

0	7	8	9	1	3	5	2	4	6
6	1	7	8	9	2	4	3	5	0
5	0	2	7	8	9	3	4	6	1
4	6	1	3	7	8	9	5	0	2
0	6	5	4	9	8	7	1	2	3
7	1	0	6	5	9	8	2	3	4
8	7	2	1	0	6	9	3	4	5
9	8	7	3	2	1	0	4	5	6
1	9	8	7	4	3	2	5	6	0
3	2	9	8	7	5	4	6	0	1
5	4	3	9	8	7	6	0	1	2
2	3	4	5	6	0	1	7	8	9
4	5	6	0	1	2	3	8	9	7
6	0	1	2	3	4	5	9	7	8

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CHROMATIC POLYNOMIALS

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1. INTRODUCTION

Consider a graph G with edge-set E and vertex-set V , both finite. A λ -colouring of G , where λ is a positive integer, is a mapping of V into the set I_λ of integers from 1 to λ , with the property that the two ends of any edge are mapped onto distinct integers.

The integers 1 to λ are commonly called "colours". The number of λ -colourings of G will be denoted by $P(G, \lambda)$. Recursion formulae are known which permit the determination of the function $P(G, \lambda)$ of λ for any graph G . We proceed to list some simple observations, including the statements of these recursion formulae.

1.1. If G is an edgeless graph with $k > 0$ vertices, then

$$P(G, \lambda) = \lambda^k.$$

For in this case, every mapping of V into I_λ satisfies the definition of a colouring.

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It is convenient to postulate a null graph with no edges and no vertices. For this graph, we make the obvious extension of formula 1.1 and postulate that $P(G, \lambda) = 1$.

1.2. Let k be a positive integer and let K_k be a complete k -graph. This means that K_k has exactly k vertices, that each pair of vertices are joined by exactly one edge, and that there are no loops. Then

$$P(G, \lambda) = \lambda(\lambda-1)(\lambda-2) \dots (\lambda-k+1) .$$

We can extend this rule to the case $k = 0$. K_0 is interpreted as the null graph, so that $P(K_0, \lambda)$ is, by convention, unity. The above formula gives $P(K_0, \lambda)$ as an empty product, and such a product is usually taken as 1.

1.3. If G has a loop, then

$$P(G, \lambda) = 0 \quad \text{for all } \lambda .$$

This is because the two ends of a loop coincide and so cannot have different colours.

Suppose G has a complete k -graph K_k as a subgraph ($k \geq 1$). Each λ -colouring Λ of G induces a λ -colouring Λ_k of K_k , and may be called an extension of Λ_k . We note that

1.4. Each λ -colouring of K_k has the same number of extensions.

To prove this, consider two λ -colourings A and B of K_k . Since K_k is complete, its vertices are all of different colours in A , and also in B . It is, therefore, possible to find a permutation θ of I_λ that transforms the colour in A to the colour in B for each vertex of K_k . Evidently θ transforms each extension of A into an extension of B ,

and θ^{-1} transforms each extension of B into an extension of A. Thus θ sets up a 1 - 1 correspondence between the extensions of A and the extensions of B. The theorem follows.

We note the corollary

1.4.1. The number of extensions of any given λ -colouring of K_k is

$$\frac{P(G, \lambda)}{P(K_k, \lambda)} .$$

1.5. Let G be the union of two subgraphs H and K such that $H \cap K$ is a complete ℓ -graph ($\ell \geq 0$). Then

$$P(G, \lambda) = \frac{P(H, \lambda)P(K, \lambda)}{P(H \cap K, \lambda)} .$$

PROOF: Let us deal first with the case in which $\ell = 0$. Then H and K have no edge or vertex in common. Each λ -colouring of $P(G, \lambda)$ is then obtained by combining a λ -colouring of H with a λ -colouring of K, and each combination of a λ -colouring of H with a λ -colouring of K yields a λ -colouring of G. The theorem follows for this case, since $H \cap K$ and $P(H \cap K, \lambda) = 1$.

For $\ell > 0$, the argument is similar. We start by constructing a λ -colouring Λ_H of H. This induces a λ -colouring Λ_0 of $H \cap K$. To complete a λ -colouring of G, we have to combine Λ_H with an extension Λ_K of Λ_0 in K. Clearly every λ -colouring of G can be obtained in this way. But we can choose Λ_H in $P(H, \lambda)$ ways, and for each of these we can choose Λ_K in

$$\frac{P(K, \lambda)}{P(H \cap K, \lambda)}$$

ways, by 1.4.1. The theorem follows.

Consider an edge A of G , not a loop. Let its ends be x and y . Let G'_A be the graph obtained from G by deleting A but leaving x and y . Let G''_A be the graph obtained from G by contracting A to a single vertex z . More precisely, we delete A and replace x and y by a single new vertex z . An edge incident with x or y in G is made incident with z in G''_A . Incidences of an edge of G''_A with vertices other than z are the same as in G . In particular, an edge, other than A , joining x and y in G becomes a loop on z in G''_A .

$$1.6. \quad P(G, \lambda) = P(G'_A, \lambda) - P(G''_A, \lambda) .$$

PROOF: The λ -colourings of G can be identified with those λ -colourings of G'_A in which x and y have different colours. Moreover, there is a 1 - 1 correspondence between the λ -colourings of G''_A and those λ -colourings of G'_A in which x and y have the same colour. This 1 - 1 correspondence is as follows: vertices common to the two graphs receive the same colour in corresponding colourings, and z in G''_A receives the same colour as x and y in G'_A . The theorem follows.

1.7. Let two distinct vertices x and y of G be joined by two distinct edges A and B . Then

$$P(G, \lambda) = P(G'_A, \lambda) .$$

This follows at once from the definition of a λ -colouring. It can also be regarded as a consequence of 1.3 and 1.6.

1.8. Let G be a non-null loopless graph. Then $P(G, \lambda)$ can be expressed as a polynomial in λ with the following properties.

(1) The coefficient a_j of λ^j is an integer (all j).

(ii) $a_j \neq 0$ if and only if $p_0(G) \leq j \leq |V|$, where $p_0(G)$ is the number of components of G , and $|V|$ is the cardinality of V .

(iii) $a_{|V|} = 1$,

(iv) The non-zero coefficients a_j alternate in sign.

PROOF: We proceed by induction over $|E|$. If $|E| = 0$, we have

$$P(G, \lambda) = \lambda^{|V|}$$

by 1.1. Moreover, $p_0(G) = |V|$. Hence the four conditions are satisfied, trivially in the case of (iv).

Assume as an inductive hypothesis that the theorem holds whenever $|E|$ is less than some positive integer q , and consider the case $|E| = q$. Choose an edge A of G . It is not a loop, by hypothesis. By 1.6, we have

$$P(G, \lambda) = P(G'_A, \lambda) - P(G''_A, \lambda).$$

By the inductive hypothesis, the theorem is true for $P(G'_A, \lambda)$. It is also true for $P(G''_A, \lambda)$ unless G''_A has a loop, in which case $P(G''_A, \lambda) = 0$ by 1.3.

We deduce at once that $P(G, \lambda)$ can be expressed as a polynomial in λ satisfying (i). We note also that (iii) holds for G , since G'_A has one more vertex than G''_A .

The non-zero coefficients of G'_A extend from that of $\lambda^{|V|}$ to that of λ^t , $t = p_0(G'_A)$, with alternation of sign, the coefficient of $\lambda^{|V|}$ being positive. There are two possibilities for $p_0(G'_A)$; it is either $p_0(G)$ or $p_0(G) + 1$ according as the deletion of A fails or does not fail to separate its ends. In the latter alternative, G''_A has no loop.

If G''_A has no loop, its non-zero coefficients extend from that of $\lambda^{|V|-1}$ to that of λ^ℓ , where $\ell = p_0(G''_A) = p_0(G)$, with alternation in sign, the coefficient of $\lambda^{|V|-1}$ being positive.

It now follows from the above equation that the non-zero coefficients of $P(G, \lambda)$ extend from that of $\lambda^{|V|}$ to that of $\lambda^{p_0(G)}$, with alternation in sign. The theorem thus holds when $|E| = q$. It follows, in general, by induction.

When G has a loop, we can think of $P(G, \lambda)$ as a zero polynomial in λ . In all cases, therefore, we refer to $P(G, \lambda)$ as the *chromatic polynomial* of G .

2. CHROMATIC POLYNOMIALS OF SOME SPECIAL GRAPHS

A k -arc L_k is a graph having $k+1$ vertices $a_0, a_1, a_2, \dots, a_k$, and k edges A_1, A_2, \dots, A_k , the ends of A_j being a_{j-1} and a_j ($1 \leq j \leq k$). We identify L_0 with the edgeless graph of one vertex.

$$2.1. \quad P(L_k, \lambda) = \lambda(\lambda-1)^k.$$

PROOF: We have λ choices for the colour of a_0 . If $k > 0$, we then have $\lambda-1$ choices for the colour of a_1 . If $k > 1$, we have $\lambda-1$ choices next for the colour of a_2 , and so on.

Let k be a positive integer. A k -circuit C_k is a graph having k vertices a_0, a_1, \dots, a_{k-1} , and k edges A_1, A_2, \dots, A_k , the ends of A_j being a_{j-1} and a_j if $j < k$, and a_{k-1} and a_0 if $j = k$. We note that C_k has a loop if $k = 1$ but not if $k > 1$.

$$2.2. \quad P(C_k, \lambda) = (\lambda-1)^k + (-1)^k(\lambda-1).$$

PROOF: If $k = 1$, then $P(C_k, \lambda) = 0$, by 1.3, and so the above formula holds.

Assume as an inductive hypothesis that the formula holds when k is less than some positive integer $q \geq 2$, and consider the case $k = q$. Putting $A = A_k$, we have

$$P(C_k, \lambda) = P(L_{k-1}, \lambda) - P(C_{k-1}, \lambda),$$

by 1.6,

$$= \lambda(\lambda-1)^{k-1} - (\lambda-1)^{k-1} - (-1)^{k-1}(\lambda-1),$$

by 2.1 and the inductive hypothesis,

$$= (\lambda-1)^k + (-1)^k(\lambda-1).$$

The theorem follows, by induction.

We note that $P(C_2, \lambda) = P(L_1, \lambda)$ by 1.7, and $C_3 = K_3$.

The wheel W_k of k spokes, where $k \geq 1$, is formed from C_k by adjoining a new vertex v called the *hub* and joining v to each vertex of C_k by an edge called a *spoke*.

We can obtain $P(W_k, \lambda)$ by using the following theorem.

2.3. Let a graph G be obtained from a graph H by adjoining a new vertex v and then joining v to each vertex of H by a new edge. Then

$$P(G, \lambda) = \lambda P(H, \lambda-1).$$

PROOF: When v has been coloured, in any one of the λ colours, only $\lambda-1$ colours are available for the vertices of H . On the other hand, any colouring of H in these $\lambda-1$ colours can be used to complete a λ -colouring of G . The theorem is now evident.

$$**2.4.** \quad P(W_k, \lambda) = \lambda\{(\lambda-2)^k + (-1)^k(\lambda-2)\}.$$

To prove this, we apply 2.3 with $G = W_k$ and v the hub of W_k . Then $H = C_k$, and we obtain $P(H, \lambda-1)$ from 2.2.

Having obtained the chromatic polynomial for the wheel W_k , with hub v we can carry the construction one step further as follows. We adjoin a new vertex w and we join w to each vertex of W_k , including v , by a new edge. Let us denote the resulting double wheel by U_k . By 2.3 and 2.4, we have

$$\begin{aligned} 2.5. \quad P(U_k, \lambda) &= \lambda P(W_k, \lambda-1) \\ &= \lambda(\lambda-1)\{(\lambda-3)^k + (-1)^k(\lambda-3)\}. \end{aligned}$$

The double wheel U_k has an "axle" vw . When we delete this edge in U_k , we obtain a *bipyramid* B_k . This is conveniently drawn on the sphere with v and w at the North and South poles respectively. The circuit C_k occupies the equator. v and w are joined to the vertices on the equator by great-circular segments. When $k \geq 2$, this representation of B_k defines a map on the sphere whose faces are all triangles. Such a map we call a *triangulation* of the sphere.

$$2.6. \quad P(B_k, \lambda) = \lambda\{(\lambda-1)(\lambda-3)^k + (\lambda-2)^k + (-1)^k(\lambda^2 - 3\lambda + 1)\}.$$

PROOF: We can identify B_k with $(U_k)'_A$ where A is the axle vw . The graph $(U_k)''_A$ resembles a wheel, with a hub resulting from the identification of v and w , except that each spoke is doubled. However, $P((U_k)''_A, \lambda) = P(W_k, \lambda)$, by 1.7. Hence

$$\begin{aligned} P(B_k, \lambda) &= P(U_k, \lambda) + P(W_k, \lambda), && \text{by 1.6} \\ &= \lambda(\lambda-1)\{(\lambda-3)^k + (-1)^k(\lambda-3)\} \end{aligned}$$

$$\begin{aligned}
& + \lambda\{(\lambda-2)^k + (-1)^k(\lambda-2)\} \\
= & \lambda\{(\lambda-1)(\lambda-3)^k + (\lambda-2)^k + (-1)^k(\lambda^2 - 3\lambda + 1)\}.
\end{aligned}$$

3. GENERAL λ

As a polynomial, $P(G, \lambda)$ has a definite value when any given real or complex number is substituted for λ . From now on, therefore, we shall regard λ as a complex variable, though usually we shall be concerned only with its values on the real axis.

Polynomial identities that are valid for an integral variable remain valid when this integral variable is replaced by a complex one. (Otherwise, we could construct a polynomial with infinitely many zeros but not identically zero.) Thus the polynomial identities 1.3, 1.5, 1.6, 1.7 and 2.3 extend to complex λ . In other words, we can apply our basic recursion formulae without requiring that λ be an integer.

3.1. Let G be a loopless graph. Let λ be real and negative. Then $P(G, \lambda)$ is non-zero. It is positive or negative according as the number $|V|$ of vertices of G is even or odd.

PROOF: This is clearly true when G is null and therefore $P(G, \lambda) = 1$. When G is non-null, the theorem is a consequence of 1.8.

A stronger result can be obtained by considering the quotient:

$$\frac{P(G, \lambda)}{\lambda}.$$

When G is non-null, this quotient is a polynomial in λ and as such can be assigned a value even when $\lambda = 0$ (see 1.8).

3.2. Let G be a connected non-null loopless graph. Let λ be real and less than 1. Then

$$\frac{P(G, \lambda)}{\lambda}$$

is non-zero. It is positive or negative according as the number $|V|$ of vertices of G is odd or even.

PROOF: We first dispose of two very simple cases.

Suppose that G is the edgeless graph of one vertex. Then $P(G, \lambda) = \lambda$ and

$$\frac{P(G, \lambda)}{\lambda} = 1.$$

The theorem is thus verified.

Next suppose $G = K_2$, i.e., $G = L_1$. Then $P(G, \lambda) = \lambda(\lambda-1)$ by 1.2 or 2.1, and so

$$\frac{P(G, \lambda)}{\lambda} = \lambda-1 < 0.$$

Again the theorem holds.

We proceed by induction over the number $|E|$ of edges. If $|E| = 0$, then G is the loopless graph of one vertex, for which we have already verified the theorem. Assume the theorem to hold for $|E| < q$ and consider the case $|E| = q$. (q is a positive integer.)

Suppose first that G is the union of two subgraphs H and K with only a single vertex v in common, and that each of H and K has an edge. Then H and K must be loopless, non-null and connected. Each has fewer than q edges, and, therefore, each satisfies the theorem. By 1.5,

$$\frac{P(G, \lambda)}{\lambda} = \frac{P(H, \lambda)}{\lambda} \cdot \frac{P(K, \lambda)}{\lambda}.$$

We deduce that

$$\frac{P(G, \lambda)}{\lambda}$$

is non-zero, and has the same sign as

$$(-1)^{h+k} = (-1)^{|V|-1},$$

where h and k are the numbers of vertices of H and K respectively.

Thus G satisfies the theorem.

In general, choose an edge A of G .

It may happen that A is an *isthmus*, that is G'_A has two distinct components each containing one of the ends of A in G . If each of these components consists of a single vertex, then $G = K_2$, and we have already dealt with this graph. In every other case, G is a union of two sub-graphs H and K of the kind just considered.

It remains to consider the case in which A is not an isthmus. Then G'_A and G''_A are both connected non-null graphs, G'_A being loopless. But

$$P(G, \lambda) = P(G'_A, \lambda) - P(G''_A, \lambda)$$

by 1.6. If G''_A has a loop, then $P(G''_A, \lambda) = 0$, by 1.3. Accordingly

$$\frac{P(G, \lambda)}{\lambda} = \frac{P(G'_A, \lambda)}{\lambda}.$$

But the theorem holds for G'_A , which has only $q-1$ edges. Hence it holds for G .

If G''_A has no loop, the theorem holds for both G'_A and G''_A . Then

$$\frac{P(G, \lambda)}{\lambda} = \frac{P(G'_A, \lambda)}{\lambda} - \frac{P(G''_A, \lambda)}{\lambda}.$$

But G''_A has one fewer vertex than G'_A . Hence

$$\frac{P(G'_A, \lambda)}{\lambda} \quad \text{and} \quad \frac{P(G''_A, \lambda)}{\lambda}$$

have opposite signs. We deduce that the theorem is true for G .

The theorem now follows in general, by induction.

At the University of Waterloo, the zeros, both real and complex, of the chromatic polynomials of some graphs triangulating the sphere have been determined. This work was, of course, done with a computer. Polynomials previously calculated by Ruth Basi and Dick Wick Hall were used.

The most striking regularity appearing in the results was that each of the graphs studied has a zero close to

$$\frac{3 + \sqrt{5}}{2}.$$

This is the number $\tau^2 = \tau + 1$, τ being the usual symbol for the "golden section" $(1 + \sqrt{5})/2$. Another way of describing the regularity is to say that $P(G, \tau^2)$ tends to be small when G is a triangulating graph of the sphere.

An example of the effect is provided by the bipyramid B_k whose chromatic polynomial is given by 2.6. When we substitute $\lambda = \tau^2$ in 2.6, the quadratic form $\lambda^2 - 3\lambda + 1$ takes the value zero. Since τ^2 is approximately 2.618, the numbers $\lambda - 2$ and $\lambda - 3$ become less than 1 in absolute value. We deduce that

$$3.3. \quad \lim_{k \rightarrow \infty} P(B_k, \tau^2) = 0.$$

4. THE VERTEX-ELIMINATION THEOREM

When a connected graph is drawn on the sphere, without crossings, we obtain a spherical, or "planar" map. The connected regions into which the graph separates the rest of the sphere are the "faces" of this map. The map is *proper* if the boundary of each face is a simple closed curve. A

proper map in which each face is a triangle, i.e., is bounded by a 3-circuit of the graph, is a triangulation of the sphere.

The simplest example of a triangulation has two faces, three vertices and three edges. Another example is the bipyramid B_k , with $k \geq 2$.

There is an interesting theory of chromatic polynomials of planar triangulation which so far applies only to the special value $\lambda = \tau^2$. It was constructed in an attempt to explain the computer results described in §3. Its basic theorem is the subject of the present section.

A planar map M is *triangular* at a vertex v if all the faces incident with v are triangles. The sides of these triangles opposite v make up a connected graph L_v , though not necessarily a simple closed curve. The graph obtained from a graph G by deleting a vertex v and its incident edges will be denoted by G_v . In the present case, G_v is connected and defines a map M_v on the sphere. M_v is obtained from M by uniting the triangles incident with v so as to form a single new face. It is, of course, not necessarily a triangulation.

When a map M is determined by a graph G , we refer to the chromatic polynomial of G also as the chromatic polynomial of M , and denote it by $P(M, \lambda)$.

4.1. Let M be a map on the sphere, triangular at some vertex v .

Then

$$P(M, \tau^2) = (-1)^m \tau^{1-m} P(M_v, \tau^2),$$

where m is the number of edges incident with v .

PROOF: If possible, choose M so that the theorem fails and the number $|E|$ of edges has the least value consistent with this condition.

Suppose first that the graph G of M is separable, that is it can be represented as the union of two graphs H and K having only a vertex x in common, and each having at least one edge. Then H and K are connected graphs, by the connection of G , and x is distinct from v by the connection of the graph L_v . Suppose v to be a vertex of K . Then K defines a map N that is triangular at v , and G_v is the union of H and K_v . By the choice of M , the map N satisfies the theorem. Hence

$$\begin{aligned} P(G, \tau^2) &= \tau^{-2} P(H, \tau^2) (-1)^m \tau^{1-m} P(K_v, \tau^2), \\ P(G_v, \tau^2) &= \tau^{-2} P(H, \tau^2) P(K_v, \tau^2), \end{aligned}$$

by 1.5. Combining these results, we find

$$P(G, \tau^2) = (-1)^m \tau^{1-m} P(G_v, \tau^2).$$

From now on we may assume that M is non-separable. Hence it has no isthmus.

Suppose next that M has an edge A that is not the side of a triangle incident with v . If A is a loop, the theorem is trivially true, by 1.3. From now on, therefore, we may assume M to be loopless. If the ends x and y of A are joined by a second edge B , the theorem is again trivially true for M , by 1.7. We may, therefore, assume M to have no 2-circuits whose edges do not both belong to the boundaries of triangles incident with v .

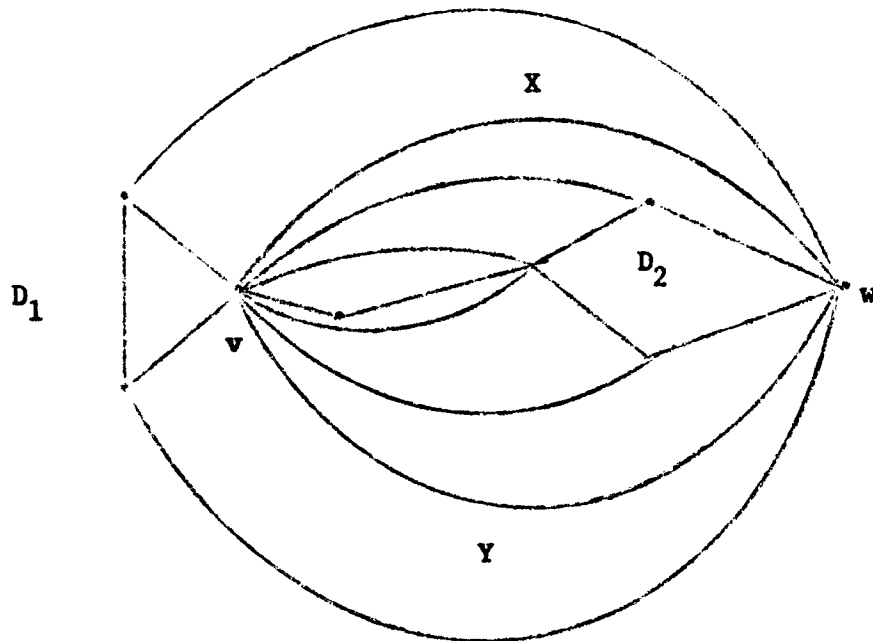
Consider the graphs G'_A and G''_A . They are both connected, since A is not an isthmus, and they define maps M'_A and M''_A on the sphere. We form M'_A from M by fusing the two faces of M incident with A into a single new face, and we form M''_A from M by identifying all the points of the edge A to form a single new point. Both M'_A and M''_A are triangular at v , and both satisfy the theorem, by the choice of M . Using 1.6, we find

$$\begin{aligned}
P(M, \tau^2) &= P(M'_A, \tau^2) - P(M''_A, \tau^2) \\
&= (-1)^m \tau^{1-m} \{P((M'_A)_v, \tau^2) - P((M''_A)_v, \tau^2)\} \\
&= (-1)^m \tau^{1-m} \{P((M_v)_A', \tau^2) - P((M_v)_A'', \tau^2)\} \\
&= (-1)^m \tau^{1-m} P(M_v, \tau^2),
\end{aligned}$$

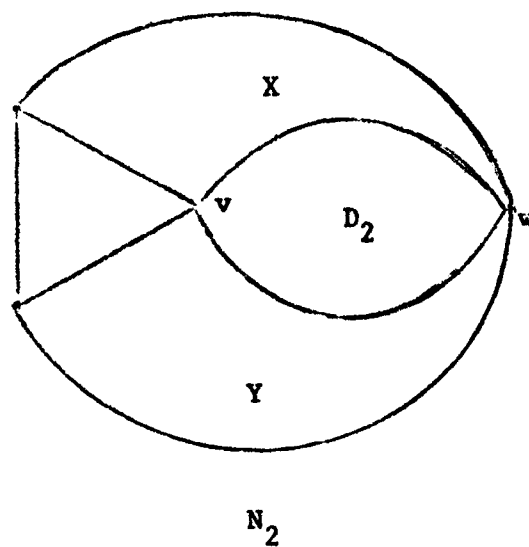
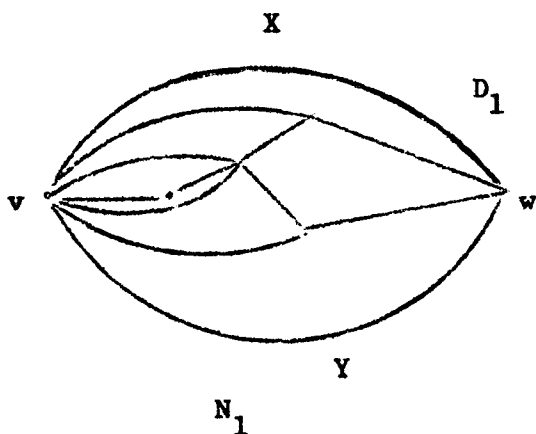
by 1.6. Thus M satisfies the theorem, a contradiction.

From now on we may assume that the only edges of M are the sides of the triangles incident with v .

Suppose two distinct edges X and Y of M incident with v have the same second end w . They make up a simple closed curve separating the sphere into two Jordan domains, each containing some of the triangles incident with v . Call these domains D_1 and D_2 .

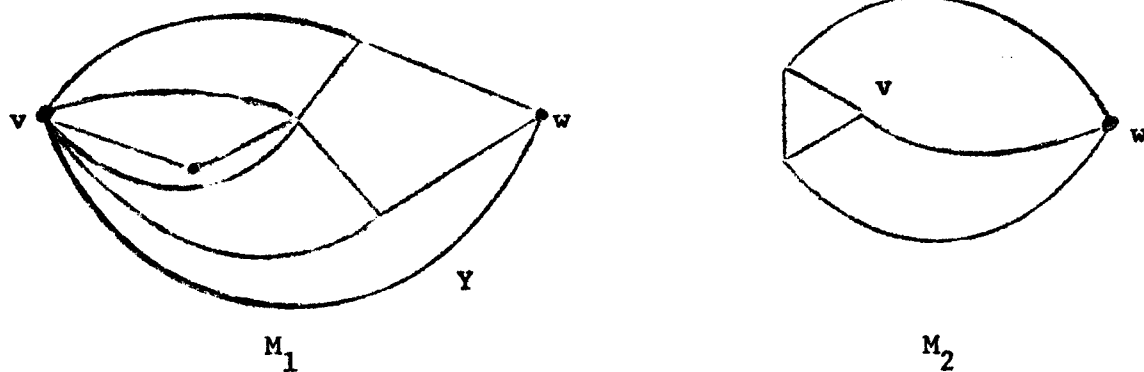


If we delete from the graph the edges and vertices inside D_1 and treat D_1 as a single new face, we obtain a map N_1 having a 2-gon incident with v



Apart from the 2-gon, v is incident only with triangular faces in N_1 and N_2 . Let us suppose it is incident with m_1 triangular faces in N_1 , and m_2 in N_2 . Then $m = m_1 + m_2$.

Let M_1 be formed from N_1 by deleting X from its graph and fusing the digon with the triangle incident with X so as to give a new triangle.



The maps M_1 and M_2 are triangular at v . They satisfy the theorem by the choice of M .

Let N be the map derived from M by deleting X from its graph and fusing the two adjacent faces into a single new face. Then

$$\begin{aligned}
 P(M, \tau^2) &= P(N, \tau^2), \quad \text{by 1.7,} \\
 &= \frac{P(M_1, \tau^2) P(M_2, \tau^2)}{\tau^2 \cdot \tau}, \quad \text{by 1.5,} \\
 &= \tau^{-3} (-1)^{m_1} \tau^{1-m_1} P((M_1)_v, \tau^2) (-1)^{m_2} \tau^{1-m_2} P((M_2)_v, \tau^2) \\
 &= (-1)^m \tau^{-1-m} P((M_1)_v, \tau^2) P((M_2)_v, \tau^2).
 \end{aligned}$$

But the graph of M_v is the union of the graphs of $(M_1)_v$ and $(M_2)_v$, and these have only the vertex w in common. Hence

$$P(M_v, \tau^2) = \tau^{-2} P((M_1)_v, \tau^2) P((M_2)_v, \tau^2).$$

Combining the above results, we find

$$P(M, \tau^2) = (-1)^m \tau^{1-m} P(M_v, \tau^2).$$

We may now assume that all the m edges radiating from v in M have distinct ends. Hence L_v is either a circuit or a complete 2-graph; the graph G of M is either a wheel of m spokes, or a complete 3-graph.

If $M = W_m$, we have $M_v = C_m$. Hence

$$\begin{aligned} P(M, \tau^2) &= \tau^2 (\tau^{-m} + (-1)^m \tau^{-1}), \quad \text{by 2.4} \\ &= (-1)^m \tau^{1-m} (\tau^m + (-1)^m \tau) \\ &= (-1)^m \tau^{1-m} P(M_v, \tau^2), \quad \text{by 2.2.} \end{aligned}$$

If $M = K_3$, we have $M_v = K_2$. Hence, by 1.2,

$$\begin{aligned} P(M, \tau^2) &= \tau^2 \cdot \tau \cdot \tau^{-1} = \tau^{-1} P(M_v, \tau^2) \\ &= (-1)^2 \tau^{1-2} P(M_v, \tau^2). \end{aligned}$$

In every case, we have shown that M satisfies the theorem, contrary to its definition. We conclude that the theorem is true.

5. THE MAGNITUDE OF $P(M, \tau^2)$

In this section, we offer a theory to explain the empirical observation that $P(M, \tau^2)$ is small when M is a triangulation of the sphere. We use the vertex-elimination theorem to establish the following result.

5.1. If M is a triangulation of the sphere with vertex-set V , then

$$|P(M, \tau^2)| \leq \tau^{5-|V|}.$$

Here $|V|$ is the cardinality of V as usual, but the vertical bars enclosing $P(M, \tau^2)$ denote absolute value.

PROOF: If possible, choose a triangulation M so that the theorem fails and $|V|$ has the least value consistent with this.

Suppose first that M has two edges X and Y with the same ends v and w . M being a triangulation, it has no loops. The circuit made up by X and Y separates the rest of the sphere into two residual domains D_1 and D_2 . As in the proof of the vertex elimination theorem, we define the maps N_1, N_2, M_1 and M_2 and have

$$P(M, \tau^2) = \frac{P(M_1, \tau^2)P(M_2, \tau^2)}{\tau^3}.$$

Let V_1 be the vertex set of M_1 . We note that

$$|V| = |V_1| + |V_2| - 2.$$

But M_1 and M_2 satisfy the theorem, by the choice of M . Hence

$$\begin{aligned} |P(M, \tau^2)| &= \tau^{-3} |P(M_1, \tau^2)| \cdot |P(M_2, \tau^2)| \\ &\leq \tau^{-3} \cdot \tau^{5-|V_1|} \cdot \tau^{5-|V_2|} \\ &= \tau^{5-|V|}. \end{aligned}$$

From now on, we can assume that the graph G of M has no 2-circuit.

It is a well-known consequence of the Euler polyhedron formula that a triangulation of the sphere has a vertex v of valency $m \leq 5$. The least possible value of m is of course 2. Choose such a vertex v in M .

If $m = 2$, the two triangles incident with v have all three sides in common, since there is no 2-circuit. Hence G is a complete 3-graph, and

$$\begin{aligned} P(M, \tau^2) &= \tau^2 \cdot \tau \cdot \tau^{-1} \\ &= \tau^2 = \tau^{5-|V|}. \end{aligned}$$

If $m = 3$, then M_V is also a triangulation, with $|V|-1$ vertices. It satisfies the theorem by the choice of M . Hence

$$\begin{aligned} |P(M, \tau^2)| &= \tau^{-2} |P(M_V, \tau^2)|, \quad \text{by 4.1,} \\ &= \tau^{-2} \cdot \tau^{5-(|V|-1)} \\ &= \tau^{4-|V|} < \tau^{5-|V|}. \end{aligned}$$

If $m = 4$, we consider the map M_V . It is a proper map (since M has no 2-circuit). It has one quadrilateral Q and its other faces are triangular faces of M . It has $|V|-1$ vertices.

Let the vertices of Q be, in their cyclic order a, b, c, d . Since M is a map on the sphere, the pairs (a, c) and (b, d) cannot both be joined by edges of M_V . We may assume that a and c are not so joined. We construct a triangulation T by subdividing Q into two triangles by a new diagonal edge $A = (ac)$. We note that $M_V = T'_A$ and that T''_A is also a triangulation of the sphere, except for the doubling of two edges. We make it a true triangulation T_0 by retaining only one of the edges ab and bc of M_V , and only one of cd and da . We have $P(T_0, \lambda) = P(T''_A, \lambda)$, by 1.7. Hence

$$\begin{aligned} |P(M, \tau^2)| &= \tau^{-3} |P(M_V, \tau^2)|, \quad \text{by 4.1,} \\ &= \tau^{-3} |P(T, \tau^2) + P(T_0, \tau^2)|, \quad \text{by 1.6,} \\ &\leq \tau^{-3} \{ |P(T, \tau^2)| + |P(T_0, \tau^2)| \}. \end{aligned}$$

But T and T_0 satisfy the theorem, by the choice of M . Hence

$$\begin{aligned}
|P(M, \tau^2)| &\leq \tau^{-3} (\tau^{5-(|V|-1)} + \tau^{5-(|V|-2)}) \\
&= \tau^{5-|V|} (\tau^{-2} + \tau^{-1}) \\
&= \tau^{5-|V|} .
\end{aligned}$$

Note that $|P(M_V, \tau^2)| \leq \tau^{8-|V|}$ by the above results.

It remains to consider the case $m = 5$. In this case, M_V is a proper map. It has one pentagon Q and its other faces are triangular faces of M .

Let the vertices of Q be, in their cyclic order a, b, c, d, e . Again, the pairs (a, c) and (b, d) cannot both be joined by an edge of M_V and we assume that a and c are not so joined. We construct a map N by subdividing Q into a triangle and a quadrilateral by a new diagonal edge $A = (ac)$. We can identify N with the map denoted by M_V in the preceding argument, thereby deducing that

$$|P(N, \tau^2)| \leq \tau^{8-|V|} .$$

If, in the map N_A'' , we retain only one of the edges ab and bc of M_V , we obtain a triangulation T . It has only $|V|-2$ vertices and so satisfies the theorem. Thus

$$\begin{aligned}
|P(M, \tau^2)| &= \tau^{-4} |P(M_V, \tau^2)|, \quad \text{by 4.1,} \\
&= \tau^{-4} |P(N, \tau^2) + P(T, \tau^2)|, \quad \text{by 1.6,} \\
&\leq \tau^{-4} |P(N, \tau^2)| + \tau^{-4} |P(T, \tau^2)| \\
&\leq \tau^{4-|V|} \tau^{3-|V|} \\
&= \tau^{5-|V|} .
\end{aligned}$$

We find that in all cases M satisfies the theorem, contrary to the choice of M . We conclude that the theorem is true.

6. RECURSION FORMULAE FOR TRIANGULATIONS

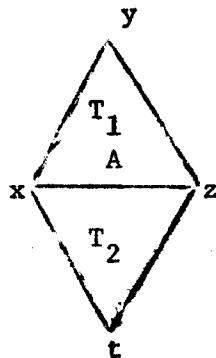
The recursion formulae of Section 1 enable us to calculate the chromatic polynomial of any graph G . But if G is fairly complicated, they require us to work out the chromatic polynomials of many smaller graphs before we get to that of G .

The graphs whose chromatic polynomials have aroused most interest so far are those of the triangulations of the sphere. If G is such a graph and A is one of its edges, not a loop, then G'_A does not usually represent a triangulation, though G''_A does so unless it has a loop. It is desirable to find a recursion formula that works with triangulations only.

One such formula is 1.5, in the case $\ell = 3$, and this is very helpful in practice. Another, concerned with a separating 2-gon, has already been used in the proofs of 4.1 and 5.1.

There is another formula for triangulations that can be derived from 1.6.

Suppose A is an edge of a triangulation M . Let T_1 and T_2 be its incident triangles. We denote the ends of A by x and z , the third vertex of T_1 by y and the third vertex of T_2 by t .



We assume that x , y , z and t are distinct.

From M we can derive a number of other maps. We write N for the map obtained by deleting A from the graph so as to replace T_1 and T_2 by a quadrilateral Q . We write $\theta_A M$ for the graph obtained from N by subdividing Q into two triangles by the diagonal $B = (yt)$. It is often said that $\theta_A M$ is obtained from M by "twisting" A .

In M we can contract A to a single vertex and delete the edge yz of T_1 and the edge zt of T_2 . This operation yields a map $\phi_A M$, which is a triangulation unless the contraction of A introduces a loop. We denote the map $\phi_B(\theta_A M)$ also by $\psi_A M$.

Using 1.6 and 1.7, we find

$$(i) \quad P(M, \lambda) + P(\phi_A M, \lambda) = P(N, \lambda).$$

Similarly

$$(ii) \quad P(\theta_A M, \lambda) + P(\psi_A M, \lambda) = P(N, \lambda).$$

Hence we have the recursion formula

$$6.1. \quad P(M, \lambda) - P(\theta_A M, \lambda) = P(\psi_A M, \lambda) - P(\phi_A M, \lambda).$$

This formula involves triangulations only, except that loops may appear in $\psi_A M$ or $\phi_A M$. In theory, we can use it to construct a list of chromatic polynomials of triangulations with vertex-valences ≥ 3 . Suppose we have listed the polynomials up to a particular value of $|V|$, say $|V| = q$. Choose a map M with $|V| = q+1$ and let $\beta \geq 3$ be its smallest vertex valency. If $\beta = 3$, we use 1.2 or 1.5 to calculate $P(M, \lambda)$. If $\beta = 4$, we apply 6.1 with A incident with a tetravalent vertex. Then $\beta = 3$ in $\theta_A M$ and so $P(M, \lambda)$ is expressed in terms of known polynomials. Similarly, if $\beta = 5$ in M , we can arrange that $\beta = 4$ in $\theta_A M$. Of course, if M has a separating 2-gon or 3-gon, we can apply a simpler formula.

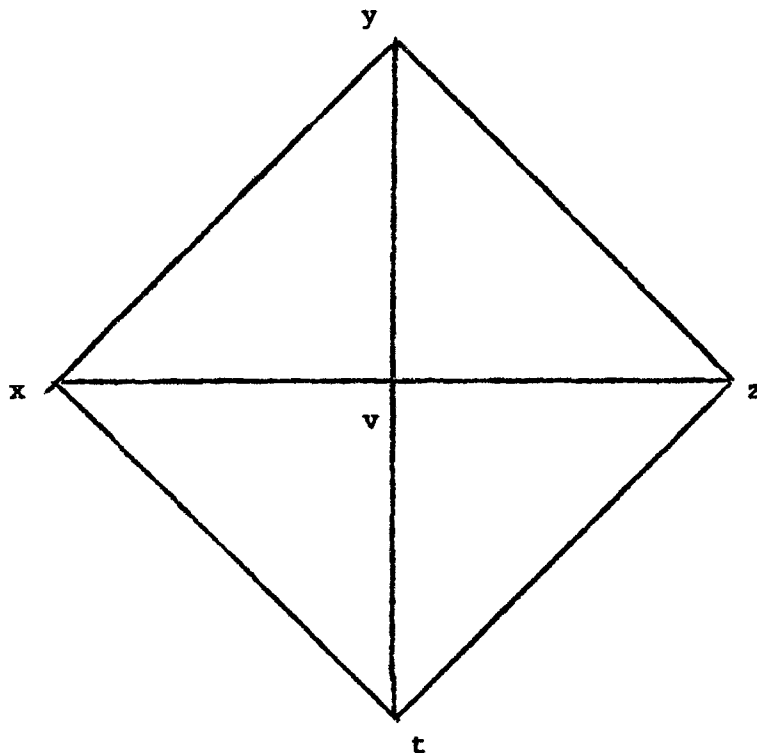
The recursion formulae used in practice are based on 6.1.

It is a curious fact that when $\lambda = \tau^2$ the four polynomials of 6.1 satisfy a second linear relation

$$6.2. \quad P(M, \tau^2) + P(\theta_A M, \tau^2) = \tau^{-3} \{P(\psi_A M, \tau^2) + P(\phi_A M, \tau^2)\}$$

For $\lambda = \tau^2$, we can, of course, eliminate $P(\theta_A M, \tau^2)$ from 6.1 and 6.2, and so obtain $P(M, \tau^2)$ directly in terms of triangulations with fewer vertices.

PROOF OF 6.2: We can obtain from N a map N_1 by subdividing the quadrilateral Q into four triangles by introducing a new central vertex v and joining it by new edges to x , y , z , and t .



The map $(N_1)''_B$, where $B = (xv)$, has the same chromatic polynomial as M , by 1.7. The map $((N_1)'_B)''_C$, where $C = (vt)$ has likewise the same

chromatic polynomial as $\theta_A M$. The chromatic polynomial of $((N_1)_B')'_C$ can be computed from 1.5 as

$$(\lambda-2)P(N, \lambda).$$

Hence

$$\begin{aligned} P(N_1, \lambda) &= P((N_1)_B', \lambda) - P((N_1)_B'', \lambda) \\ &= P((N_1)_B')'_C, \lambda) - P((N_1)_B')''_C, \lambda) - P((N_1)_B'', \lambda) \\ &= (\lambda-2)P(N, \lambda) - \{P(M, \lambda) + P(\theta_A M, \lambda)\}. \end{aligned}$$

Now, by the vertex-elimination theorem,

$$P(N_1, \tau^2) = \tau^{-3} P(N, \tau^2).$$

So by substituting $\lambda = \tau^2$ in the above equation, we have

$$(\tau^{-1} - \tau^{-3})P(N, \tau^2) = P(M, \tau^2) + P(\theta_A M, \tau^2),$$

that is

$$P(M, \tau^2) + P(\theta_A M, \tau^2) = \tau^{-2} P(N, \tau^2).$$

But by (i) and (ii)

$$2P(N, \lambda) = P(M, \lambda) + P(\theta_A M, \lambda) + P(\psi_A M, \lambda) + P(\phi_A M, \lambda)$$

Eliminating $P(N, \tau^2)$, we find

$$(2 - \tau^{-2})\{P(M, \lambda) + P(\theta_A M, \lambda)\} = \tau^{-2} \{P(\psi_A M, \lambda) + P(\phi_A M, \lambda)\}.$$

But $2 - \tau^{-2} = \tau$. The theorem follows.

7. THE "GOLDEN IDENTITY"

For a triangulation M , there is a curious relation between $P(M, \tau^2)$ and $P(M, \sqrt{5}\tau)$. (We note that $\tau^2 = \tau + 1$ and $\sqrt{5}\tau = \tau + 2$.)

$$7.1. \quad P(M, \sqrt{5}\tau) = \sqrt{5} \tau^{3|V|-9} P^2(M, \tau^2).$$

PROOF: If possible, choose M so that the theorem fails, $|V|$ has the smallest value consistent with this condition, and then the minimum vertex-valency q of M is as small as possible. We have, of course,

$$(i) \quad 2 \leq q \leq 5.$$

Suppose M has a 2-circuit with edges X and Y . We define N_1, N_2, M_1 and M_2 as in the proof of 4.1. Using 1.5 and 1.7, we find

$$(ii) \quad P(M, \lambda) = \frac{P(M_1, \lambda)P(M_2, \lambda)}{\lambda(\lambda-1)}.$$

Since M_1 and M_2 are triangulations with fewer vertices than M , they satisfy the theorem. Denoting the vertex set of M_1 by V_1 , we have

$$|V_1| + |V_2| = |V| + 2.$$

Hence

$$\begin{aligned} P(M, \sqrt{5}\tau) &= \frac{(\sqrt{5})^2 \tau^{3|V|-12} P^2(M_1, \tau^2) P^2(M_2, \tau^2)}{\sqrt{5}\tau \cdot \tau^2} \\ &= \sqrt{5} \tau^{3|V|-15} P^2(M_1, \tau^2) P^2(M_2, \tau^2). \end{aligned}$$

But, by (ii)

$$P(M, \tau^2) = \tau^{-3} P(M_1, \tau^2) P(M_2, \tau^2).$$

Hence

$$P(M, \sqrt{5}\tau) = \sqrt{5} \tau^{3|V|-9} P^2(M, \tau^2).$$

From now on, we can assume that M has no 2-circuit. Let x be a vertex of valency q .

If $q = 2$, then, in the absence of a 2-circuit, the two triangles incident with x have all their sides in common. Hence the graph G of M is K_3 . By 1.2,

$$P(K_3, \sqrt{5}\tau) = \sqrt{5} \tau \cdot \tau^2 \cdot \tau = \sqrt{5} \tau^4$$

$$P(K_3, \tau^2) = \tau^2 \cdot \tau \cdot \tau^{-1} = \tau^2$$

$$P(M, \sqrt{5}\tau) = \sqrt{5} \tau^{3|V|-9} \cdot P^2(M, \tau^2).$$

Suppose next that $q > 2$. Let A be an edge incident with x and let its other end be z . Let T_1 and T_2 be the triangles incident with A and let their third vertices be y and t respectively. Then $y \neq t$, since there is no 2-circuit. We accordingly use the notation of §6.

$$\begin{aligned} P(M, \sqrt{5}\tau) - P(\theta_A M, \sqrt{5}\tau) &= P(\psi_A M, \sqrt{5}\tau) - P(\phi_A M, \sqrt{5}\tau) \\ &= \sqrt{5} \tau^{3|V|-12} \{P^2(\psi_A M, \tau^2) - P^2(\phi_A M, \tau^2)\}, \end{aligned}$$

since the theorem holds for $\psi_A M$ and $\phi_A M$, by the choice of M ,

$$\begin{aligned} &= \sqrt{5} \tau^{3|V|-12} \{P(\psi_A M, \tau^2) + P(\phi_A M, \tau^2)\} \cdot \\ &\quad \cdot \{P(\psi_A M, \tau^2) - P(\phi_A M, \tau^2)\} \\ &= \sqrt{5} \tau^{3|V|-9} \{P(\theta_A M, \tau^2) + P(M, \tau^2)\} \cdot \\ &\quad \cdot \{P(M, \tau^2) - P(\theta_A M, \tau^2)\}, \end{aligned}$$

by 6.1 and 6.2

$$= \sqrt{5} \tau^{3|V|-9} \{P^2(M, \tau^2) - P^2(\theta_A M, \tau^2)\}.$$

But the theorem holds for $\theta_A M$ since this has a vertex x of valency $q-1$. That is

$$P(\theta_A M, \sqrt{5}\tau) = \sqrt{5} \tau^{3|V|-9} P^2(\theta_A M, \tau^2) .$$

Combining this with the preceding result, we obtain

$$P(M, \sqrt{5}\tau) = \sqrt{5} \tau^{3|V|-9} P^2(M, \tau^2) .$$

In every case, M satisfies the theorem, contrary to its definition. The theorem follows.

Combining Theorems 3.2 and 7.1, we can establish

7.2. $P(M, \sqrt{5}\tau)$ is positive.

PROOF: Write $\tau^* = \frac{1-\sqrt{5}}{2}$. Then the number $1+\tau^*$ lies between 0 and 1. Hence $P(M, 1+\tau^*)$ is non-zero, by 3.2. Since $P(M, \lambda)$ is a polynomial with integer coefficients, we must have also

$$P(M, 1+\tau) \neq 0,$$

i.e.,

$$P(M, \tau^2) \neq 0 .$$

The required result now follows, by 7.1.

The number $\sqrt{5}\tau$ is approximately

$$3.61803398874989484820 .$$

It has been remarked that Theorem 7.2 is of interest because it shows $P(M, \lambda)$ to be positive at a value of λ near 4, and many people are specially interested in the value $\lambda = 4$. There is a long-standing conjecture that $P(M, 4)$ is positive, for every triangulation M . The obvious suggestion that $P(M, 4)$ is always positive from $\sqrt{5}\tau$ to 4 inclusive turns out to be wrong. At Waterloo, a few cases have been found of triangular polynomials having two real zeros between $\sqrt{5}\tau$ and 4.

8. AN EXPLICIT FORMULA FOR $P(M, \tau^2)$

An independent set of vertices in a graph G is a set W of vertices such that each edge has one end in W and one not.

One way of constructing a λ -colouring of G is to begin by choosing an independent set W of vertices and assigning the colour λ to its members. The other vertices are to be coloured in the remaining $\lambda-1$ colours. Clearly in any λ -colouring, the vertices of any one colour must constitute an independent set.

Let the graph obtained from G by deleting W and its incident edges be G_W . By the above considerations, we have

$$8.1. \quad P(G, \lambda) = \sum_W P(G_W, \lambda-1).$$

This is a polynomial identity: it remains valid when we substitute $\lambda = \tau^2 + 1 = \sqrt{5} \tau$. Hence

$$P(G, \sqrt{5}\tau) = \sum_W P(G_W, \tau^2).$$

We can use the vertex-elimination formula to express $P(G_W, \tau^2)$ in terms of $P(G, \tau^2)$, in the case in which G is the graph of a triangulation M . For G_W is formed from G by eliminating one by one the vertices of W , and at each stage the map is triangular at the vertex being eliminated. So if $m(W)$ is the sum of the valences in G of the vertices of W we have

$$\begin{aligned} P(M, \sqrt{5}\tau) &= \sum_W P(G_W, \tau^2) \\ &= \sum_W (-1)^{m(W)} \tau^{m(W)-|W|} P(M, \tau^2), \end{aligned}$$

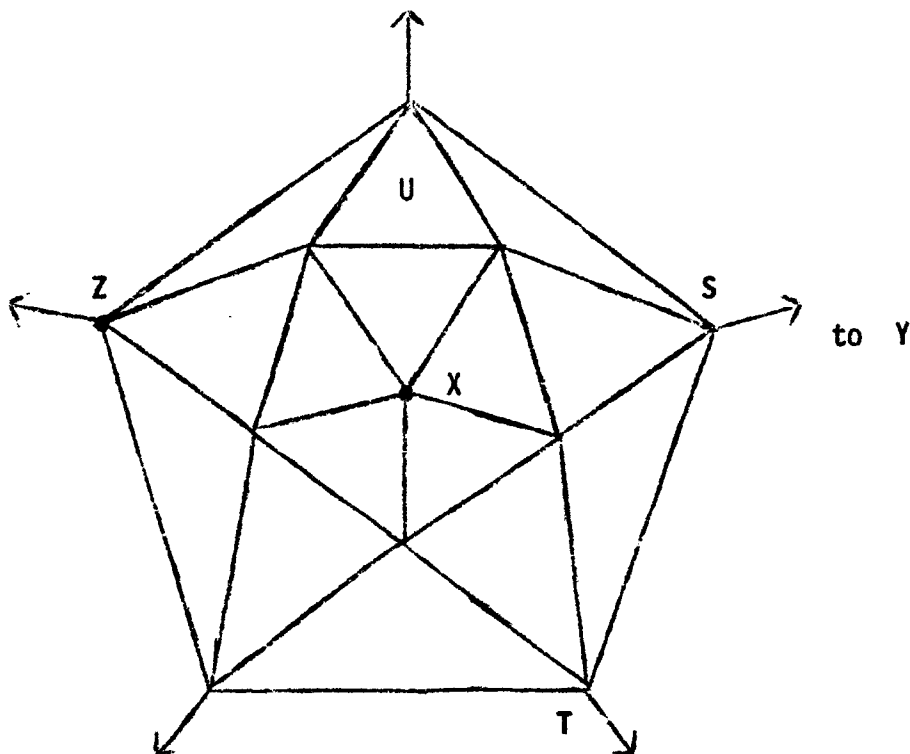
by 4.1. But

$$P(M, \sqrt{5}\tau) = \sqrt{5} \tau^{3|V|-9} P^2(M, \tau^2).$$

Hence

$$8.2. \quad P(M, \tau^2) = \frac{1}{\sqrt{5}} \tau^{9-3|V|} \sum_W (-1)^{m(W)} \tau^{m(W)-|W|}.$$

As an example, let us calculate $P(M, \tau^2)$ for the regular icosahedron. In this triangulation, there is as usual one null independent set. There are 12 independent sets of one vertex, and 6 independent sets consisting of two diametrically opposite vertices. For other independent sets of two vertices if we choose one member, such as X in the diagram, we have five choices for the other. Hence the number of independent sets of two non-opposite vertices is $\frac{1}{2}(12 \times 5) = 30$.



It is evident that an independent set of 3 vertices cannot contain two diametrically opposite ones. It is thus symmetrically equivalent to a set containing the vertices X and Z of the diagram. The third member must be S or its symmetrical equivalent T. A set W of three independent

vertices can thus be characterized by giving a triangle U and postulating that the members of W are the remote vertices of the three triangles sharing sides with U . The number of such sets of 3 vertices is thus 20. Clearly a set of three independent vertices cannot be extended as a set of four.

We tabulate this information about independent sets W as follows.

$ W $	$m(W)$	No. of sets W
0	0	1
1	5	12
2	10	36
3	15	20.

So by 8.2, we have

$$P(M, \tau^2) = \{1 - 12\tau^4 + 36\tau^8 - 20\tau^{12}\} \cdot \frac{1}{\sqrt{5}} \tau^{9-36}.$$

We now need a table of powers of τ . This is given below. We put

$$\tau^n = \frac{1}{2}(A_n + \sqrt{5} B_n)$$

and tabulate A_n and B_n . Each row of the table is obtained as the sum of the two preceding ones, (except for the first two).

n	A _n	B _n
0	2	0
1	1	1
2	3	1
3	4	2
4	7	3
5	11	5
6	18	8
7	29	13
8	47	21
9	76	34
10	123	55
11	199	89
12	322	144
13	521	233
14	843	377

therefore

$$36\tau^8 = \frac{1}{2} (1692 + 756 \sqrt{5})$$

$$1 + 36\tau^8 = \frac{1}{2} (1694 + 756 \sqrt{5}) ,$$

$$12\tau^4 = \frac{1}{2} (84 + 36 \sqrt{5})$$

$$20\tau^{12} = \frac{1}{2} (6440 + 2880 \sqrt{5})$$

$$12\tau^4 + 20\tau^{12} = \frac{1}{2} (6524 + 2916 \sqrt{5})$$

therefore

$$1 - 12\tau^4 + 36\tau^8 - 20\tau^{12} = \frac{-1}{2} (4830 + 2160 \sqrt{5})$$

$$= -5(483 + 216\sqrt{5})$$

$$= -15(161 + 72\sqrt{5})$$

$$= -15\tau^{12}.$$

Hence
$$P(M, \tau^2) = \frac{1}{\sqrt{5}} \tau^{-27} \cdot (-15\tau^{12})$$

$$= -3\sqrt{5} \tau^{-15}.$$

(This has been checked by substitution in the known polynomial $P(M, \lambda)$.)