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THE ASYMPTOTIC DISTRIBUTIONS OF THE STATISTICS BASED ON THE COMPLEX GAUSSIAN DISTRIBUTION.

by

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ABSTRACT

This paper considers the asymptotic distributions of the statistics based on the complex Gaussian distribution. To treat this problem, we define the hermitian differential operator matrix and give the fundamental formulas of the zonal polynomials.

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A LIST OF KEY WORDS AND PHRASE.

(i) a zonal polynomial.

$\tilde{C}_\kappa(S)$. κ: kappa

(ii) hermitian matrix, hermitian differential operator matrix.

(iii) unitary group. $\overline{U}(n)$, $\overline{U}(m_2)$, etc.

(iv) $\delta_{\alpha\beta}$: Kronecker's delta.

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1. Introduction and Notations.

Recently the asymptotic distribution of the statistics based on the multivariate normal samples were derived by the use of the fundamental formulas of the series of the zonal polynomials [1], [2], [7]. The purpose of this paper is to give the asymptotic distributions of the statistics based on the complex multivariate Gaussian distribution which was developed by Goodman, N.R. [3], James, A.T. [4] and Khatri, C.G. [5], [6]. To obtain these distributions, we need also the fundamental formulas of the series of the zonal polynomials of the positive definite hermitian matrix. If we do not notice in this paper, we assume that all the matrices are $m \times m$ positive definite hermitian matrices.

Let S be a positive definite hermitian matrix whose characteristic roots are $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1 > \cdots > \lambda_m > 0$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ be a diagonal matrix whose diagonal elements are $\lambda_1, \ldots, \lambda_m$ in a descending order. Let $\tilde{C}_\kappa(S)$ be a zonal polymonial of S, which corresponds to the partition κ of k into not more than m parts. It can be represented

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by

$$\tilde{C}_\kappa(S) = \chi_{[\kappa]}(1)\chi_{\{\kappa\}}(S) ,$$

where $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group and $\chi_{\{\kappa\}}(S)$ is the character of the representation $\{\kappa\}$ of the general linear group [4].

Let

$${}_p\tilde{F}_q^{(m)}(a_1,\ldots,a_p, b_1,\ldots,b_q; S,T) = \sum_{k=0}^{\infty}\sum_{\kappa} \frac{[a_1]_\kappa\cdots[a_p]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa} \frac{\tilde{C}_\kappa(S)\tilde{C}_\kappa(T)}{k!\tilde{C}_\kappa(I_m)},$$

where

$$[a]_\kappa = \prod_{\alpha=1}^{m} (a-\alpha+1)_{k_\alpha}, \quad (a)_x = a(a+1)\cdots(a+x-1),$$

$$k = k_1 + \cdots + k_m, \quad k_1 \geq \cdots \geq k_m \geq 0.$$

We denote as ${}_p\tilde{F}_q(\ldots, \ldots, S) = {}_pF_q^{(m)}(\ldots, \ldots, S, I)$ if $T=I_m$.

Let S and R be a positive definite hermitian matrices and T be also a hermitian matrix, then

$$(1.1) \qquad \int_{\bar{R}'=R>0} \operatorname{etr}(-RS)(\det R)^{a-m}\tilde{C}_\kappa(RT)\,dR = \tilde{\Gamma}_m(a,\kappa)(\det S)^{-a}\tilde{C}_\kappa(TS^{-1}),$$

where $\tilde{\Gamma}_m(a,\kappa) = \pi^{\frac{1}{2}m(m-1)} \Pi_{\alpha=1}^m \Gamma(a+k_\alpha-(\alpha-1))$, and

$$(1.2) \qquad \int_{I>\bar{R}'=R>0} (\det R)^{a-m}\det(I-R)^{b-m}\tilde{C}_\kappa(RS)\,dR = \frac{\tilde{\Gamma}_m(a,\kappa)\tilde{\Gamma}_m(b)}{\tilde{\Gamma}_m(a+b,\kappa)} \tilde{C}_\kappa(S),$$

where $\tilde{\Gamma}_m(a) = \pi^{\frac{1}{2}m(m-1)} \Pi_{\alpha=1}^m \Gamma(a-(\alpha-1)) = \tilde{\Gamma}_m(a,\kappa)/[a]_\kappa$.

Let X be an n×n arbitrary complex matrix and U be a unitary matrix on the unitary group U(n) of order n, then

$$(1.3) \qquad \int_{U(n)} \operatorname{etr}(XU+\bar{U}'\bar{X}')\,d(U) = {}_0\tilde{F}_1(n,\ X\bar{X}'),$$

where d(U) is the unitary invariant measure of the unitary group with total volume unity.

We use the following notations. Let X be an m×n (m≤n) complex matrix which has a complex Gaussian distribtuion with mean $M_{m\times n}$ and covariance matrix Σ, then we denote as $X \sim CN_m(M,\Sigma)$. Let S be an hermitian matrix which has a complex Wishart distribution of n degrees of freedom with a non-central matrix Ω, then we denote as $S \sim CW(\Sigma,\ n,\ \Omega)$.

2. The fundamental formulas of the sum of the zonal polynomials.

In this section, we consider only m×m positive definite hermitian matrices. Let Σ be hermitian matrix and

$$\Sigma = \Sigma^R + i\Sigma^I, \quad \Sigma^R = (\sigma^R_{\alpha\beta}), \quad \Sigma^I = (\sigma^I_{\alpha\beta}), \quad \alpha,\beta = 1,2,\ldots,m,$$

where $\Sigma^{R'} = \Sigma^R$ and $\Sigma^{I'} = -\Sigma^I$. We here define the hermitian differential operator matrix ∂ as follows.

$$(2.1) \qquad \partial = \partial_R + i\partial_I, \quad \partial = (\partial_{\alpha\beta}), \quad \partial_R = (\partial^R_{\alpha\beta}), \quad \partial_I = (\partial^I_{\alpha\beta}),$$

and

$$\partial^R_{\alpha\beta} = \frac{1+\delta_{\alpha\beta}}{2}\frac{\partial}{\partial\sigma^R_{\alpha\beta}} \quad \text{and} \quad \partial^I_{\alpha\beta} = \frac{1-\delta_{\alpha\beta}}{2}\frac{\partial}{\partial\sigma^I_{\alpha\beta}} .$$

From the symmetry of Σ^R and the skew symmetry of Σ^I, we can see $\partial^R_{\alpha\beta} = \partial^R_{\beta\alpha}$ and $\partial^I_{\alpha\beta} = -\partial^I_{\beta\alpha}$. Hence ∂_R and ∂_I are a symmetric and a skew symmetric differential operator matrices, respectively.

Let $f(\Sigma)$ be a real valued function of an hermitian matrix Σ, and it belongs to C^∞, then we have a Taylor series expansion of $f(\Sigma)$ in the neighborhood at $\Sigma = \Sigma_0$ as follows.

$$(2.2) \qquad f(\Sigma) = \operatorname{etr}((\Sigma-\Sigma_0)\partial)f(\Sigma)\big|_{\Sigma=\Sigma_0} .$$

We can show easily that (2.2) is same as (2.3) if S is an hermitian matrix.

$$(2.3) \qquad f(S) = \operatorname{etr}((S-\Sigma_0)\partial)f(\Sigma)\big|_{\Sigma=\Sigma_0} .$$

The following lemmas are fundamental.

<u>Lemma 1</u>. Let κ be a partition of k into not more than m parts, i.e., $\kappa = (k_1,\ldots,k_m)$, $k = k_1+\cdots+k_m$, $k_1 \geq \ldots \geq k_m \geq 0$ and let $\tilde{a}_1(\kappa) = \Sigma^m_{\alpha=1} k_\alpha(k_\alpha-2\alpha)$ and $\tilde{a}_2(\kappa) = 2\Sigma^m_{\alpha=1} k_\alpha(k_\alpha^2-3\alpha k_\alpha^2+\alpha^2)$, then

$$(2.4) \qquad (\tilde{a}_1(\kappa)+k)\tilde{C}_\kappa(\Sigma) = \operatorname{tr}(\Lambda\partial)^2\tilde{C}_\kappa(\Sigma)\big|_{\Sigma=\Lambda}$$

$$(2.5) \qquad \{3\tilde{a}_1^2(\kappa) - 2\tilde{a}_2(\kappa) + 6k\tilde{a}_1(\kappa) - 6\tilde{a}_1(\kappa) + 3k^2 - 2k\}\tilde{C}_\kappa(\Sigma)$$
$$= [8(\operatorname{tr}(\Lambda\partial)^3) + 3(\operatorname{tr}(\Lambda\partial)^2)^2]\tilde{C}_\kappa(\Sigma)\big|_{\Sigma=\Lambda} ,$$

where $\Lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_m)$ is a diagonal matrix of latent roots of Σ.

<u>Proof</u>. From (1.1), we have

$$(2.6) \qquad \frac{n^{mn}}{\tilde{\Gamma}_m(n)(\det\Sigma)^n} \int_{\bar{R}'=R>0} \operatorname{etr}(-n\Sigma^{-1}R)(\det R)^{n-m}\tilde{C}_\kappa(R)\,dR = [n]_\kappa(\tfrac{1}{n})^k\tilde{C}_\kappa(\Sigma).$$

We can see easily that the L.H.S. of (2.6) is invariant under the transformation $R = U W \bar{U}'$ such that $\Sigma = U \Lambda \bar{U}'$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ is a diagonal latent roots matrix and $U \in U(m)$. Hence we can rewrite L.H.S. of (2.6) as (2.7),

$$(2.7) \qquad \frac{n^{mn}}{\tilde{\Gamma}_m(n)(\det\Lambda)^n} \int_{\bar{W}'=W>0} \mathrm{etr}(-n\Lambda^{-1}W)(\det W)^{n-m}\tilde{C}_\kappa(W)\,dW.$$

Here we expand $\tilde{C}_\kappa(W)$ into a Taylor series expansion in the neighborhood at $W = \Lambda$ by the use of (2.3). Then we have a following asymptotic expansion

$$(2.8) \qquad \frac{n^{mn}}{\tilde{\Gamma}_m(n)(\det\Lambda)^n} \int_{\bar{W}'=W>0} \mathrm{etr}(-n\Lambda^{-1}W)(\det W)^{n-m}\mathrm{etr}((W-\Lambda)\partial)\,dW\tilde{C}_\kappa(\Sigma)\big|_{\Sigma=\Lambda}$$

$$= \mathrm{etr}(-\Lambda\partial)\det(I - \frac{1}{n}\Lambda\partial)^{-n}\tilde{C}_\kappa(\Sigma)\big|_{\Sigma=\Lambda}$$

$$= \{1 + \frac{1}{2n}\mathrm{tr}(\Lambda\partial)^2 + \frac{1}{24n^2}[8\mathrm{tr}(\Lambda\partial)^3 + 3(\mathrm{tr}(\Lambda\partial)^2)^2]$$

$$+ 0(1/n^3)\}\ \tilde{C}_\kappa(\Sigma)\big|_{\Sigma=\Lambda}.$$

On the other hand, R.H.S. of (2.6) also have an asymptotic expansion such that

$$(2.9) \qquad \{1 + \frac{1}{2n}(\tilde{a}_1(\kappa)+k) + \frac{1}{24n^2}\{3\tilde{a}_1^2(\kappa)-2\tilde{a}_2(\kappa)+6k\tilde{a}_1(\kappa)-6\tilde{a}_1(\kappa)$$

$$+ 3k^2-2k\} + 0(1/n^3)\}\ \tilde{C}_\kappa(\Sigma).$$

Hence by comparing with the both side of order $1/n$ and $1/n^2$, we have Lemma 1.

Lemma 2.

$$\sum_{k=r}^{\infty} \sum_{\kappa} \frac{x^k \tilde{C}_\kappa(\Sigma)}{(k-r)!} = x^r \cdot (\mathrm{tr}\Sigma)^r \mathrm{etr}(x\Sigma) \tag{2.10}$$

(2.10) holds for all integers.

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_1(\kappa) \tilde{C}_\kappa(\Sigma)}{k!} = (x^2 \mathrm{tr}\Sigma^2 - x\mathrm{tr}\Sigma)\mathrm{etr}(x\Sigma). \tag{2.11}$$

$$\sum_{k=r}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_1(\kappa) \tilde{C}_\kappa(\Sigma)}{(k-r)!} = \{x^{r+2} \mathrm{tr}\Sigma^2 (\mathrm{tr}\Sigma)^r - x^{r+1} (\mathrm{tr}\Sigma)^{r+1} + 2rx^{r+1} \mathrm{tr}\Sigma^2 (\mathrm{tr}\Sigma)^{r-1} - rx^r (\mathrm{tr}\Sigma)^r + r(r-1)x^r \mathrm{tr}\Sigma^2 (\mathrm{tr}\Sigma)^{r-2}\} \mathrm{etr}(x\Sigma). \tag{2.12}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_1^2(\kappa) \tilde{C}_\kappa(\Sigma)}{k!} = \{x^4 (\mathrm{tr}\Sigma^2)^2 + 4x^3 \mathrm{tr}\Sigma^3 - 2x^3 \mathrm{tr}\Sigma \mathrm{tr}\Sigma^2 + 3x^2 (\mathrm{tr}\Sigma)^2 - 4x\mathrm{tr}\Sigma + x\mathrm{tr}\Sigma\} \mathrm{etr}(x\Sigma). \tag{2.13}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_2(\kappa) \tilde{C}_\kappa(\Sigma)}{k!} = \{2x^3 \mathrm{tr}\Sigma^3 + 3x^2 (\mathrm{tr}\Sigma)^2 - 3x^2 \mathrm{tr}\Sigma^2 + 2x\mathrm{tr}\Sigma\} \mathrm{etr}(x\Sigma). \tag{2.14}$$

Proof. Since $\mathrm{etr}(x\Sigma) = \Sigma_{k=0} \Sigma_\kappa (x^k \tilde{C}_\kappa(\Sigma)/k!)$, we have (2.10) by differentiation or integration on both sides, successively. From Lemma 1, we know

$$\tilde{a}_1(\kappa) \tilde{C}_\kappa(\Sigma) = \mathrm{tr}(\Lambda\partial)^2 \tilde{C}_\kappa(\Sigma)\big|_{\Sigma=\Lambda} - k\tilde{C}_\kappa(\Sigma) .$$

Multiply $x^k/k!$ on both sides and sum form $k=0$ to infinite, we have

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_1(\kappa)\tilde{C}_\kappa(\Sigma)}{k!} = \mathrm{tr}(\Lambda\partial)^2 \mathrm{etr}(x\Sigma)\big|_{\Sigma=\Lambda} - x\,\mathrm{tr}\Sigma\,\mathrm{etr}(x\Sigma).$$

From the definition of ∂, the first term of R.H.S. becomes

$$\begin{aligned} \mathrm{tr}(\Lambda\partial)^2 \mathrm{etr}(x\Sigma)\big|_{\Sigma=\Lambda} &= \sum_{\alpha,\beta=1}^{m} \lambda_\alpha \lambda_\beta \partial_{\alpha\beta} \bar{\partial}_{\alpha\beta} \exp(x \sum_{\alpha=1}^{m} \sigma_{\alpha\alpha})\big|_{\Sigma=\Lambda} \\ &= \left\{ \sum_{\alpha=1}^{m} \lambda_\alpha^2 \frac{\partial^2}{\partial \sigma_{\alpha\alpha}^2} + \frac{1}{2} \sum_{\alpha<\beta} \left(\frac{\partial^2}{\partial \sigma_{\alpha\beta}^{R^2}} + \frac{\partial^2}{\partial \sigma_{\alpha\beta}^{I^2}} \right) \right\} \mathrm{etr}(x\Sigma)\big|_{\Sigma=\Lambda} \\ &= x^2 \mathrm{tr}\Lambda^2 \mathrm{etr}(x\Lambda) . \end{aligned}$$

Hence we obtain (2.11). (2.12) can be obtained by applying the Leibnitz formula of differentiation to (2.11). As we can show (2.13) and (2.14) by the same way as one of [7], we will omit.

3. The asymptotic distribution of the statistics based on the non-central complex Wishart matrix.

Recently Fujikoshi [1], [2] has obtained the asymptotic distributions of a generalized variance and a trace of a non-central Wishart Matrix. In this section, we give the asymptotic distribution of these statistics based on a complex non-central Wishart matrix by the completely same way as [1] and [2].

Theorem 1. Let nS be distributed with $CW(\Sigma,n,\Omega)$ and let's assume that Ω is a constant matrix with respect to n. Put

(3.1) $$\lambda = \sqrt{n/m}\,\log\{\det S/\det\Sigma\}.$$

Then we have

(3.2) $$\Pr\{\lambda\leq x\} = \Phi(x) + \frac{G_1}{\sqrt{mn}} + \frac{G_2}{mn} + \frac{G_3}{mn\sqrt{mn}} + O(1/n^2).$$

where

$$G_1 = -\Phi'(x) - \frac{m^2}{2}\Phi^{(2)}(x) + \frac{1}{6}\Phi^{(3)}(x)$$

$$G_2 = \frac{\Phi^{(2)}(x)}{2}\{\frac{m^2(m^2+2)}{4} + (\mathrm{tr}\Omega)^2\} + \frac{\Phi^{(3)}(x)}{2}m^2\mathrm{tr}\Omega$$
$$+ \frac{\Phi^{(4)}(x)}{12}(m^2+1-2\mathrm{tr}\Omega) + \frac{1}{72}\Phi^{(6)}(x)$$

$$G_3 = \frac{\Phi'(x)}{12}\{m^2(2m^2-1) + 6m\mathrm{tr}\Omega^2\}$$
$$+ \frac{\Phi^{(3)}(x)}{48}\{m^2(m^2+2)(m^2+4) - 6m^2(m^2+2)\mathrm{tr}\Omega - 8(\mathrm{tr}\Omega)^3\}$$
$$+ \frac{\Phi^{(5)}(x)}{240}\{5m^4 + 20m^2 + 12 - 10(m^2+1)\mathrm{tr}\Omega + 10(\mathrm{tr}\Omega)^2\}$$
$$+ \frac{\Phi^{(7)}(x)}{144}(m^2 + 2 - 2\mathrm{tr}\Omega) + \frac{\Phi^{(9)}(x)}{1296},$$

where $\Phi^{(k)}(x)$ denotes the k-th derivative of the standard normal distribution function $\Phi(x)$.

Proof. We can easily obtain the characteristic function of λ as follows:

(3.3) $$\mathrm{etr}(-\Omega)\left(\frac{1}{n}\right)^{it\sqrt{mn}}\frac{\tilde{\Gamma}_m(n+it\sqrt{n/m})}{\tilde{\Gamma}_m(n)}\;{}_1\tilde{F}_1(n+it\sqrt{n/m},\, n,\, \Omega).$$

Hence by expanding (3.3) as the series of order $1/\sqrt{n}$ and by applying Lemma

2, we have the asymptotic expansion of $\phi(t)$. Therefore, by inverting this series, we obtain the result (3.2).

Theorem 2. Let nS be distributed with $CW(\Sigma,n,\Omega)$.

Case 1. Ω is a constant matrix with respect to n.

Put $\hat{\lambda} = (\sqrt{n}/\tau)(\mathrm{tr}S-\mathrm{tr}\Sigma)$, where $\tau^2 = \mathrm{tr}\Sigma^2$, then

$$\Pr\{\hat{\lambda}\le x\} = \Phi(x) - \frac{1}{\sqrt{n}} T_1 + \frac{1}{n} T_2 - \frac{1}{n\sqrt{n}} T_3 + O(1/n^2) ,$$

where

$$T_1 = \Phi'(x) \cdot \frac{\mathrm{tr}\Sigma}{\tau} + \frac{\mathrm{tr}\Sigma^3}{3\tau^2} \Phi^{(3)}(x)$$

$$T_2 = \frac{\Phi^{(2)}(x)}{\tau^2} (\mathrm{tr}\Sigma^2\Omega + \tfrac{1}{2}(\mathrm{tr}\Omega)^2)$$

$$+ \frac{\Phi^{(4)}(x)}{12\tau^2} \{3\mathrm{tr}\Sigma^4 + 3\mathrm{tr}\Sigma\mathrm{tr}\Sigma^4\} + \frac{\Phi^{(6)}(x)}{18\tau^6} (\mathrm{tr}\Sigma^3)^2 .$$

$$T_3 = \frac{\Phi^{(3)}(x)}{\tau^3} \{\mathrm{tr}\Sigma^3\Omega + \mathrm{tr}\Sigma\mathrm{tr}\Sigma^2\Omega + \frac{1}{6} (\mathrm{tr}\Sigma)^3\}$$

$$+ \frac{\Phi^{(5)}(x)}{\tau^5} \{\frac{\mathrm{tr}\Sigma^5}{5} + \frac{\mathrm{tr}\Sigma\mathrm{tr}\Sigma^4}{4} + \frac{(\mathrm{tr}\Sigma)^2\mathrm{tr}\Sigma^2}{6}\}$$

$$+ \frac{\Phi^{(7)}(x)}{10\tau^7} \{\frac{1}{12} \mathrm{tr}\Sigma^4\mathrm{tr}\Sigma^3 + \frac{1}{18} \mathrm{tr}\Sigma(\mathrm{tr}\Sigma^3)^2\}$$

$$+ \frac{\Phi^{(9)}(x)}{162\tau^9} (\mathrm{tr}\Sigma^3)^3 .$$

Case 2. $\Omega = n\theta$, where θ is a constant matrix.

Put

$$\tilde{\lambda} = \frac{\sqrt{n}}{\sigma} (\mathrm{tr}\Sigma - \mathrm{tr}(I+2\theta)\Sigma), \qquad \sigma^2 = \mathrm{tr}(I+2\theta)\Sigma^2.$$

Then we have

$$(3.4) \qquad \Pr\{\tilde{\lambda}\le x\} = \Phi(x) - \frac{M_1}{\sqrt{n}} + \frac{M_2}{n} - \frac{M_3}{n\sqrt{n}} + O(1/n^2),$$

where

$$M_1 = \frac{\Phi^{(3)}(x)}{3\sigma^3}\,\mathrm{tr}(I+3\theta)\Sigma^3$$

$$M_2 = \frac{\Phi^{(4)}(x)}{4\sigma^4}\,\mathrm{tr}(I+5\theta)\Sigma^4 + \frac{\Phi^{(6)}(x)}{18\sigma^6}\,\{\mathrm{tr}(I+3\theta)\Sigma^3\}^2$$

$$M_3 = \frac{\Phi^{(5)}(x)}{5\sigma^5}\,\mathrm{tr}(I+5\theta)\Sigma^5 + \frac{\Phi^{(7)}(x)}{12\sigma^7}\,\mathrm{tr}(I+3\theta)\Sigma^3\mathrm{tr}(I+4\theta)\Sigma^4 + \frac{\Phi^{(9)}(x)}{54\sigma^9}\,(\mathrm{tr}(I+3\theta)\Sigma^3)^3 .$$

Proof. Case 1. As the characteristic function of $\hat{\lambda}$ is given by

$$\mathrm{etr}(-it\frac{\sqrt{n}}{\tau}\Sigma)\mathrm{etr}(-\Omega)\det(I - \frac{it}{\sqrt{n}\tau}\Sigma)^{-n}\,\mathrm{etr}\{\Omega(I - \frac{it}{\sqrt{n}\tau}\Sigma)^{-1}\} ,$$

we expand this as the series of order $1/\sqrt{n}$ by using the formulas such that

$$\det(I-cA)^{-d} = \mathrm{etr}(cdA)\{1 + \frac{c^2 d}{2}\,\mathrm{tr}A^2 + \ldots\} ,$$

$$(I-X)^{-1} = I + X + X^2 + \cdots .$$

Hence by inverting this series, we obtain (3.3).

Case 2. This is obtained by the same way as Case 1.

4. The likelihood ratio criterion in a linear model.

In the linear hypothesis we have a following canonical model. Let the each column vectors of $Y = [\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_N]_{m\times n}$ be independently distributed with the complex normal distribution with the same covariance matrix Σ. The hypothesis H and the alternative K are specified by

$$(4.1)\quad \begin{array}{lll} H: & E(\underline{y}_\alpha) = \underline{0}, & \alpha = 1,2,\ldots,q_1 \text{ and } q_2+1,\ldots,N. \quad (q_1 \le q_2). \\ K: & E(\underline{y}_\alpha) \ne \underline{0}, & \text{for some } \alpha \ (1 \le \alpha \le q_1) \text{ and} \\ & E(\underline{y}_\alpha) = 0 & \text{for } \alpha = q_2+1,\ldots,N. \end{array}$$

The likelihood ratio test for this model is expressed by

$$(4.2)\qquad \Lambda = \{\det A/\det(A+B)\}^N,$$

where $A = \Sigma_{\alpha=q_2+1}^{N} \underline{Y}_\alpha \overline{\underline{Y}}_\alpha'$ and $B = \Sigma_{\alpha=1}^{q_1} \underline{Y}_\alpha \overline{\underline{Y}}_\alpha'$. Under K, A is distributed according to the complex Wishart matrix with $N-q_2$ degrees of freedom and B is distributed with according to the non-central complex Wishart distribution with q_1 degrees of freedom and non-centrality parameter $\Omega = \Gamma\,\overline{\Gamma}'\,\Sigma^{-1}$, where $\Gamma = E[\underline{y}_1,\ldots,\underline{y}_{q_1}]$.

Put $\lambda = -\rho\log\Lambda$, where $\rho N = 2n = 2N-2q_2+q_1-m$. Then the characteristic function $\phi(t)$ of λ is given by

$$(4.3)\qquad \frac{\tilde{\Gamma}_m(n+\frac{1}{2}(q_1+m))\tilde{\Gamma}_m(n(1-2it)-\frac{1}{2}(q_1-m))}{\tilde{\Gamma}_m(n-\frac{1}{2}(q_1-m))\tilde{\Gamma}_m(n(1-2it)+\frac{1}{2}(q_1+m))}\ \mathrm{etr}(-\Omega)\ \cdot$$

$$\cdot\ {}_1\tilde{F}_1(n+\tfrac{1}{2}(q_1+m),\ n(1-2it) + \tfrac{1}{2}(q_1+m),\ \Omega)$$

$$= \phi_1(t)\ \phi_2(t).$$

Note that (4.3) is not the same form as (1.26) in [7], because we can't use the Kummer's formula since we have not a Kummer's formula for a hermitian matrix. We expand (4.3) as before.

$$\phi_1(t) = \frac{1}{(1-2it)^{q_1 m}}\left\{1 - \frac{mq_1}{48n^2}(5m^2+q_1^2-6)\left(\frac{1}{(1-2it)^2} - 1\right) + O\left(\frac{1}{n^3}\right)\right\} \operatorname{etr}(-\Omega)$$

$$\phi_2(t) = \operatorname{etr}\left(\frac{\Omega}{1-2it}\right)\Big[1 + \frac{1}{2n}\left\{\frac{\operatorname{tr}\Omega^2}{(1-2it)^3} + \frac{(q_1+m)\operatorname{tr}\Omega - \operatorname{tr}\Omega^2}{(1-2it)^2} - \frac{(q_1+m)\operatorname{tr}\Omega}{1-2it}\right\}$$

$$+ \frac{1}{24n^2}\sum_{\alpha=1}^{6}\frac{A_\alpha}{(1-2it)^\alpha} + O(1/n^3)\Big],$$

where

(4.4) $$A_1 = -8B^2\operatorname{tr}\Omega$$

$$A_2 = -8B^2\operatorname{tr}\Omega - (6B+7)(\operatorname{tr}\Omega)^2 + (4B^2 + 14B + 5)\operatorname{tr}\Omega^2$$

$$A_3 = 16B^2\operatorname{tr}\Omega + 2\operatorname{tr}\Omega^3 + 2(6B+1)(\operatorname{tr}\Omega)^2 - 2(2B+1)(2B+5)\operatorname{tr}\Omega^2 + 6B\operatorname{tr}\Omega\operatorname{tr}\Omega^2.$$

$$A_4 = -(6B-5)(\operatorname{tr}\Omega)^2 + (4B+5)(B+1)\operatorname{tr}\Omega^2 - 12\operatorname{tr}\Omega^3 - 12B\operatorname{tr}\Omega\operatorname{tr}\Omega^2 + 3(\operatorname{tr}\Omega^2)^2.$$

$$A_5 = 4(\operatorname{tr}\Omega)^3 + 6\operatorname{tr}\Omega^3 + 6B\operatorname{tr}\Omega\operatorname{tr}\Omega^2 - 6(\operatorname{tr}\Omega^2)^2$$

$$A_6 = 3(\operatorname{tr}\Omega^2)^2.$$

$$B = \frac{1}{2}(q_1 + m).$$

Therefore, we obtain the asymptotic expansion of $\phi(t)$ with respect to the order $1/n$ as follows.

(4.5) $$\phi(t) = \frac{1}{(1-2it)^{q_1 m}} \operatorname{etr}\left(\frac{2it}{1-2it}\Omega\right)$$

$$\left[1 + \frac{1}{2n}\left\{\frac{\mathrm{tr}\Omega^2}{(1-2it)^3} + \frac{(q_1+m)\mathrm{tr}\Omega-\mathrm{tr}\Omega^2}{(1-2it)^2} - \frac{q_1+m}{1-2it}\mathrm{tr}\Omega\right\}\right.$$

$$\left. + \frac{1}{24n^2}\left\{\sum_{\alpha=1}^{6}\frac{A_\alpha}{(1-2it)^\alpha} + \frac{2q_1m(5m^2+q_1^2-6)}{(1-2it)^2} - 2q_1m(5m^2+q_1^2-6)\right\} + O(1/n^3)\right] .$$

Since we know that $\exp((2it/1-2it)\mathrm{tr}\Omega)(1/(1-2it)^{q_1m+\beta})$ is a characteristic function of the χ^2 variable with $2(q_1m+\beta)$ degrees of freedom and with non-central parameter $\delta^2 = \mathrm{tr}\Omega$, by inverting (4.5) we have a following theorem of the asymptotic distribution of the likelihood ratio criterion Λ as follows.

Theorem 4. In the linear statistical testing hypothesis model (4.1), we have the asymptotic distribution of Λ under the alternative K as follows.

$$(4.6) \qquad \Pr\{-\rho\log\Lambda\le x\} = \Pr\{\chi^2_{2q_1m}(\delta^2)\le x\}$$

$$+ \frac{1}{2n}\left\{\mathrm{tr}\Omega^2\Pr\{\chi^2_{2q_1m+6}(\delta^2) \le x\}\right.$$

$$+ ((q_1+m)\mathrm{tr}\Omega-\mathrm{tr}\Omega^2)\Pr\{\chi^2_{2q_1m+4}(\delta^2) \le x\}$$

$$\left. - (q_1+m)\mathrm{tr}\Omega\Pr\{\chi^2_{2q_1m+2}(\delta^2) \le x\}\right\}$$

$$+ \frac{1}{24n^2}\left\{\sum_{\alpha=1}^{6} A_\alpha\Pr\{\chi^2_{2q_1m+2\alpha}(\delta^2) \le x\}\right.$$

$$+ 2q_1m(5m^2+q_1^2-6)\Pr\{\chi^2_{2q_1m+4}(\delta^2) \le x\}$$

$$\left. - 2q_1m(5m^2+q_1^2-6)\Pr\{\chi^2_{2q_1m}(\delta^2) \le x\}\right\} + O(1/n^3),$$

where $\rho N = 2n = 2N - 2q_2 + q_1 - m$, and A_α and B are given in (4.4).

5. The likelihood ratio test for the independence.

Let S be distributed with $CW(\Sigma,N)$ and let's partition S and Σ into m_1 and m_2 rows and columns $(m_1 \le m_2)$ as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ \bar{S}'_{12} & S_{22} \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \bar{\Sigma}'_{12} & \Sigma_{22} \end{bmatrix}.$$

The likelihood ratio test for the independence $H: \Sigma_{12} = 0$ against all alternatives $K: \Sigma_{12} \neq 0$ is given by

$$(5.1) \qquad \Lambda = (\det S/\det S_{11}\det S_{22})^N .$$

<u>Lemma 5.</u> Under the alternative K, the moment of Λ^h is expressed as

$$(5.2) \qquad \frac{\tilde{\Gamma}_{m_1}(N)\tilde{\Gamma}_{m_1}(N-m_2+Nh)}{\tilde{\Gamma}_{m_1}(N-m_2)\tilde{\Gamma}_{m_1}(N+Nh)} \det(I - \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}'_{12})^N \cdot$$

$$\cdot\ {}_2\tilde{F}_1(N,\ N,\ N+Nh,\ \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}'_{12}).$$

<u>Proof</u>. The proof of this lemma is different from [7]. Let $\Sigma^{-1} = D$ and let's partition D as

$$D = \begin{bmatrix} D_{11} & D_{12} \\ \bar{D}'_{12} & D_{22} \end{bmatrix}.$$

Then the expectation of Λ^h is written as follows

$$E(\Lambda^h) = \frac{1}{\tilde{\Gamma}_m(N)(\det\Sigma)^N} \int_{\bar{S}'=S>0} \operatorname{etr}(-\Sigma^{-1}S)(\det S)^{N-m} \left(\frac{\det S}{\det S_{11} \det S_{22}}\right)^{Nh} dS$$

$$= \frac{1}{\tilde{\Gamma}_m(N)(\det\Sigma)^N} \int_{\bar{S}'_{11}=S_{11}>0} dS_{11} \operatorname{etr}(-D_{11}S_{11})(\det S_{11})^{N-m} \cdot$$

$$\cdot \int_{\bar{S}'_{22}=S_{22}>0} dS_{22} \operatorname{etr}(-D_{22}S_{22})(\det S_{22})^{N-m} \cdot$$

$$\cdot \int_{S_{12}} \operatorname{etr}(-(\bar{D}'_{12}S_{12}+D_{12}\bar{S}'_{12}))\det(I-S_{11}^{-\frac{1}{2}}S_{12}S_{22}^{-1}\bar{S}'_{12}S_{11}^{-\frac{1}{2}})^{N+Nh-m} dS_{12} .$$

Let $S_{12} = S_{11}WS_{22}$, then $dS_{12} = (\det S_{11})^{m_2}(\det S_{22})^{m_1}dW$. Hence

$$(5.3) \quad E(\Lambda^h) = \frac{1}{\tilde{\Gamma}_m(N)(\det\Sigma)^N} \int_{\bar{S}'_{11}=S_{11}>0} dS_{11} \operatorname{etr}(-D_{11}S_{11})(\det S_{11})^{N-m_1} \cdot$$

$$\cdot \int_{\bar{S}'_{22}=S_{22}>0} dS_{22} \operatorname{etr}(-D_{22}S_{22})(\det S_{22})^{N-m_2} \cdot$$

$$\cdot \int_W \operatorname{etr}\{-(S_{22}^{\frac{1}{2}}\bar{D}'_{12}S_{11}^{\frac{1}{2}}W+S_{11}^{\frac{1}{2}}D_{12}S_{22}^{\frac{1}{2}}\bar{W}')\}\det(I_{m_1}-W\bar{W}')^{N+Nh-m} dW.$$

Since $\det(I-W\bar{W}')$ is invariant under the transformation W to WU, $U\in U(m_2)$, we first project $\operatorname{etr}\{-(S_{22}^{\frac{1}{2}}\bar{D}'_{12}S_{11}^{\frac{1}{2}}W + S_{11}^{\frac{1}{2}}D_{12}S_{22}^{\frac{1}{2}}\bar{W}')\}$ into the space of $\phi(W\bar{W}')$ and we integrate $\phi(W\bar{W}')$ on the whole space such that $W\bar{W}' > 0$. Therefore, by using (1.3),

$$\int_W \operatorname{etr}\{-(S_{22}^{\frac{1}{2}}\bar{D}'_{12}S_{11}^{\frac{1}{2}}W + S_{11}^{\frac{1}{2}}D_{12}S_{22}^{\frac{1}{2}}\bar{W}')\} \det(I - W\bar{W}')^{N+Nh-m} dW$$

$$= \int_W dW \int_{U(m_2)} \mathrm{etr}\{-(S_{22}^{\frac{1}{2}}\bar{D}_{12}'S_{11}^{\frac{1}{2}}WU + S_{11}^{\frac{1}{2}}D_{12}S_{22}^{\frac{1}{2}}\bar{U}'\bar{W}')\}\det(I - W\bar{W}')^{N+Nh-m}d(U)$$

$$= \int_W \det(I - W\bar{W}')^{N+Nh-m}\, {}_0\tilde{F}_1(m_2,\ S_{11}^{\frac{1}{2}}D_{12}S_{22}\bar{D}_{12}'S_{11}^{\frac{1}{2}}W\bar{W}')\ dW.$$

Hence by applying the Hsu's lemma in a complex case and (1.2), the above integral can be written as

$$(5.4)\quad \frac{\pi^{m_1m_2}}{\tilde{\Gamma}_{m_1}(m_2)} \int_{\bar{R}'=R>0} (\det R)^{m_2-m_1} \det(I-R)^{N+Nh-m}\, {}_0\tilde{F}_1(m_2, S_{11}^{\frac{1}{2}}D_{12}S_{22}\bar{D}_{12}'S_{11}^{\frac{1}{2}}R)\,dR$$

$$= \pi^{m_1m_2} \frac{\tilde{\Gamma}_{m_1}(N-m_2+Nh)}{\tilde{\Gamma}_{m_1}(N+Nh)}\ {}_0\tilde{F}_1(N+Nh,\ S_{11}D_{12}S_{22}\bar{D}_{12}').$$

Thus inserting (5.4) into (5.3), we have the moments of Λ by integration with respect to S_{11} and S_{22} as follows

$$\pi^{m_1m_2} \frac{\tilde{\Gamma}_{m_1}(N)\tilde{\Gamma}_{m_2}(N)\tilde{\Gamma}_{m_1}(N-m_2+Nh)}{\tilde{\Gamma}_m(N)\tilde{\Gamma}_{m_1}(N+Nh)(\det\Sigma)^N}(\det D_{11})^{-N}(\det D_{22})^{-N}\ \cdot$$

$$\cdot\ {}_2\tilde{F}_1(N,\ N;\ N+Nh,\ D_{11}^{-1}D_{12}D_{22}^{-1}\bar{D}_{12}')\ .$$

Since

$$\det\Sigma = (\det D_{11})^{-1}(\det D_{22})^{-1}\det(I - \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}_{12}')$$

$$\tilde{\Gamma}_m(N) = \pi^{m_1m_2}\tilde{\Gamma}_{m_1}(N-m_2)\tilde{\Gamma}_{m_2}(N), \qquad m = m_1 + m_2\ ,$$

we have (5.2).

Theorem 5. The power function of the likelihood ratio criterion for the testing of independence between two sets of variates is expressed asymptotically in the following way.

Let $\lambda = -(\rho/\tau\sqrt{n})\{\log\Lambda - \log\{\det\Sigma/(\det\Sigma_{11})(\det\Sigma_{22})\}\}$, where Λ is (5.1), $2n = \rho N = N-(m_1+m_2)$ and $\tau = 2\sqrt{2\mathrm{tr}P^2}$, $P = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}'$.

Then

$$\Pr\{\lambda \le x\} = \Phi(x) - \frac{1}{3\sqrt{n}}\left\{\frac{m_1 m_2}{\tau}\Phi'(x) + \Phi^{(2)}(x) + 16\frac{\Phi^{(3)}(x)}{\tau^3}\mathrm{tr}P^4\right\} + O(1/n) .$$

Proof. The proof is done completely same way as before.

6. References.

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