COMBINATORIAL MATHEMATICS YEAR

February 1969 - June 1970

SOME PROPERTIES AND APPLICATIONS OF HERMITIAN VARIETIES IN A FINITE PROJECTIVE SPACE $PG(N, q^2)$ IN THE CONSTRUCTION OF STRONGLY REGULAR GRAPHS (TWO-CLASS ASSOCIATION SCHEMES) AND BLOCK DESIGNS

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Institute of Statistics Mimeo Series No. 600.23

MARCH 1970

This research was partially supported by the Army Research Office, Durham, under Grant No. DA-ARO-D-31-124-G910 and the U.S. Air Force Office of Scientific Research under Contract No. AFOSR-68-1406.

SOME PROPERTIES AND APPLICATIONS OF HERMITIAN VARIETIES IN A FINITE PROJECTIVE SPACE $PG(N,q^2)$ IN THE CONSTRUCTION OF STRONGLY REGULAR GRAPHS (TWO-CLASS ASSOCIATION SCHEMES) AND BLOCK DESIGNS

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1. Introduction and Summary. If h is any element of a Galois field $GF(q^2)$, where q is a prime or a power of a prime, then $\bar{h} = h^q$ is defined to be conjugate to h. Since $h^{q^2} = h$, h is conjugate to $\bar{h}$. A square matrix $H = ((h_{ij}))$, $i,j = 0,1,2,\ldots,N$, with elements from $GF(q^2)$ is called Hermitian if $h_{ij} = \bar{h}_{ji}$ for all i,j. The set of all points in $PG(N,q^2)$ whose row vectors $\underline{x}^T = (x_0,x_1,\ldots,x_N)$ satisfy the equation $\underline{x}^T H\underline{x}^{(q)} = 0$ are said to form a Hermitian variety V_{N-1}, if H is Hermitian and $\underline{x}^{(q)}$ is the column vector whose transpose is $(x_0^q,x_1^q,\ldots,x_N^q)$. The variety V_{N-1} is said to be non-degenerate if H has rank $N+1$.

The Hermitian form $\underline{x}^T H\underline{x}^{(q)}$ of order $N+1$ and rank r can be reduced to the canonical form $y_0\bar{y}_0 + y_1\bar{y}_1 + \ldots + y_r\bar{y}_r$ by a suitable non-singular linear transformation $\underline{x} = A\underline{y}$ [4]. The equation of a non-degenerate Hermitian variety V_{N-1} in $PG(N,q^2)$ can then be taken in the canonical form

$$(1.1) \qquad x_0^{q+1} + x_1^{q+1} + \ldots + x_N^{q+1} = 0.$$

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The properties of the non-degenerate Hermitian variety V_1 in $PG(2,q^2)$ whose equation in canonical form is $x_0^{q+1}+x_1^{q+1}+x_2^{q+1} = 0$, were studied by R.C. Bose in [1]. The geometry of the Hermitian variety V_{N-1} in $PG(N,q^2)$ was studied in some detail by R.C. Bose and by the author in [4]. In this last paper, the authors developed the theory of tangent and polar hyperplanes of Hermitian varieties and characterized the sections of these varieties by tangent hyperplanes and derived an algebraic expression for the number of points on a Hermitian variety. It was also shown there, that if $N = 2t+1$ or $2t+2$, a non-degenerate Hermitian variety V_{N-1} contains flats of dimension t and no higher. The number of such subspaces was derived. A study of the geometry of the surface $x_0^{q+1}+x_1^{q+1}+x_2^{q+1}+x_3^{q+1} = 0$, provided a new geometric interpretation of some statistical designs, in particular, the balanced incomplete block design $v = b = 45$, $r = k = 12$, $\lambda = 3$.

In [3], R.C. Bose made a special study of the Hermitian variety V_2 with equation $x_0^3+x_1^3+x_2^3+x_3^3 = 0$ in $PG(3,2^2)$ and derived a representation of $PG(3,3)$ in terms of external points, secant planes and self-conjugate tetrahedra with respect to the Hermitian variety. We shall assume the results of the two papers [1] and [4] to which the reader may refer for detailed proofs.

In this paper, we derive the section of a non-degenerate Hermitian variety V_{N-1} in $PG(N,q^2)$ by a *polar hyperplane* (not necessarily a tangent hyperplane) and the number of u-flats contained in a non-degenerate V_{N-1} in $PG(N,q^2)$ for $o \le u \le [\frac{N-1}{2}]$, where $[x]$ denotes the greatest integer less than or equal to x. The number of points and the number of $[\frac{N-1}{2}]$-flats in V_{N-1} were enumerated already in [4]. Next, considering the geometry of external points, tangent lines and secant lines with respect to a non-degenerate Hermitian variety V_1 in $PG(2,q^2)$ we get a family of two-class association schemes (or a strongly regular graph) with parameters, $v = q^2(q^2-q+1)$, $n_1 = (q^2-1)(q+1)$, $p^1_{11} = q^2-2+q^2$ and $p^2_{11} = (q+1)^2$. Then, the configuration of external points, tangent lines and secant lines with respect to a non-degenerate V_2 in $PG(3,q^2)$ is used to derive a family of two-class association schemes with parameters, $v = q^3(q^2+1)(q-1)$, $n_1 = (q^3+1)(q^2-1)$, $n_2 = q^2(q^2-q+1)(q^2-q-1)$, $p^1_{11} = q^2-2+q^4$ and $p^2_{11} = (q^2-q)(q+1)^2$.

Again, considering the geometry of generators in a non-degenerate V_2 in $PG(3,q^2)$, we get a family of two-class association schemes (equivalently, a strongly regular graph) with parameters $v = (q^3+1)(q+1)$, $n_1 = q(q^2+1)$, $n_2 = q^4$, $p^1_{11} = q-1$ and $p^2_{11} = q^2+1$. Taking the tangent planes as blocks we get a class of partially balanced incomplete block design with parameters $b = (q^3+1)(q^2+1)$, $v = (q^3+1)(q+1)$, $n_1 = q(q^2+1)$, $n_2 = q^4$, $p^1_{11} = q-1$, $p^2_{11} = q^2+1$, $r = q^2+1$, $k = q+1$, $\lambda_1 = 1$, $\lambda_2 = 0$.

2. <u>Definitions</u>. Consider a Hermitian variety V_{N-1} in $PG(N,q^2)$, with equation $\underline{x}^T H \underline{x}^{(q)} = 0$. A point C in $PG(N,q^2)$, with row vector $\underline{c}^T = (c_0, c_1, \ldots, c_N)$ is called a *singular point* of V_{N-1} if $\underline{c}^T H = \underline{0}$ or equivalently $H\underline{c}^{(q)} = 0$. A point of V_{N-1} which is not singular is called a *regular point* of V_{N-1}. Thus a non-singular point may be a regular point of V_{N-1} or a point not lying on V_{N-1}, in which case we shall call it an *external point* of $PG(N,q^2)$ with respect to V_{N-1}. It is clear that a non-degenerate Hermitian variety cannot possess a singular point. On the other hand, if V_{N-1} is degenerate and rank $H = r < N+1$, the singular points of V_{N-1} will constitute a $(N-r)$-flat called the *singular space* of V_{N-1}.

Let C be a point with row vector $\underline{c}^T$. Then the *polar space* of C with respect to the Hermitian variety V_{N-1} with equation $\underline{x}^T H \underline{x}^{(q)} = 0$ is defined to be the set of points of $PG(N,q^2)$ which satisfy the equation

$$\underline{x}^T H \underline{c}^{(q)} = 0. \tag{2.1}$$

When C is a singular point of V_{N-1}, the polar space of C is identical with the whole space $PG(N,q^2)$. When, however, C is a non-singular point of $PG(N,q^2)$, that is either a regular point of V_{N-1} or an external point, the equation (2.1) is the equation of a hyperplane which will be called the *polar hyperplane* of C with respect to V_{N-1}. Let C and D be two points of $PG(N,q^2)$. If the polar hyperplane of C passes through D, then the polar hyperplane of D passes through C. Two such points are defined to be *conjugate* to each other with respect to V_{N-1}. Thus the points lying in the polar hyperplane of C are all the points which are conjugate to C. If C is a regular point of V_{N-1}, the polar hyperplane of C passes through C, that is, C is self-conjugate and in this case, the polar hyperplane is called the tangent hyperplane to V_{N-1} at C.

When V_{N-1} is non-degenerate, there is no singular point. To every point there corresponds a unique polar hyperplane, and at every point of V_{N-1} there is a unique tangent hyperplane.

3. Sections of Hermitian varieties with polar hyperplanes which are not tangents. Let the equation of a non-degenerate Hermitian variety V_{N-1} in $PG(N,q^2)$ be $\underline{x}^T H \underline{x}^{(q)} = 0$ and let D be an external point with row vector $\underline{d}^T$. Then the polar hyperplane S_{N-1} of D has the equation $\underline{x}^T H \underline{d}^{(q)} = 0$.

Let $F_0, F_1, \ldots, F_{N-1}$ be N points of S_{N-1}, whose row vectors are linearly independent and let $F_N = D$. Then the row vector $\underline{d}^T$ of F_N must be independent of the row vectors of $F_0, F_1, \ldots, F_{N-1}$. Hence, we can find a non-singular linear transformation $\underline{y} = A\underline{x}$ such that $F_0, F_1, \ldots, F_N$ become the fundamental points of the reference system with row vectors $(1,0,\ldots,0)$, $(0,1,\ldots,0)$, ... and $(0,0,\ldots,1)$. Thus the equation of S_{N-1} becomes $y_N = 0$ and the equation of V_{N-1} becomes $\underline{y}^T G \underline{y}^{(q)} = 0$, where $G = ((g_{ij}))$ $i,j = 0,1,\ldots,N$ is Hermitian of rank $N+1$. Since F_N is conjugate to $F_0, F_1, \ldots, F_{N-1}$ and is not self-conjugate, we have

$$(3.1) \qquad g_{NN} \neq 0, \quad g_{Ni} = 0, \quad g_{iN} = 0, \quad i = 0,1,\ldots,N-1.$$

We can therefore write

$$(3.2) \qquad G = \left[\begin{array}{ccc|c} g_{00} & \cdots & g_{0N-1} & 0 \\ \hline g_{0N-1} & & g_{N-1N-1} & 0 \\ \hline 0 & \cdots & 0 & g_{NN} \end{array}\right] = \left[\begin{array}{c|c} G^* & 0 \\ \hline 0 & g_{NN} \end{array}\right] .$$

Now, $\det G = g_{NN}|G^*| \neq 0$, $g_{NN} \neq 0$, $|G^*| \neq 0$ and thus rank $G^* = N$.

Regarding S_{N-1} as a projective space of $(N-1)$ dimensions the section of $y_N = 0$ and $\underline{y}^T G \underline{y}^{(q)} = 0$ can be expressed as $\underline{y}^{*T} G^* \underline{y}^{*(q)} = 0$ where $\underline{y}^{*T} = (y_0, y_1, \ldots, y_{N-1})$ and G^* is the matrix obtained from G by retaining only the first N rows and N columns of G. But rank G^*, as shown above, is N. Hence the Hermitian variety V_{N-2} with equation $\underline{y}^{*T} G^* \underline{y}^{*(q)} = 0$ is a non-degenerate Hermitian variety of rank N. Thus we have the

THEOREM 3.1: Given a non-degenerate Hermitian variety V_{N-1} in $PG(N,q^2)$, the polar hyperplane S_{N-1} of an external point D intersects V_{N-1} in a non-degenerate Hermitian variety V_{N-2} of rank N contained in S_{N-1}.

Let V_{N-1} be a non-degenerate Hermitian variety in $PG(N,q^2)$ and $D_1, D_2, \ldots, D_{N-m}$ be a set of $N-m$ external points whose row vectors $\underline{d}_1^T, \ldots, \underline{d}_{N-m}^T$ are linearly independent. The intersection of the polar hyperplanes of these points is an m-flat Σ_m whose points satisfy the equations

$$(3.3) \qquad \underline{x}^T H \underline{d}_1^{(q)} = 0, \ \ldots, \ \underline{x}^T H \underline{d}_{N-m}^{(q)} = 0.$$

Let $F_0, F_1, \ldots, F_m$ be a set of $m+1$ points belonging to Σ_m, whose row vectors are linearly independent and let $F_{m+1} = D_1, \ \ldots, \ F_N = D_{N-m}$. We can find a non-singular linear transformation $\underline{y} = A\underline{x}$ such that $F_0, F_1, \ldots, F_m, \ \ldots \ F_N$ become the fundamental points of the reference system. The points of Σ_m will then satisfy the equations

$$(3.4) \qquad y_{m+1} = \ldots = y_N = 0,$$

and the equation of V_{N-1} will be

$$(3.5) \qquad \underline{y}^T G \underline{y}^{(q)} = 0,$$

where G is Hermitian of rank N+1.

Now $F_0, F_1, \ldots, F_m$ are each conjugate to $F_{m+1}, \ldots, F_N$. Hence, we have

$$(3.6) \qquad G = \left[\begin{array}{c|c} G^* & 0 \\ \hline 0 & \tilde{G} \end{array}\right]$$

where G^* is a $(n+1) \times (m+1)$ matrix and $\tilde{G}$ is a $(N-m) \times (N-m)$ matrix.

But $|G| = |G^*||\tilde{G}| \neq 0$, hence both $|G^*|$ and $|\tilde{G}|$ must be distinct from zero. Thus the equation of the section V_{m-1} of V_{N-1} with Σ_m can be expressed as

(3.7) $$\underline{y}^{*T} G^{*} \underline{y}^{*(q)} = 0$$

where $\underline{y}^{*T} = (y_0, y_1, \ldots, y_m)$ is a point in m-dimensional projective space Σ_m and G^* is Hermitian of rank m+1. Hence we have the

THEOREM 3.2: The section of a non-degenerate Hermitian variety V_{N-1} in $PG(N,q^2)$ and an m-flat Σ_m which is the intersection of the polar planes of (N-m) linearly independent external points is a non-degenerate Hermitian variety V_{m-1} of rank m+1 contained in Σ_m.

We quote here, without proof, some theorems proved in [4] as we need them for proving some new results.

THEOREM 3.3: If the Hermitian variety V_{N-1} with equation $\underline{x}^T H \underline{x}^{(q)} = 0$ is degenerate with rank $r < N+1$ and Σ_{r-1} is a flat space of dimensions r-1 disjoint with the singular space Σ_{N-r} of V_{N-1}, then V_{N-1} and Σ_{r-1} intersect in a non-degenerate Hermitian variety V_{r-2} contained in Σ_{r-1}.

This is the Theorem 7.1 in [4].

THEOREM 3.4: The tangent spaces at two distinct regular points A and B of a Hermitian variety V_{N-1} are identical if and only if the line joining A and B meets the singular space of V_{N-1} in a point. In particular, if V_{N-1} is non-degenerate then the tangent spaces at A and B must be distinct.

This is Theorem 7.3 in [4].

THEOREM 3.5: Given a non-degenerate Hermitian variety V_{N-1}, the tangent space at a point C of V_{N-1} intersects V_{N-1} in a degenerate Hermitian variety V_{N-2} of rank N-1 contained in Σ_{N-1}. The singular space of V_{N-2} consists of the single point C.

This is Theorem 7.4 in [4].

Let Σ_{N-1} be the tangent space at a point C on a non-degenerate Hermitian variety V_{N-1} and let $\underline{c}^T$ be the row-vector of C. Then the equation of Σ_{N-1} is $\underline{x}^T H \underline{c}^{(q)} = 0$, where $\underline{x}^T H \underline{x}^{(q)} = 0$ is the equation of V_{N-1}. By Theorem 3.5, the intersection of Σ_{N-1} and V_{N-1} is a degenerate Hermitian variety V_{N-2} contained in Σ_{N-1} and its singular space consists of the single point C. Consider an external point D with row vector $\underline{d}^T$. Suppose C is not conjugate to D. Then the polar space S_{N-1} of D, with equation $\underline{x}^T H \underline{d}^{(q)} = 0$, will intersect Σ_{N-1} in a (N-2)-flat Σ_{N-2} disjoint with C. Hence by Theorem 3.4, Σ_{N-2} and V_{N-2} will intersect in a non-degenerate Hermitian variety V_{N-3} contained in Σ_{N-2}. Thus we have the

THEOREM 3.6: Given a non-degenerate Hermitian variety V_{N-1}, a (N-2)-flat Σ_{N-2} which is the intersection of the polar hyperplane of an external point D and the tangent hyperplane at a point C on V_{N-1}, meets V_{N-1} in a non-degenerate Hermitian variety V_{N-3} contained in Σ_{N-2}, provided C and D are not conjugates.

4. <u>Number of u-flats contained in a non-degenerate Hermitian variety</u> V_{N-1} <u>in</u> $PG(N,q^2)$. In this section, we derive an expression for the number of u-flats contained in a non-degenerate Hermitian variety V_{N-1} in $PG(N,q^2)$, $0 \le u \le [\frac{N-1}{2}]$, where $[x]$ denotes the greatest integer less than or equal to x. It was shown in [4], that if $N = 2t+1$ or $2t+2$, then a non-degenerate Hermitian variety V_{N-1} contains flat spaces of dimension t and no higher. The expressions for the number of points (u=0) and the number

of $[\frac{N-1}{2}]$-flats $(u=[\frac{N-1}{2}])$ contained in V_{N-1} were given in [4]. The same method of reasoning as in [4], is used here to derive expressions for every integer u in $0 \le u \le [\frac{N-1}{2}]$.

We quote, without proof, a lemma and a corollary to it, proved in [4] which we need here.

Lemma 4.1: The line joining two points C and D on a Hermitian variety V_{N-1} is completely contained in V_{N-1} if and only if C and D are conjugate with respect to V_{N-1}. This is Lemma 9.1 in [4].

Corollary. The necessary and sufficient condition for any t-flat Σ_t to be completely contained in V_{N-1} is that any two points of Σ_t are conjugate with respect to V_{N-1}. If Σ_t is contained in V_{N-1} and a point C of Σ_t is a regular point of V_{N-1}, then Σ_t is contained in the tangent space to V_{N-1} at C.

This is the corollary to Lemma 9.1 in [4].

Let $\psi(N,u,q^2)$ denote the number of u-flats contained in a non-degenerate Hermitian variety V_{N-1} in $PG(N,q^2)$ and let $f(u,q^2)$ denote the number of points in a u-flat in $PG(N,q^2)$. Then it was shown in [4] that

(4.1) $\psi(N,0,q^2) = \phi(N,q^2) =$ number of points in a non-degenerate

$$V_{N-1} = \frac{(q^{N+1}-(-1)^{N+1})(q^N-(-1)^N)}{(q^2-1)},$$

(4.2) $\psi(2t+1,u,q^2) = \psi(2t+2,u,q^2) = 0$ for $u > t$,

and

(4.3) $$\psi(2t+1,t,q^2) = (q^{2t+1}+1)(q^{2t-1}+1)\ \dots\ (q+1),$$

(4.4) $$\psi(2t+2,t,q^2) = (q^{2t+3}+1)(q^{2t+1}+1)\ \dots\ (q^3+1).$$

Here, we find the value of $\psi(N,u,q^2)$ for all u, $0 \le u \le [\frac{N-1}{2}]$.

Let C be any point on a non-degenerate Hermitian variety V_{N-1}. Then the tangent space at C Σ_{N-1} cuts V_{N-1} in a degenerate Hermitian variety V_{N-2} contained in Σ_{N-1}, for which C is the singular space (Theorem 3.5). By Theorem 3.3, we can find a (N-2)-flat Σ_{N-2} contained in Σ_{N-1} and disjoint from C, which intersects V_{N-2} in a non-degenerate Hermitian variety V_{N-3} contained in Σ_{N-2}.

Now V_{N-3} contains $\psi(N-1,u-1,q^2)$ (u-1)-flats, any one of which together with C determines a u-flat contained in V_{N-1} and passing through C. Conversely, if Σ_u is a u-flat contained in V_{N-1} and passing through C, then it is contained in Σ_{N-1} (Corollary to Lemma 4.1) and intersects Σ_{N-2} in a (u-1)-flat contained in V_{N-3}. Hence the number of u-flats contained in V_{N-1} and passing through a fixed point C on V_{N-1} is $\psi(N-1,u-1,q^2)$. But the number of points on V_{N-1} is $\phi(N,q^2)$. We thus obtain $\psi(N-2,u-1,q^2)\ \phi(N,q^2)$ u-flats by considering all points of V_{N-1}. Here each u-flat has been counted $f(u,q^2) = ((q^{2(u+1)}-1)/(q^2-1))$ times. Hence,

(4.5) $$\psi(N,u,q^2) = \frac{\psi(N-2,u-1,q^2)\phi(N,q^2)}{f(u,q^2)}.$$

Thus by successive reduction,

(4.6) $$\psi(N,u,q^2) = \frac{\phi(N,q^2)\phi(N-2,q^2)\ \dots\ \phi(N-2u,q^2)}{f(u,q^2)f(u-1,q^2)\ \dots\ f(0,q^2)}$$

$$= \frac{(q^{N+1}-(-1)^{N+1})(q^{N}-(-1)^{N}) \ldots (q^{N-2u+1}-(-1)^{N-2u+1})(q^{N-2u}-(-1)^{N-2u})}{(q^{2(u+1)}-1) \ldots (q^{2}-1)}$$

$$= \frac{\prod_{i=0}^{2u+1} (q^{N-2u+i}-(-1)^{N-2u+i})}{\prod_{i=0}^{u} (q^{2(1+i)}-1)} .$$

5. Association schemes, strongly regular graphs and block designs from Hermitian varieties. Following Bose and Shimamoto [6], we define a two-class association scheme as relations between v objects as follows:

(i) any two objects are either first associates or second associates,

(ii) each object has n_i i-th associates, $i = 1,2$,

(iii) if two objects are i-th associates, then the number of objects which are j-th associates of the first and k-th associates of the second is p^i_{jk} and is independent of the pair of objects with which we start and $p^i_{jk}=p^i_{kj}$.

Bose and Clatworthy showed [5] that the constancy of n_1, n_2, p^1_{11} and p^2_{11} implies the constancy of p^1_{12}, p^1_{21}, p^1_{22}, p^2_{12}, p^2_{21} and p^2_{22} and the equalities $p^1_{12} = p^1_{21}$, $p^2_{12} = p^2_{21}$. Also it follows from their proof that

$$p^1_{12} = n_1 - p^1_{11} - 1 = p^1_{21}, \quad p^1_{22} = n_2 - n_1 + p^1_{11} + 1$$

$$p^2_{12} = n_1 - p^2_{11} = p^2_{21}, \quad p^2_{22} = n_2 - n_1 + p^2_{11} - 1.$$

A finite connected graph G (without loops or multiple edges) is called strongly regular [2], if it is regular of valence n_1 and any pair of adjacent vertices is adjacent to exactly p^1_{11} other vertices and any pair of non-adjacent vertices is adjacent to exactly p^2_{11} vertices.

If v is the number of vertices, $(v, n_1, p_{11}^1, p_{11}^2)$ are called the parameters of the strongly regular graph G. Given a two-class association scheme defined on v objects, the graph we obtain by taking the objects as vertices and joining two vertices if the corresponding objects are first associates and leaving them unjoined if they are second associates, is a strongly regular graph and conversely [2]. Hence a strongly regular graph $(v, n_1, p_{11}^1, p_{11}^2)$ is equivalent to a two-class association scheme $(v, n_1, p_{11}^1, p_{11}^2)$.

A block design B defined on the v objects of a two-class association scheme $(v, n_1, p_{11}^1, p_{11}^2)$, is called a partially balanced block design if it is an arrangement of v objects in b blocks such that every block consists of k objects, every object occurs in exactly r blocks, any two objects which are first associates occur in exactly λ_1 blocks and any two objects which are second associates occur together in exactly λ_2 blocks. The parameters of such a partially balanced incomplete block design are then $(b, v, r, k, \lambda_1, \lambda_2, n_1, p_{11}^1, p_{11}^2)$. If $\lambda_1 = \lambda_2$, the block design is called a balanced incomplete block design.

5.1. Consider a non-degenerate Hermitian variety V_1 in $PG(2,q^2)$. The number of points in $PG(2,q^2)$ is q^4+q^2+1. The number of points in V_1 is (q^3+1) and hence the number of external points is $v = (q^4+q^2+1) - (q^3+1) = q^2(q^2-q+1)$. V_1 cannot contain any lines. A line of $PG(2,q^2)$ meets V_1 either exactly at one point or intersects V_1 in a non-degenerate V_0 which consists of $(q+1)$ points of the line. In the former case, the line is called a tangent line at the point and in the latter it is a secant line. Thus any two external points will be either on a tangent line or on a secant line. Let us define two external points as first associates if they are on a tangent line and as second associates if they are on a secant line. Consider an external point P. Its polar line meets V_1 at q+1 points. The q+1 tangents at these q+1 points must pass through P, since these points are conjugate to P. Thus through every external point P pass (q+1) tangent lines and q^2-q secant lines. The number of first associates of a given point P is then,

$$(5.1) \qquad n_1 = (q^2-1)(q+1),$$

and the number of second associates

$$(5.2) \qquad n_2 = (q^2-q)(q^2-q-1).$$

Let P and Q be two external points on a tangent line ℓ. Through P pass q tangent lines other than ℓ and these are distinct from the q tangent lines passing through Q other than ℓ. A tangent line through P meets a tangent line through Q exactly at one external point. Hence number of external points which are first associates to both P and Q is

$$(5.3) \qquad p_{11}^{1} = q^2 - 2 + q \cdot q = q^2 - 2 + q^2.$$

Consider two external points R and S which are on a secant line m. Then the (q+1) tangent lines passing through R intersect the (q+1) tangent lines passing through S at $(q+1)^2$ distinct external points. Thus we have

$$(5.4) \qquad p_{11}^{2} = (q+1)^2.$$

Hence we have the

THEOREM 5.1: The association scheme defined on points external to a non-degenerate V_1 in $PG(2,q^2)$ by considering two external points as first associates if they are incident with a tangent line and as second associates if they are incident with a secant line is a two-class association scheme with the parameters $v = q^2(q^2-q+1)$, $n_1 = (q^2-1)(q+1)$, $p_{11}^1 = q^2-2+q^2$ and $p_{11}^2 = (q+1)^2$. Hence this is also a strongly regular graph.

For $q=2$, the result was derived by Bose in [3]. Let us define a $v \times v$ matrix B_i $i = 1,2$ on the v objects of a two-class association scheme as follows. The rows and the columns of B_i $i = 1,2$ correspond to the v objects and the entry $b_{i\alpha}^{\beta}$ at the intersection of row α and column β of B_i is 1 if α and β are i-th associates, zero otherwise. Then B_i's are called the association matrices. On the other hand, B_1 is the adjacency matrix of the strongly regular graph $(v,n_1,p_{11}^1,p_{11}^2)$ and B_2 is the adjacency matrix of the complementary graph.

If $p_{11}^1 = p_{11}^2$, B_1 becomes the incidence matrix of a balanced incomplete block design $v = b$, $r = k = n_1$, $\lambda = p_{11}^1 = p_{11}^2$.

In theorem 5.1, for $q = 3$, B_1 is the incidence matrix of a balanced incomplete block design with $v = b = 63$, $r = k = 32$, $\lambda = 16$ and B_2 is the incidence matrix of the balanced incomplete block design $v = b = 63$, $r = k = 31$, $\lambda = 15$.

5.2. Consider a non-degenerate Hermitian variety V_2 in $PG(3,q^2)$. It was shown in [4] that the lines of $PG(3,q^2)$ fall into three classes with respect to V_2; (i) secant line which intersects V_2 in a non-degenerate Hermitian variety V_0 consisting of $q+1$ points, (ii) tangent line which meets V_2 at a single point and (iii) generator which is contained entirely in V_2. We quote here, without proof, some results proved in [4] as lemmas which we need to derive some new results.

LEMMA 5.2.1: Through any point C of V_2 there pass exactly $q+1$ generators which constitute the intersection with V_2 of the tangent plane at C.

LEMMA 5.2.2: A line passing through a given point C on V_2, and not contained in the tangent plane at C is a secant line.

LEMMA 5.2.3: Any plane through a generator is tangent to V_2 at some point on the generator and intersects V_2 in a set of $q+1$ generators through the point of contact.

LEMMA 5.2.4: Given a generator ℓ of V_2 and a point P of V_2 not on ℓ there passes through P exactly one generator which meets ℓ at one point.

LEMMA 5.2.5: If two points C_0 and C_1 on V_2 lie on a secant line ℓ, then there are exactly $q+1$ points D_i $i = 1,2..$ $q+1$, on V_2 such that the tangent planes at D_i $i = 1,2..$ $q+1$ pass through the secant line ℓ. Thus the lines D_iC_0 and D_iC_1 $i = 1,2..$ $q+1$ are generators.

Next we prove

LEMMA 5.2.6: Through a tangent line ℓ meeting a non-degenerate V_2 in $PG(3,q^2)$, at a point Q on V_2, pass q^2+1 planes of which one is a tangent plane at Q and the remaining q^2 planes are secant planes.

PROOF. Let π_0 be the tangent plane at Q passing through ℓ. Suppose contrary to the hypothesis, that another plane π passing through ℓ is a tangent plane at a point R on V_2. Then it follows that both π_0 and π will contain QR, a generator of V_2. This is impossible, because π_0 and π contain already ℓ.

Now in $PG(3,q^2)$ there are $(q^4+1)(q^2+1)$ points and in V_2 there are $(q^3+1)(q^2+1)$ points. Hence the number of points in $PG(3,q^2)$ external to V_2 is $v = q^3(q^2+1)(q-1)$.

Let P be an external point. The polar plane of P intersects V_2 in a non-degenerate V_1 consisting of q^3+1 points. Each one of these points are conjugate to P.

Hence the number of tangent planes passing through P is q^3+1 and this is also the number of tangent lines through P. A secant line through P meets V_2 at $q+1$ points. Hence the number of secant lines through P is equal to the number of points in V_2 which are joined to P by a non-tangent line divided by $q+1$. This number is $q^2(q^2-q+1)$.

Let us define two external points as first associates if they are on a tangent line and as second associates if they are on a secant line. Then the number of first associates of a given external point P is

$$(5.5) \qquad n_1 = (q^3+1)(q^2-1),$$

and the number of second associates of P

$$(5.6) \qquad n_2 = q^2(q^2-q+1)(q^2-q-1).$$

Suppose P_1 and P_2 are two external points on a tangent line ℓ meeting V_2 at the point Q. There are q^2+1 planes passing through ℓ, of which only one is a tangent plane at Q and the remaining q^2 planes are secant planes. Now there are q^2-2 external points on the tangent line ℓ which are common first associates of P_1 and P_2. Again a secant plane passing through ℓ intersects V_2 in a non-degenerate V_1 contained in the plane. From Section 5.1, we have that there are $q+1$ lines in this plane, passing through P_1 which are tangent lines to V_1 and similarly

$q+1$ lines through P_2 which are tangent lines to V_1. But ℓ is common to both the sets of $q+1$ tangents. Hence the q tangents passing through P_1 will intersect the q tangents passing through P_2 at q^2 distinct external points. But there are q^2 secant planes passing through ℓ. Thus there are $q^2 \cdot q^2 = q^4$ external points in q^2 secant planes, which are first associates of both P_1 and P_2. Hence

(5.7) $$p_{11}^1 = q^2 - 2 + q^4.$$

Suppose Q_1 and Q_2 are two external points on a secant line m. Now through m pass q^2+1 planes of which q^2-q are secant planes and $q+1$ are tangent planes. Thus the number of external points which are at the intersection of tangent lines passing through Q_1 and Q_2 is equal to the number of secant planes passing through m times the number of intersections of tangent lines passing through P_1 and P_2 in a given secant plane. Hence

(5.8) $$p_{11}^2 = (q^2-q)(q+1)^2.$$

Hence we have the

THEOREM 5.2. The association scheme defined on points external to a non-degenerate V_2 in $PG(3,q^2)$ by considering two external points as first associates if they are incident with the same tangent line and as second associates if they are incident with the same secant line is a two-class association scheme with the parameters

(5.9) $$v = q^3(q^2+1)(q-1)$$

$$n_1 = (q^3+1)(q^2-1)$$

$$n_2 = q^2(q^2-q+1)(q^2-q-1)$$

$$p_{11}^1 = q^2 - 2 + q^4$$

$$p_{11}^2 = (q^2-q)(q+1)^2.$$

Hence this is also a strongly regular graph.

For $q=2$, we have $v = 40$, $n_1 = 27$, $n_2 = 12$, $p_{11}^1 = 18 = p_{11}^2$. Hence in this case, B_1 is the incidence matrix of a balanced incomplete block design $v = 40$, $r = k = 27$, $\lambda = 18$. The two-class association scheme on 40 points of $PG(3,2^2)$ external to V_2 was earlier obtained by R.C. Bose in [3].

5.3. A non-degenerate V_2 in $PG(3,q^2)$ has $(q^3+1)(q^2+1)$ points and $(q^3+1)(q+1)$ generators. Through each point of V_2 passes a tangent plane and two tangent planes at two distinct points on V_2 are distinct.

Consider the association scheme where two generators are first associates if they are incident with the same tangent plane, otherwise they are second associates.

Through a generator ℓ pass q^2+1 planes which are tangent planes at q^2+1 points of the generator. The tangent plane at a given point Q on ℓ will contain besides ℓ, q other generators meeting ℓ at Q. Hence the number of first associates of ℓ is

(5.10) $$n_1 = q(q^2+1),$$

and the number of second associates

(5.11) $$n_2 = (q^3+1)(q+1) - q(q^2+1) - 1 = q^4.$$

Consider two generators ℓ_1 and ℓ_2 which lie in the same tangent plane π. π also contains q-1 generators other than ℓ_1 and ℓ_2. Hence ℓ_1 and ℓ_2 which are first associates have q-1 generators as common first associates. Thus

(5.12) $$p_{11}^1 = q - 1.$$

Suppose ℓ_1 and ℓ_3 are two generators which do not lie on the same tangent plane.

By Lemma 5.2.4, there is a unique generator passing through a given point P on ℓ_3, which meets ℓ_1 at a point. Hence there are q^2+1 generators which are first associates of both ℓ_1 and ℓ_3. Hence

(5.13) $$p_{11}^2 = q^2 + 1.$$

We thus have the

THEOREM 5.3: The association scheme defined on $v = (q^3+1)(q+1)$ generators of a non-degenerate V_2 in $PG(3,q^2)$ by considering two generators as first associates if they are incident with the same tangent plane and as second associates if they are skew, define a two-class association scheme with the parameters

(5.14) $$v = (q^3+1)(q+1)$$

$$n_1 = q(q^2+1)$$

$$n_2 = q^4$$

$$p_{11}^1 = q - 1, \quad p_{11}^2 = q^2 + 1.$$

This is also a strongly regular graph.

Now if we define the tangent planes as blocks and the generators as treatments (or objects) we get a block design with parameters

$$(5.15)\qquad b = (q^3+1)(q^2+1), \quad v = (q^3+1)(q+1), \quad k = q+1,$$
$$r = q^2+1, \quad n_1 = q(q^2+1), \quad n_2 = q^4,$$
$$p_{11}^1 = q-1, \quad p_{11}^2 = q^2+1, \quad \lambda_1 = 1, \quad \lambda_2 = 0.$$

This is a partially balanced incomplete block design based on a two-class association scheme defined in (5.14) . For $q=2$, we get the design:

$$(5.16)\qquad b = 45, \quad v = 27, \quad r = s, \quad k = 3, \quad n_1 = 10,$$
$$n_2 = 16, \quad p_{11}^1 = 1, \quad p_{11}^2 = 5, \quad \lambda_1 = 1, \quad \lambda_2 = 0.$$

The design (5.16) can be obtained from the famous Steiner configuration of 45 triangles formed by 27 straight lines contained in a general cubic surface. We quote from Jordan [7]: "Steiner a fait connaître (Journal de M. Borchardt, t. LIII) les theorèmes suivants:

Toute surface du troisième degré contient vingt-sept droites;

L'une quelconque d'entre elles, a, en rencontre dix autres, se coupant elles-mêmes deux à deux, et formant ainsi avec a cinq triangles. Le nombre total des triangles ainsi formés sur la surface par les vingt-sept droites est de quarante-cinq. Si deux triangles a b c, a' b' c' n'ont aucun côté commun, on peut leur en associer un troisième a" b" c" tel, que les côtés correspondants de ces trois triangles se coupent, et forment trois nouveaux triangles a a' a", b b' b", c c' c". Les trois triangles associés a b c, a' b' c', a" b" c" s'appellent le trièdre de Steiner. D'après cela, désignons par les lettres a, b, c, d, e, f, g, h, i, k, ℓ, m, n, p, q, r, s, t, u, m', n', p', q', r', s', t', u' les vingt-sept droites: on formera sans peine le tableau suivant des quarante-cinq triangles, où la désignation des droites reste seule arbitraire:

(5.17)

$$\begin{array}{llll} abc, & ade, & afg, & ahi, \quad ak\ell, \\ bmn, & bpq, & brs, & btu, \\ cm'n', & cp'q', & cr's', & ct'u', \\ dmm', & dpp', & drr', & dtt', \\ enn', & eqq', & ess', & euu', \\ fmq', & fpn', & fst', & fur', \\ gnp', & gqm', & gru', & gts', \\ hms', & hrn', & hqt', & hup', \\ inr', & ism', & itq', & ipu', \\ kmu', & ktn', & kqr', & ksp', \\ \ell nt', & \ell um', & \ell rq', & \ell ps'." \end{array}$$

Consider the block design (5.15). Let P be a point in $PG(3,q^2)$ external to V_2. Then the points on V_2 which are conjugate to P are q^3+1 in number and they constitute a non-degenerate V_1 which is the intersection of the polar plane of P with V_2. No two of the q^3+1 tangent planes at these q^3+1 points will contain the same generator.

Hence these q^3+1 tangent planes regarded as blocks and $q+1$ generators in each tangent plane as objects, provide us a parallel class of $v = (q^3+1)(q+1)$ objects in q^3+1 blocks or equivalently, a set of q^3+1 maximal cliques in the strongly regular graph.

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