|\}/\tau < z) &= \phi(z) - \frac{1}{\sqrt{n}} \{a_3 \tau^{-3} \phi^{(3)}(z) \\ &+ a_1 \tau^{-1} \phi^{(1)}(z)\} + \frac{1}{n} \sum_{j=1}^3 b_{2j} \tau^{-2j} \phi^{(2j)}(z) \\ &- \frac{1}{n\sqrt{n}} \sum_{j=1}^5 c_{2j-1} \tau^{-2j+1} \phi^{(2j-1)}(z) + o(n^{-2}), \end{aligned}$$

where the coefficients a_j, b_j, c_j , are given by (2.4) and $\phi^{(j)}(z)$ means the j -th derivative of the standard normal distribution function $\phi(z)$. If the null hypothesis H is true, the same formula (2.5) holds by simply putting $\log|\tau| = 0$.

Applying the general inverse expansion formula of Hill and Davis [6] to (2.5) with $\log|\tau| = 0$, we get the following asymptotic formula for the upper $\alpha\%$ point of λ_1/τ , in terms of the upper $\alpha\%$ point u of the standard normal distribution function.

$$(2.6) \quad u + \frac{1}{\sqrt{n}} \{ \tilde{a}_3 H_3 + \tilde{a}_1 \} - \frac{1}{n} \left[\sum_{j=1}^3 \tilde{b}_{2j} H_{2j} + (\tilde{a}_3 H_3 + \tilde{a}_1) u \{ \tilde{a}_3 (H_3 - 2) + \tilde{a}_1 \} \right] \\ + \frac{1}{n\sqrt{n}} \left[\sum_{j=1}^5 \tilde{c}_{2j-1} H_{2j-1} + \sum_{j=1}^3 \tilde{b}_{2j} H_{2j} u \{ \tilde{a}_3 (H_3 - 2) + \tilde{a}_1 \} \right. \\ \left. + \frac{1}{3} (u^2 - 1) (\tilde{a}_3 H_3 + \tilde{a}_1)^3 - (3u^2 - 1) \tilde{a}_3 (\tilde{a}_3 H_3 + \tilde{a}_1)^2 \right. \\ \left. + (\tilde{a}_3 H_3 + \tilde{a}_1) (4\tilde{a}_3^2 u^2 - \sum_{j=1}^3 \tilde{b}_{2j} H'_{2j}) \right] + O(n^{-2}),$$

where $\tilde{a}_j = a_j \tau^{-j}$, $\tilde{b}_j = b_j \tau^{-j}$, $\tilde{c}_j = c_j \tau^{-j}$ and H'_j means the derivative of Hermite polynomial $H_j = H_j(u)$ defined by $\phi^{(j)}(u) = H_j(u) \phi^{(1)}(u)$.

For $j = 1, 2, \dots, 9$

(2.7)

$$\begin{aligned} H_1 &= 1, & H_2 &= -u, & H_3 &= u^2 - 1, & H_4 &= -u^3 + 3u, \\ H_5 &= u^4 - 6u^2 + 3, & H_6 &= -u^5 + 10u^3 - 15u, \\ H_7 &= u^6 - 15u^4 + 45u^2 - 15, & H_8 &= -u^7 + 21u^5 - 105u^3 + 105u, \\ H_9 &= u^8 - 28u^6 + 210u^4 - 420u^2 + 105. \end{aligned}$$

3. ASYMPTOTIC DISTRIBUTION OF $|S_2(S_1+S_2)^{-1}|$. Unlike the previous section, we cannot get a simple expression for the characteristic function of $\lambda_2 = -\sqrt{n} \log |(n/n_2)S_2(S_1+S_2)^{-1}|$ under K . However, by applying the lemma in Sugiura [11], which is a direct extension of Siotani and Hayakawa [10], to the case of several Wishart matrices, to $f(S_1, S_2) = \log|S_2| - \log|S_1+S_2|$, we get the limiting distribution of λ_2 under K .

THEOREM 3.1. Under alternatives K , the statistic $\lambda_2 - \sqrt{n} \log|\rho_1\Gamma + \rho_2I|$ has asymptotically normal distribution with mean zero and variance $2(\rho_1^{-1} + \rho_2^{-1})\text{tr}(I + \rho_2\rho_1^{-1}\Gamma^{-1})^{-2}$.

Now we shall consider the sequence of alternatives $K_\delta: \Gamma = I + n^{-\delta}\Theta(\delta \geq 0)$, under which the characteristic function of λ_2 can be expressed for large n such that all positive elements of diagonal matrix $n^{-\delta}\Theta$ are less than one, as

$$(3.1) \quad \rho_2^{\sqrt{np}it} \Gamma_p(\frac{1}{2}n_2 - \sqrt{nit})\Gamma_p(\frac{1}{2}n) / \{\Gamma_p(\frac{1}{2}n_2)\Gamma_p(\frac{1}{2}n - \sqrt{nit})\} \\ \cdot {}_2F_1(-\sqrt{nit}, \frac{1}{2}n_1; \frac{1}{2}n - \sqrt{nit}; I - \Gamma).$$

The above formula (3.1) can be obtained by the similar argument as in Sugiura ([12], Section 5), based on the hypergeometric function of matrix argument due to Constantine [4] and James [7]. The first factor gives the characteristic function under H , which can be expanded asymptotically in a similar way as in Section 2, having

(3.2)

$$\exp \left[-p(\rho_2^{-1} - 1)t^2 + \sum_{r=1}^{\infty} (2it)^r p n^{-r/2} \{2(it)^2(r+1)^{-1}(r+2)^{-1}(\rho_2^{-r-1} - 1) + (p+1)(\rho_2^{-r} - 1)/(4r)\} - \sum_{r=1}^{\infty} (-2/n)^r B_{r+1} r^{-1}(r+1)^{-1} \cdot \{1 - (1-2itn^{-\frac{1}{2}})^{-r} + (\rho_2 - 2itn^{-\frac{1}{2}})^{-r} - \rho_2^{-r}\} \right],$$

the first four terms of which is expressed by (2.3) after putting $\log|\Gamma| = 0$ and replacing τ, a_j, b_j, c_j with $\tau' = \{2p(\rho_2^{-1} - 1)\}^{\frac{1}{2}}$ a'_j, b'_j, c'_j respectively, given by

$$(3.3) \quad \begin{aligned} a'_1 &= \frac{1}{2}p(p+1)(\rho_2^{-1} - 1), & a'_3 &= \frac{2}{3}p(\rho_2^{-2} - 1), \\ b'_2 &= \frac{1}{2}a_1'^2 + \frac{1}{2}p(p+1)(\rho_2^{-2} - 1), \\ b'_4 &= a_1'a_3' + \frac{2}{3}p(\rho_2^{-3} - 1), & b'_6 &= \frac{1}{2}a_3'^2, \\ c'_1 &= \frac{1}{12}p(2p^2 + 3p - 1)(\rho_2^{-2} - 1), \\ c'_3 &= p(p+1)\left\{\frac{2}{3}(\rho_2^{-3} - 1) + \frac{1}{2}(\rho_2^{-2} - 1)a_1'\right\} + \frac{1}{6}a_1'^3, \\ c'_5 &= \frac{4}{5}p(\rho_2^{-4} - 1) + \frac{2}{3}p(\rho_2^{-3} - 1)a_1' + \frac{1}{2}p(p+1)(\rho_2^{-2} - 1)a_3' + \frac{1}{2}a_1'^2 a_3', \\ c'_7 &= \frac{1}{2}a_1'a_3'^2 + \frac{2}{3}p(\rho_2^{-3} - 1)a_3', & c'_9 &= \frac{1}{6}a_3'^3. \end{aligned}$$

Inverting the characteristic function, we get

THEOREM 3.2. Under the null hypothesis H , the distribution of $\lambda_2 = -\sqrt{n} \tau'^{-1} \log |(n/n_2)S_2(S_1+S_2)^{-1}|$ is expanded asymptotically by the formula (2.5) after replacing τ, a_j, b_j, c_j with $\tau' = \{2p(\rho_2^{-1} - 1)\}^{\frac{1}{2}}$ a'_j, b'_j, c'_j respectively.

Asymptotic formula for the upper $\alpha\%$ point of λ_2/τ^j is also given by (2.6) after replacing $\tilde{a}_j, \tilde{b}_j, \tilde{c}_j$ with $\tilde{a}_j^j = a_j^j/\tau^j, \tilde{b}_j^j = b_j^j/\tau^j, \tilde{c}_j^j = c_j^j/\tau^j$ respectively.

When $\delta = \frac{1}{2}$, the second factor of (3.1) is written by

$$(3.4) \quad {}_2F_1 = \sum_{k=0}^{\infty} \sum_{(\kappa)} (-\sqrt{n}it)_{\kappa} \left(\frac{1}{2}n_1\right)_{\kappa} c_{\kappa}(-n^{-\frac{1}{2}}\theta) / \{(\frac{1}{2}n - \sqrt{n}it)_{\kappa} k!\} .$$

By Fujikoshi [5] and Sugiura [12], we know already that

$$(3.5) \quad (-\sqrt{n}it)_{\kappa} = (-\sqrt{n}it)^k \left[1 - \frac{a_1(\kappa)}{2it\sqrt{n}} + \frac{1}{24(it)^2n} \{3a_1(\kappa)^2 - a_2(\kappa) + k\} + O(n^{-3/2}) \right] ,$$

$$\left(\frac{1}{2}n_1\right)_{\kappa} = \left(\frac{1}{2}n\rho_1\right)^k \left[1 + \frac{1}{n\rho_1} a_1(\kappa) + O(n^{-2}) \right] ,$$

$$\left(\frac{1}{2}n - \sqrt{n}it\right)_{\kappa} = \left(\frac{1}{2}n\right)^k \left[1 - \frac{2it}{\sqrt{n}} k + \frac{1}{n} \{a_1(\kappa) + 2(it)^2k(k-1)\} + O(n^{-3/2}) \right] ,$$

where $a_1(\kappa) = \sum_{\alpha=1}^p k_{\alpha} (k_{\alpha} - \alpha)$ and $a_2(\kappa) = \sum_{\alpha=1}^p k_{\alpha} (4k_{\alpha}^2 - 6\alpha k_{\alpha} + 3\alpha^2)$. Hence the right-hand side of (3.4) is equal to

$$(3.6) \quad \sum_{k=0}^{\infty} \sum_{(\kappa)} c_{\kappa}(\rho_1 it\theta) (k!)^{-1} \left[1 + \frac{1}{\sqrt{n}} \{2itk - a_1(\kappa)/2it\} \right. \\ \left. + \frac{1}{n} \left\{ \frac{1}{8} a_1(\kappa)^2 (it)^{-2} + (\rho_1^{-1} - 1 + k) a_1(\kappa) - \frac{1}{24} a_2(\kappa) (it)^{-2} \right. \right. \\ \left. \left. + k(4(it)^2 + \frac{1}{24}(it)^{-2}) + 2(it)^2k(k-1) \right\} + O(n^{-3/2}) \right] .$$

It follows from the lemma by Sugiura and Fujikoshi [13] that

$$(3.7) \quad {}_2F_1 = \{etr(\rho_1 it\theta)\} \left[1 + \frac{1}{\sqrt{n}} \{2\rho_1(it)^2 tr\theta - \frac{1}{2} \rho_1^2(it) tr\theta^2\} \right]$$

$$\begin{aligned}
& + \frac{1}{n} \{ 2\rho_1^2(it)^4(\text{tr}\theta)^2 + (it)^3(4\rho_1\text{tr}\theta - \rho_1^3\text{tr}\theta\text{tr}\theta^2) \\
& + (it)^2(\rho_1[1-3\rho_1]\text{tr}\theta^2 + \frac{1}{8}\rho_1^4[\text{tr}\theta^2]^2) + \frac{1}{3}it\rho_1^3\text{tr}\theta^3 \} + o(n^{-3/2}) \Big].
\end{aligned}$$

Multiplying (3.7) with the first factor of (3.1) given by (2.3) with a'_j, b'_j, c'_j, τ' instead of a_j, b_j, c_j, τ , we finally get the asymptotic formula of the characteristic function of λ_2 under $K_\delta(\delta = \frac{1}{2})$ as

$$\begin{aligned}
(3.8) \quad & \exp[-\frac{1}{2}\tau'^2t^2 + \rho_1it\theta] \cdot [1 + \frac{1}{\sqrt{n}} \{ a'_3(it)^3 + 2\rho_1(it)^2\text{tr}\theta \\
& + it(a'_1 - \frac{1}{2}\rho_1^2\text{tr}\theta^2) \} + \frac{1}{n} \sum_{j=1}^6 g_j(it)^j + o(n^{-3/2})] ,
\end{aligned}$$

where

$$\begin{aligned}
(3.9) \quad & g_1 = -\frac{1}{3}\rho_1^3\text{tr}\theta^3 , \\
& g_2 = b'_2 + \rho_1(1 - 3\rho_1 - \frac{1}{2}a'_1\rho_1)\text{tr}\theta^2 + \frac{1}{8}\rho_1^4(\text{tr}\theta^2)^2 , \\
& g_3 = \rho_1\text{tr}\theta(\rho_1^2\text{tr}\theta^2 - 2a'_1 - 4) , \\
& g_4 = 2\rho_1^2(\text{tr}\theta)^2 - \frac{1}{2}a'_3\rho_1^2\text{tr}\theta^2 + b'_4 , \\
& g_5 = 2a'_3\rho_1\text{tr}\theta , \\
& g_6 = b'_6
\end{aligned}$$

and a'_j, b'_j are given by (3.3). Hence we can get

THEOREM 3.3. Under the sequence of alternatives

$K_\delta(\delta = \frac{1}{2})$: $\Gamma = I + n^{-\frac{1}{2}}\theta$, the following asymptotic expansion of the distribution of $\lambda_2 = -\sqrt{n} \log |(n/n_2)S_2(S_1+S_2)^{-1}|$ holds, for large n such that all positive elements of the diagonal matrix $n^{-\frac{1}{2}}\theta$ are less than one.

$$\begin{aligned}
(3.10) \quad P(\{\lambda_2 - \rho_1 \text{tr}\theta\}/\tau' < z) &= \phi(z) + \frac{1}{\sqrt{n}} \{-a_3' \tau'^{-3} \phi^{(3)}(z) \\
&+ 2\rho_1 \text{tr}\theta \tau'^{-2} \phi^{(2)}(z) + (\frac{1}{2}\rho_1^2 \text{tr}\theta^2 - a_1') \tau'^{-1} \phi^{(1)}(z)\} \\
&+ \frac{1}{n} \sum_{j=1}^6 (-1/\tau')^j g_j \phi^{(j)}(z) + o(n^{-3/2}),
\end{aligned}$$

where $\tau' = \{2p(\rho_2^{-1} - 1)\}^{\frac{1}{2}}$

From (3.7) and (3.5), it is easy to see that the limiting distribution of λ_2 under $K_\delta (\delta > \frac{1}{2})$ is normal with mean zero and variance τ'^2 . Hence the limiting distributions of λ_1/τ and λ_2/τ' under $k_\delta (\delta > \frac{1}{2})$ are the same. Even when $\delta = \frac{1}{2}$, they are the same, because their asymptotic means given Theorem 2.1 and Theorem 3.3 are equal. The power of the test λ_1 will be almost the same as that of λ_2 , when alternative K is near to the null hypothesis. This is verified numerically in the following special cases.

4. NUMERICAL EXAMPLES. We shall consider two cases, namely,

Case 1: $p=2$ and $n_1=13$, $n_2=63$, Case 2: $p=5$ and $n_1=50$, $n_2=150$.

Asymptotic formula (2.6) for the upper percentage points gives

	5% points of λ_1/τ .		5% points of λ_2/τ' .	
	Case 1	Case 2	Case 1	Case 2
first term	1.6449	1.6449	1.6449	1.6449
term of $O(n^{-\frac{1}{2}})$	-0.3576	-0.4020	0.2365	0.2450
term of $O(n^{-1})$	0.0791	0.0409	0.0206	0.0272
term of $O(n^{-3/2})$	-0.0262	-0.0195	0.0044	0.0051
approx. value	<u>1.340</u>	<u>1.264</u>	<u>1.906</u>	<u>1.922</u>

All these examples show that the powers of the λ_1 -test and the λ_2 -test are almost the same when alternative is near to the null hypothesis.

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