This research was partially supported by the Army Research Office, Durham, under Grant No. DA-ARO-D-31-124-G910 and the U.S. Air Force Office of Scientific Research under Contract AFOSR-68-1406.

COMBINATORIAL MATHEMATICS YEAR

February 1969 - June 1970

STEINER TRIEDERS AND STRONGLY REGULAR GRAPHS

by

Elisabeth Carrière
Department of Statistics *and* *Faculté des Sciences*
University of North Carolina *Université de Paris*

University of North Carolina at Chapel Hill
Institute of Statistics Mimeo Series No. 600.26

May 1970

STEINER TRIEDERS AND STRONGLY REGULAR GRAPHS

by ELISABETH CARRIÈRE

Department of Statistics
University of North Carolina at Chapel Hill

and

Faculté des Sciences
Université de Paris

1. INTRODUCTION

A finite, undirected linear graph G is strongly regular if it is regular of valence n_1, each adjacent pair of vertices is adjacent to exactly p^1_{11} other vertices, and each non-adjacent pair of vertices is adjacent to exactly p^2_{11} other vertices.

We shall consider the Steiner configuration of 45 triangles formed by 27 straight lines contained in a general cubic surface. The graph associated with this configuration will be proved to be strongly regular.

The Steiner configuration has an interesting property about its triangles, and we shall see under which conditions graphs satisfying this property are strongly regular.

2. THE STEINER CONFIGURATION

The Steiner configuration, determined by the 27 straight lines of any cubic surface has the following properties (see [2] and [4]):

This research was partially supported by the Army Research Office, Durham, under Grant No. DA-ARO D-31-124- G910 and the U.S. Air Force Office of Scientific Research under Contract AFOSR-68-1406.

P.2.1. Each straight line A meets ten others.

P.2.2. The ten lines which meet any line A cut in pairs to form five triangles with A.

P.2.3. Given two triangles $A_1B_1C_1$, $A_2B_2C_2$ with no side in common, there exists a unique triangle $A_3B_3C_3$, such that $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ are three more triangles, and $A_3B_3C_3$ has no side in common with $A_1B_1C_1$ and $A_2B_2C_2$.

These nine lines forming six triangles are called the Steiner trieder.

Let us consider the graph G associated with the Steiner configuration and defined as follows:

1. The vertices of the graph G are the lines of the configuration.
2. Two vertices of the graph G are adjacent if their corresponding lines intersect in the configuration.

Thus each triangle in the configuration determines a triangle in the graph (the sides of a triangle become its vertices).

Let us rewrite the properties P.2.1, P.2.2, P.2.3 for the graph G:

P.1: Each vertex is adjacent to ten others.

P.2: The ten vertices adjacent to any vertex x of the graph are associated in adjacent pairs to form five triangles with x.

P.3: Given two triangles $x_1y_1z_1$ and $x_2y_2z_2$ of G with no vertex in common, there exists a unique triangle $x_3y_3z_3$ such that $x_1x_2x_3$, $y_1y_2y_3$, $z_1z_2z_3$ are three more triangles and $x_3y_3z_3$ has no vertex in common with $x_1y_1z_1$ and $x_2y_2z_2$.

These nine vertices will also be said to form a Steiner trieder.

The graph G has 27 vertices and is regular of valance 10 by P.1.

PROPOSITION 1.

The graph G associated with the Steiner configuration is strongly regular with the parameters: $v = 27$, $n_1 = 10$, $p_{11}^1 = 1$, $p_{11}^2 = 5$.

PROOF:

First, let us notice that the property P.3 implies obviously the following property:

P.2.3 Given two triangles $x_1y_1z_1$ and $x_2y_2z_2$ of the graph G, with no vertex in common, each vertex of one triangle is adjacent at least to one vertex of the other triangle.

Given any pair x,y of adjacent vertices of G, there exists, by P.2, a vertex z such that xyz is a triangle. There are 24 vertices adjacent to x,y and z. If they are distinct, we have, with x,y and z, the 27 vertices of the graph G. If not, there exists a vertex u which is adjacent neither to x nor to y, nor to z and a triangle uvw which has no vertex in common with xyz. But, by the property P.2.3, each vertex of the triangle uvw has to be adjacent to one vertex of xyz, which contradicts our hypothesis that u is adjacent neither to x nor to y nor to z.

Then, the 24 vertices adjacent to x,y and z are distinct and z is the unique vertex which is adjacent to x and y.

Thus we have $p_{11}^1 = 1$. The ten vertices adjacent to any vertex x form with x five triangles, two of them having only x as a common vertex.

Any pair x, y of nonadjacent vertices of G determine 10 distinct triangles.

Let xx_1x_2 and yy_1y_2 be two of these triangles. They cannot have two vertices in common, since $p^1_{11} = 1$. If they have one vertex in common, say $x_1 = y_1$, and if the third vertex x_2 is adjacent to y, we will have a triangle yx_1x_2 which is impossible since $p^1_{11} = 1$. Hence, x_2 is nonadjacent to y and y_2 is nonadjacent to x for the same reason.

If they have no vertex in common, the property P.2.3 must hold, that is that at least one of x_1, x_2 is adjacent to y and one of y_1, y_2 is adjacent to x. But if both of x_1, x_2 are adjacent to y, or if both of y_1, y_2 are adjacent to x, then yx_1x_2 or xy_1y_2 will be a triangle which contradicts that $p^1_{11} = 1$. Thus each of the five triangles determined by the vertex x has one and only one vertex which is adjacent to y. That is there are exactly five vertices adjacent to x and y and we have $p^2_{11} = 5$.

3. GRAPHS SATISFYING THE PROPERTY P.3.

Consider a graph G, with at least two triangles with no vertex in common and satisfying P.3. G has at least nine distinct vertices x_i, y_i, z_i, $i = 1$ to 3 which form a Steiner trieder T_0, that is, which satisfy the following:

P.3.3 Given two triangles which are two rows of

$$\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{matrix}$$

the third row is the unique triangle such that each column is a triangle

then we have the following property:

P.3.4 for $i \neq j$, $i, j = 1$ to 3.

x_i and y_j are nonadjacent

y_i and z_j are nonadjacent

z_i and x_j are nonadjacent

PROOF:

Suppose x_1 adjacent to y_2, then $x_1x_2y_2$ and $x_1y_1y_2$ will be two triangles with no vertex in common with $z_1z_2z_3$. Then z_3 has to be adjacent to x_1, or to y_2, or to x_2 and y_1.

If z_3 is adjacent to x_1, then $z_1x_1z_3$ is a triangle and we have

$$\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & x_1 \\ x_3 & y_3 & z_3 \end{matrix}$$

where each row and each column is a triangle which contradicts P.3.3 since $x_1 \neq z_2$.

If z_3 is adjacent to y_2, then $y_2z_2z_3$ is a triangle and we have

$$\begin{matrix} x_1 & y_1 & y_2 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{matrix}$$

where each row and each column is a triangle which contradicts P.3.3 since $z_1 \neq y_2$.

If z_3 is adjacent to y_1 and x_2, then $x_2z_3z_2$ and $y_1z_3y_3$ are two triangles and we have

$$\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & z_3 & z_2 \\ x_3 & y_3 & z_3 \end{matrix}$$

where each row and each column is a triangle which contradicts P.3.3 since $y_2 \neq z_3$.

Thus x_1 is not adjacent to y_2 and we can reason similarly for any other pair and P.3.4 holds. This implies three more properties:

P.3.5 Any triangle xyz of G which is not contained in the trieder T_0 has at most one vertex in common with it.

P.3.6 Any triangle xyz of G is contained in a Steiner trieder.

P.3.7 For any pair xy of adjacent vertices of G, there is at most one vertex z adjacent to x and y.

If xyz is a triangle of G not contained in T_0, which has two vertices xy in common with T_0, say $x = x_1$, then $y_2z_2y_3z_3$ are distinct from y since they are not adjacent to x_1 Suppose $y = y_1$, then $z \neq z_1$ and $xyz = x_1y_1z$ has no vertex in common with $x_2y_2z_2$ and $x_3y_3z_3$ and P.2.3 implies that z_2 and z_3 nonadjacent to x_1 and y_1 are adjacent to z, that is zz_2z_3 is a triangle. We have

$$\begin{matrix} x_1 & y_1 & z \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{matrix}$$

Where each row and each column is a triangle which contradicts P.3.3 since $z \neq z_1$. Thus P.3.5 is satisfied.

It follows that any pair of adjacent vertices in T_0 determine a unique triangle. It also follows that for any triangle xyz of G there is a triangle of T_0 which has no vertex in common with it and by P.3 they must form with a third triangle a Steiner trieder. Thus P.3.6 and P.3.7 are satisfied.

Let us introduce now one more property:

P.3.8 Each vertex of the graph G is contained in at least one triangle.

A graph G with at least two triangles with no vertex in common, satisfying P.3.8 and P.3 also satisfies:

P.3.9 Each vertex of G is contained in a Steiner trieder.

THEOREM 1.

A finite, undirected linear graph G, containing at least two triangles with no vertex in common satisfies P.3.9 iff it is strongly regular. Its parameters are:

$$v = 3(2r-1), \quad n_1 = 2r, \quad p^1_{11} = 1, \quad p^2_{11} = r, \quad r \geq 2$$

PROOF:

Let xy be a pair of adjacent vertices of G, then P.3.7 and P.3.8 imply that there is a unique vertex z which is adjacent to x and y. Then $p^1_{11} = 1$. The valence n_x of each vertex x of G is even and x is contained in $\frac{n_x}{2}$ triangles. Let xy be a pair of nonadjacent vertices in G, and xx_1x_2, yy_1y_2 be two triangles. If they have one vertex in common, $x_1 = y_1$, and if x_2 is adjacent to y, or y_2 adjacent to x, then yx_1x_2 or xy_1y_2 will be a triangle which contradicts that $p^1_{11} = 1$. If they have no vertex in common they must form with a third triangle a Steiner trieder and P.3.4 implies, since x is not adjacent to y that one and only one of x_1,x_2 is adjacent to y, and one and only one of y_1,y_2 is adjacent to x. And this for any pair of triangles containing respectively x and y. But x is contained in $\frac{n_x}{2}$ triangles and y in $\frac{n_y}{2}$ triangles. That implies $n_x = n_y = n_{xy}$ and there are $\frac{n_{xy}}{2}$ vertices of G adjacent to xy.

Now let x be a vertex of valence $n_1 = 2r$ in G, r must be $\geq$ 2.

For any vertex y adjacent to x, there is a vertex z such that xyz is a triangle. Then, for every z_i adjacent to z, z_i is not adjacent to x nor to y and then $n_y = n_{z_i} = n_1 = 2r$. Thus any vertex adjacent or not to x has the same valence $n_1 = 2r$, and the graph G is regular. It also is strongly regular with $p_{11}^2 = \frac{n_1}{2} = r$. Then the number of vertices in G is $3(2r-1)$. The converse is obvious.

COROLLARY

A strongly regular graph G, with at least two triangles with no vertex in common satisfies P.3 iff its parameters are $v = 3(2r-1)$, $n_1 = 2r$, $p_{11}^1 = 1$, $p_{11}^2 = r$, $r \geq 2$.

Strongly regular graphs with such parameters have been proved by Seidel in [3] to exist only for $r = 1, 2, 3$, and 5.

I would like to express my thanks to I. M. Chakravarti who suggested to me the study of the property P.3. see [1].

REFERENCES

[1] Chakravarti, I. M., Some properties and applications of Hermitian varieties in a finite projective space $PG(N,q^2)$ in the construction of strongly regular graphs (two-class association schemes) and block designs. *Institute of Statistics Mimeo Series No. 600.23.* March 1970, U.N.C., Chapel Hill.

[2] Jordan, C., *Traité des Substitutions et des Equaticns Algébriques.* Gauthiers-Villars, Paris (1870).

[3] Seidel, J. J., Strongly regular graphs with (-1, 1, 0) adjacency matrix having eigenvalue 3. *Linear Algebra and Its Applications*, 1 (1968), 281-298.

[4] Steiner, J., Über eine besondere curve dritter Klasse (und vierten Grades), *Journal für die Reine und Angewandte Mathematik*, Berlin, 1857, Vol. 53, pp. 231-237.