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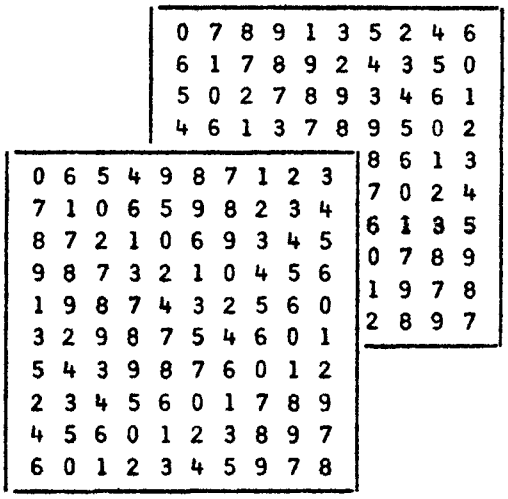

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ANTISYMMETRIC HADAMARD DIFFERENCE SETS

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1. INTRODUCTION.

In [1], we have studied binary relations (X, U) complete and antisymmetric, that is

$$\forall \quad i,j \in X: \quad (i,j) \in U \iff (j,i) \notin U$$

for which there exist a group of automorphisms verifying

1. For every pair of arcs, (i,j), $(i',j') \in U$, there exist a $g \in G$ with $(i)g = i$, $(j)g = j'$.
2. The identity alone fixes two points.
3. Let A be the set of elements of G moving all the points, there exists an $a \in A$ and two points i and j such that a is the unique element of A with $(i)a = j$.

We proved the following theorem.

The group of automorphisms of (X, U) verifies 1, 2, 3 iff X is the set of elements of a near-field K where there exists a subset $P \subset K$ verifying

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$$P \cdot P \subset P$$

$$P \cap (-P) = \{0\} \quad \text{and} \quad P \cup (-P) = K,$$

and where $\forall\ m \in P\backslash\{0,1\}$, the mapping defined by $x \mapsto x-xm$ is an automorphism of the additive group K, and U is the set of couples (α,β) with $\beta-\alpha \in P$. Moreover, in the case where X is finite, condition 3 may be dropped, the near-field has an order congruent to 3 *mod* 4 and P is the set of its squares.

In the finite case, such a group of automorphisms may be defined as a sharply two homogeneous group(*), and a short proof of that special case may be found in [2], where the theorem of Feit and Thompson is used.

We consider here a generalization of the finite case. That is, the graph (X,U) has an abelian group of automorphism G, with the elements of which its vertices are labeled and the set

$$\{(0,x) \mid (0,x) \in U\}$$

is a difference set.

That condition is necessarily fulfilled when 1 and 2 are verified but the question of knowing if it is sufficient has not been cleared out in this paper except when G is cyclic.

However, we prove that G is a p-group, and our argument leads to a proof of the fact that condition 1 added to the difference set condition implies G is elementary abelian. The case where G is cyclic is thoroughly cleared out.

(*) We are indebted to Professor Michel Jean of the College Royal de Quebec who brought that fact to our attention.

2. Antisymmetric Hadamard Difference Sets.

2.1. Definition.

Let G be an abelian group and D be a *difference set* [3], that is a subset of G such that the integer

$$\lambda_a = |\{(x,y)|x,y \in D,\ x-y = a\}| \tag{1}$$

does not depend on $a \in G\backslash\{0\}$.

The common value of the λ_a is denoted λ and D is a *Hadamard difference set* whenever $k = (v-1)/2$, where $k \equiv |D|$ and $v \equiv |G|$

Now D will be called *antisymmetric* if

$$\forall\ x \in G\backslash\{0\},\ x \in D \Rightarrow -x \notin D. \tag{2}$$

A difference set D verifying all those conditions will be referred to as a A.H.D.S.

2.2. Statement of the result.

Theorem 2.1. *An abelian group which contains an antisymmetric Hadamard difference set is a p-group with order congruent to three modulo four. When it is cyclic, the difference set is the set of the quadratic residues or the set of the quadratic non-residues of a prime.*

2.3. General properties of A.H.D.S.

2.3.1. Properties of v, k and λ.

$|D| = (v-1)/2$ and as the mapping $x \mapsto -x$ of G into itself is injective, $|-D| = (v-1)/2$.

But by (2)

$$-D \cap D = \emptyset \tag{3}$$

and since $-D \cup D \subset G \setminus \{0\}$, $|-D \cup D| = v-1$ implies

$$-D \cup D = G \setminus \{0\}. \tag{4}$$

On the other hand, the identity

$$k(k-1) = \lambda(v-1) \tag{5}$$

is verified for any difference set which implies in the present case

$$\lambda = (k-1)/2 = (v-3)/4. \tag{6}$$

Hence the order of G is congruent to 3 *mod* 4.

2.3.2. Properties of the incidence matrix of a A.H. configuration.

Let A be the incidence matrix of G versus the subsets $\{y\}+D$, $y \in G$, that is $A = (a_{x,y})$, $a_{x,y} = 1 \iff x \in y+D$. (3) and (4) are equivalent to

$$A + A^T + I = J, \tag{7}$$

and since D is a difference set,

$$AA^T = nI + \lambda J, \tag{8}$$

where, as usually, $n = k-\lambda$.

We now have

PROPERTY 2.1. *Let A be a v by v matrix with entries 0 and 1. The following conditions are pairwise equivalent.*

I. $AA^T = nI + \lambda J$

II. $A + A^T + I = J$

III. $A^2 + A = n(J-I)$.

I implies

$$JA = kJ, \tag{9}$$

and

$$A^{-1} = \frac{1}{n}(A^{T} - \frac{\lambda}{k}J). \tag{10}$$

III follows immediately from I and II and II follows from I, III, (9) and (10), hence

$$(I \ \& \ II) \iff (I \ \& \ III). \tag{11}$$

On the other hand, I follows from II and III by simple calculation.

PROPERTY 2.2. *For any difference set D, condition III on its incidence matrix is equivalent to*

$$\begin{cases} \forall\ z\in D, \ |\{x|x\in D,\ z-x\in D\}| = n-1. \\ \forall\ z\notin D, \ z\neq 0 |\{x|x\in D,\ z-x\in D\}| = n \\ -D\cap D = \emptyset. \end{cases} \tag{12}$$

Actually if $A = (a_{x,y})$ and $A^2 = (b_{x,y})$, III is equivalent to

$$\forall\ z\in G: \ \sum_{x\in G} a_{z,x}a_{x,0} + a_{z,0} = n(1-\delta_{z,0}) = b_{z,0}. \tag{13}$$

COROLLARY 2.1. *Condition (12) implies, for any difference set D, $-D\cup D = G\setminus\{0\}$.*

Remark. If D verifies conditions (12) it is a A.H.D.S. and $n-1 = k-\lambda-1 = \lambda$. Also $k-1-(n-1) = \lambda$ and

$$\forall\ z\in D, \ |\{x|x\in D,\ z-x\notin D\}| = \lambda. \tag{14}$$

2.3.3. Equations in the convolution algebra $\mathbb{Z}G$ of the abelian group G over the ring $\mathbb{Z}$ of integers.

The matrix $A = (a_{x,y})$ is a sum of permutation matrices, i.e.

$$(15) \qquad a_{x,y} = \sum_{g\in D} \delta_{x-g,y},$$

and the transpose $A^T = (a^T_{y,x})$ is given by

$$(16) \qquad a^T_{y,x} = \sum_{g\in D} \delta_{y,x-g} = \sum_{g\in D} \delta_{y+g,x} = \sum_{g\in -D} \delta_{y-g,x}.$$

Consequently, conditions I, II, III may be considered as identities in the $\mathbb{Z}$-algebra generated by the set $\{(\delta_{x-g,y})\}_{g\in G}$.

Now let us represent as in [3] by a polynomial

$$(17) \qquad \sum_{g\in G} a_g X^g$$

an element of $\mathbb{Z}G$.

In the canonical basis $\{X^g\}_{g\in G}$ of the $\mathbb{Z}$-module $\mathbb{Z}G$, the matrix of the linear mapping $S_{g'}: \sum_{g\in G} a_g X^g \mapsto \sum_{g\in G} a_g X^{g+g'}$ corresponding to the multiplication by $X^{g'}$ is precisely

$$(18) \qquad (\delta_{x-g',y}).$$

We thus have the isomorphisms

$$(19) \qquad \mathbb{Z}G \approx \mathbb{Z}[\{S_g\}_{g\in G}] \approx \mathbb{Z}[\{\delta_{x-g,y}\}].$$

Now let us use the following notations.

$$(20) \qquad D(X) = \sum_{g\in D} X^g, \quad D(X^{-1}) = \sum_{g\in -D} X^g, \quad T(X) = \sum_{g\in G} X^g.$$

Then the conditions I, II, III become by (19)

I'. $D(X)D(X^{-1}) = n + \lambda T(X)$

II'. $D(X) + D(X^{-1}) + 1 = T(X)$

III'. $D^2(X) + D(X) = n(T(X)-1)$.

Here X^0 is denoted by 1, and as usual, $n = k-\lambda$.

2.4. Proof of the main lemma leading to Theorem 2.1.

The group G may be considered as a module over $\mathbb{Z}$ so that we shall use multiplication of elements of G by integers.

LEMMA 2.4. *If $D \subset G$ is a A.H.D.S., then for every integer t with $(t,v) = 1$*

$$tD = \left(\frac{t}{v}\right)D,$$

where $(-)$ is the Jacobi symbol [4]

By the quadratic reciprocity law, since $v \equiv 3 \bmod 4$, one has for $q \neq 2$, q a prime

$$\left(\frac{v}{q}\right)\left(\frac{q}{v}\right) = (-1)^{\frac{q-1}{2}}, \tag{22}$$

hence

$$\left(\frac{q}{v}\right) = (-1)^{\frac{q-1}{2}}\left(\frac{v}{q}\right) \tag{23}$$

and if $q = 2$,

$$\left(\frac{2}{v}\right) = (-1)^{\frac{v^2-1}{8}}. \tag{24}$$

From (23), (24) and from the property

(25) $\left(\frac{t}{v}\right) = \prod_i \left(\frac{t_i}{v}\right)$, where $\prod_i t_i = t$, t_i a prime, $\forall$ i,

all we have to prove is

(26) $qD = (-1)^{\frac{q-1}{2}} \left(\frac{v}{q}\right) D$,

for q odd, and for q = 2

(27) $qD = (-1)^{\frac{v^2-1}{8}} D$.

For every prime q, we shall consider the group G* of characters of G into the group $[a] = F^*_{q^\nu}$, where F_{q^ν} is the splitting field of X^e-1 over F_q, e being the exponent of G. (i.e., ν is the smallest integer such that $q^\nu-1 \equiv 0$ *mod* e.)

We shall consider, as in [5], the isomorphism μ^{-1} of F_qG onto $F_{q^\nu}^{G^*}$. For brevity, we write D_χ for the image of D(X) under the character $\chi \in G^*$.

For every non principal character χ, one has

(28) $D_\chi^2 + D_\chi + n = 0, \quad \chi \in G^* \setminus \{0\}$.

So, for every non principal character χ, D_χ is one of the two roots of the first member of (28).

Let us first suppose that the polynomial Y^2+Y+n, (n is taken *mod* 9) does split over F_q, that is

(29) $1 - 4(\lambda+1) = -v$

is a quadratic residue *mod* 9, or in other words

(30) $(-1)^{\frac{q-1}{2}} \left(\frac{v}{q}\right) = 1$,

if q is odd, and if $q = 2$

(31) $\quad n = \lambda + 1 \equiv 0 \ mod\ 2, \quad or \quad v \equiv -1 \ mod\ 8.$

In that case, since $D_{\chi_0} = k \epsilon F_q$, every character sends $D(X)$ in F_q and we know that this is possible if and only if (see for example [5] for a proof)

(32) $\quad D(X) = D(X^q),$

since μ^{-1} is an isomorphism, which means that

(33) $\quad qD = D.$

Now, on the contrary, if for $q \neq 2$, $(-1)^{(q-1/2)}(\frac{v}{q}) = -1$, and, for $q = 2$, when $(-1)^{(v^2-1/8)} = 1$, Y^2+Y+n is irreducible over F_q.

In that case, Y^2+Y+n divides $X^{q^\nu-1}-1$, ν is even, and the Galois group of F_{q^ν} over F_{q^2} is given by all the even powers of q modulo $q^\nu-1$; every odd power of q defines an F_q automorphism of F_{q^ν} which permutes the roots of $Y^2+Y+n = (Y-b)(Y-c)$, and, in particular,

(34) $\quad b^q = c.$

But since $b+c = -1$, one has for every non principal character χ,

(35) $\quad D_\chi^q = -1 - D_\chi$

or, by II',

(36) $\quad D_\chi^q = (-D)_\chi.$

But also $D_{\chi_0} = (-D)_{\chi_0}$ for the principal character χ_0, hence, since the Fourier transform μ^{-1} is an isomorphism onto $F_{q^\nu}^{G^*}$,

(37) $\quad D(X^q) = D(X^{-1})$ C.Q.F.D.

2.5. Consequences of Lemma 2.4. for the group G.

PROPERTY 2.5.1. *If G' is a subset of G, symmetric with respect to the origin, one has for D' = D∩G'*

$$\begin{aligned} -D' \cup D' &= G'\backslash\{0\} \\ -D' \cap D' &= \emptyset. \end{aligned} \tag{38}$$

$$\begin{aligned} -D' \cup D' &= -(D\cap G') \cup (D\cap G') = (-D\cap G') \cup (D\cap G') \\ &= (-D\cup D) \cap G' = G'\backslash\{0\}. \\ -D' \cap D' &= (-D\cap G') \cap (D\cap G') = -D \cap D \cap G' = \emptyset. \end{aligned}$$

COROLLARY 2.5.1. *In the hypothesis of Lemma 2.4., v cannot have any prime factor congruent to one modulo four.*

We shall show that if v has a factor r congruent to one modulo four, there exists a non trivial subgroup G' of G such that D∩G' is empty, which is impossible for a A.H.D.S. D.

We split G into the canonical direct sum

$$G = G_1 \oplus \ldots \oplus G_\delta \tag{39}$$

where the order of G_1 is the exponent e of G. So r divides e, and a subgroup G" of G isomorphic with G_1 contains a subgroup G' of order r. Now, if (t,v) = 1, tG" = G" and

$$tG' = G'. \tag{40}$$

We write $v = v_1 r^c$, $(v_1, r) = 1$, and we choose, as it is possible, an integer t verifying

$$t \equiv 1 \ mod\ v_1, \quad t \equiv -1 \ mod\ r, \quad \text{and} \quad (t,v) = 1. \tag{41}$$

Now, let $D' = D \cap G'$. We have

$$tD' = -D', \tag{42}$$

by (41).

But also

$$tD' = t(D \cap G') = tD \cap G' \tag{43}$$

and by Lemma 2.4,

$$tD' = \left(\frac{t}{v}\right)D \cap G' = \left(\frac{t}{v}\right)(D \cap G') = \left(\frac{t}{v}\right)D'. \tag{44}$$

Now, let us compute $\left(\frac{t}{v}\right)$.

Since $t \equiv 1 \ mod \ v_1$, $\left(\frac{t}{v_1}\right) = 1$, and

$$\left(\frac{t}{v}\right) = \left(\frac{t}{v_1}\right)\left(\frac{t}{r^e}\right) = \left(\frac{t}{r^e}\right) = \left(\frac{t}{r}\right)^e = 1, \tag{45}$$

since -1 is a quadratic residue modulo r. But (42), (44) and (45) imply

$$-D' = D', \tag{46}$$

which is impossible by (38).

COROLLARY 2.5.2. *If G contains a A.H.D.S. D, let $p \equiv 3 \ mod \ 4$ be a prime factor of v and $q > v$ a prime number. Then*

$$\left(\frac{v}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{q-1}{2}}. \tag{42}$$

By (26), we have to prove that

$$\left(\frac{q}{p}\right)D = qD. \tag{48}$$

As in the lines following (39), we may find a subgroup G' of order p, contained in a cyclic subgroup G'' of order e, direct factor of G. Then

$$(49) \qquad qD' = q(D \cap G') = qD \cap G'.$$

Now, suppose (48) is not true.

Case 1: $(\frac{q}{p}) = 1$.

Then

$$(50) \qquad qD' = -D \cap G' = -(D \cap G') = -D'.$$

But q is a quadratic residue modulo p, $p \equiv 3 \ mod\ 4$, by assumption, and (50) implies that it generates in $\mathbb{Z}/(p)$ of multiplicative group of even order. This is impossible.

Case 2: $(\frac{q}{p}) = -1$.

Then

$$(51) \qquad qD' = D \cap G' = D'.$$

But this is impossible, since

$$(52) \qquad (\frac{q}{p}) = -1 \iff q^{\frac{p-1}{2}} \equiv -1 \ mod\ p$$

and the last identity means that

$$(53) \qquad q^{\frac{p-1}{2}} D' = -D'.$$

COROLLARY 2.5.3. *If G contains a A.H.D.S. D, v cannot contain two distinct prime factors congruent to three modulo four.*

Let $p \equiv r \equiv 3 \quad 4$ and pr/v. By Corollary 2, one has for every prime $q > v$,

$$(54) \qquad \left(\frac{q}{p}\right) = \left(\frac{q}{r}\right) = (-1)^{\frac{q-1}{2}} \left(\frac{v}{q}\right).$$

Now, by the quadratic reciprocity law,

$$(55) \qquad 1 = \left(\frac{q}{p}\right)\left(\frac{q}{r}\right) = \left(\frac{p}{q}\right)\left(\frac{r}{q}\right) = \left(\frac{pr}{q}\right).$$

Hence pr is a quadratic residue modulo q, for every prime q greater than v. This implies that pr is a square. (A proof of that fact may be found in [4].)

2.6. Proof of Theorem 2.1.

2.6.1. General discussion.

Let $G = G_1 \oplus \ldots \oplus G_r$ where G_i has order $\ell_i \geq 1$.

Let L Be the subgroup of order p^r of G whose elements are of the type

$$(56) \qquad g = (g_1, \ldots, g_r), \quad g_i = 0 \text{ or } g_i = \alpha p^{\ell-1}, \quad 1 \leq \alpha \leq p-1.$$

Then the quotient group G/L has order $p^{\Sigma \ell_i - r}$. We may write an element of G

$$(57) \qquad \left(\sum_{1 \leq j \leq \ell_i} s_j^i \, p^{\ell_i - j} \right)_{1 \leq i \leq r},$$

so that a class in G/L is a set

$$(58) \qquad C_s = \left\{ \left(\alpha_i p^{\ell_i - 1} + \sum_{2 \leq j \leq \ell_i} s_j^i \, p^{\ell_i - j} \right)_{1 \leq i \leq r} \right\}_{\alpha_i \in F_p}$$

$$= \left\{ \left(\alpha_i p^{\ell_i - 1} + s_i \right)_{1 \leq i \leq r} \right\}_{\alpha_i \in F_p}.$$

Now, a set $B_s = C_s \cup C_{-s}$ is symmetric with respect to the origin and by Property 2.5.1, $|D \cap B_s| = |B_s|/2 = p^r$ for $s \neq 0$. Let

$$(59) \qquad |D \cap C_s| = p^r - x_s, \quad |D \cap C_{-s}| = x_s.$$

For counting the number of differences giving elements of $C_0 \setminus \{0\}$, we may count all the differences in the sets B_s and in C_0 and subtract

$$(60) \qquad 2 \cdot \sum_{1 \le s \le p^{\ell-r}} (p^r - x_s)\, x_s,$$

with $\sum_i \ell_i = \ell$.

We obtain

$$(61) \qquad p^r(p^r-1)\,\frac{p^{\ell-r}-1}{2} + \frac{p^r-1}{2}\,\frac{p^r-3}{2} - 2 \sum_{1 \le s \le p^{\ell-r}} (p^r - x_s)\, x_s$$

and this must equal

$$(62) \qquad (p^r-1)\lambda = \frac{(p^r-1)(p^\ell-3)}{4}.$$

The sum of the first two terms of (61) minus (62) gives

$$(63) \qquad \frac{p^r-1}{2} \cdot \frac{p^\ell - p^r}{2} = \frac{p^r-1}{2} \cdot \frac{p^{\ell-r}-1}{2}\, p^r.$$

This shows that the x_s cannot be all equal if G is not elementary abelian, since in this case we would have $x_s = p^r$, and $p^\ell - p^r = 0$, $\ell = r$, $\ell_i = 1$, $i = 1, \ldots, r$.

We arrive to the conclusion that a condition to be fullfilled by the order of G and the number r of its direct factors is, by (61), (62) and (63) that there exist integers x_s, $s = 1, \ldots p^{\ell-2}$, verifying

$$(64) \qquad x_s \le p^r \qquad \sum_{1 \le s \le p^{\ell-r}} x_s^2 \equiv 0 \ mod\ p^r.$$

One sees here the reason why if there exists a group of automorphism of G in which the stabilisor of $\{0\}$ leaves D invariant and operates transitively on it, then G is elementary abelian. Indeed, in that case, all the intersections of C_s and D must have the same cardinality.

2.6.2. The cyclic case.

In the case $r = 1$, C_s must be entirely contained in D or $-D$.

First, we have

$$tC_s = \left(\frac{t}{p}\right)^{\ell} C_s = \left(\frac{t}{p}\right) C_s, \tag{65}$$

by Lemma 2.4.

Now let $\alpha p^{\ell-1}+s$ and $\alpha' p^{\ell-1}+s$ be any two distinct elements of C_s. We write $s = s'p^j$, with $(s',p) = 1$. Let

$$x \equiv (\alpha'-\alpha)/s' \ mod\ p.$$

Multiplying $\alpha p^{\ell-1}+s$ by $xp^{\ell-j-1}+1$, we obtain, since $j < \ell-1$,

$$s'xp^{\ell-1} + \alpha p^{\ell-1} + s = \alpha' p^{\ell-1} + s. \tag{66}$$

Since $xp^{\ell-j-1}+1$ is a quadratic residue mod p, the thesis is proved and it follows from the preceding discussion that $\ell = 1$.

As a last remark, we observe that the present argument proves that $x_s \equiv 0 \ mod\ p$.

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