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ON THE DISTRIBUTION OF THE LATENT ROOTS OF A COMPLEX WISHART MATRIX (NON-CENTRAL CASE)

by

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Distribution of Complex Wishart Matrix

Abstract

This paper considers the derivation of the probability density function of the latent roots of a non-central complex Wishart matrix. To treat this problem, we define the generalized Hermite polynomials of a complex matrix argument and give some properties of the generalized Hermite polynomials. By using the generating function of the generalized Hermite polynomials, we can obtain the exact probability density functions of the latent roots, the maximum latent root and of trace of the latent roots of a non-central complex Wishart matrix.

Classification number: 10.

Main words and notations

$\tilde{H}_\kappa(T)$:	generalized Hermite polynomial (g.H.p.) of a complex matrix argument T.
$\tilde{L}^\gamma_\kappa(S)$:	generalized Laguerre polynomial (g.L.p.) of a hermitian matrix S.
d(U):	The unitary invariant measure with total volume unity.
d*(U):	The unitary invariant measure.
d(L):	The semi-unitary invariant measure with total volume unity.
$\tilde{C}_\kappa(A)$:	The zonal polynomial which corresponds the partition κ of k into not more than m parts.
etr A:	exp(tr A).
det A:	determinant of A.

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1. Introduction

N. G. Goodman [3], C. G. Khatri [7], [8] and A. T. James [6] have discussed some distribution problems of the complex multivariate normal distribution which appears in time series analysis and physics. The problems of complex variates can be treated in the same way as those of the real variates in the case of normal distributions. In this paper, we consider the distribution problems of the latent roots of a positive definite random Hermitian matrix which are extensions of author [5] to the complex variates. We introduce the generalized Hermite and Laguerre polynomials with complex matrix argument to handle these problems.

2. Notations and useful results

We shall use the following notations which are given by James [6].

(1) $$\tilde{\Gamma}_m(a) = \pi^{\frac{1}{2}m(m-1)} \prod_{\alpha=1}^{m} \Gamma(a-\alpha+1).$$

(2) $$\tilde{\Gamma}_m(a,\kappa) = \pi^{\frac{1}{2}m(m-1)} \prod_{\alpha=1}^{m} \Gamma(a+k_\alpha-\alpha+1).$$

(3) $$[a]_\kappa = \prod_{\alpha=1}^{m} (a-\alpha+1)_{k_\alpha} = \tilde{\Gamma}_m(a,\kappa)/\tilde{\Gamma}_m(a).$$

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where $\kappa = (k_1,\ldots,k_m)$ is a partition of k into not more than m parts. The corresponding generalized hypergeometric functions are defined as

$$(4)\qquad {}_p\tilde{F}_q^{(m)}(a_1,\ldots,a_p,b_1,\ldots,b_q,A,B) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa \cdots [a_p]_\kappa}{[b_1]_\kappa \cdots [b_q]_\kappa}\,\frac{\tilde{C}_\kappa(A)\tilde{C}_\kappa(B)}{k!\tilde{C}_\kappa(I_m)}$$

where $\tilde{C}_\kappa(A)$ is a zonal polynomial of a Hermitian matrix A corresponding to a partition κ of k. If either A or B is I_m, then we write as ${}_p\tilde{F}_q^{(m)}(\ldots., \ldots.; A, I) = {}_p\tilde{F}_q(\ldots, \ldots; A)$.

The most fundamental properties of a zonal polynomial are of the same form as in the real case, so that

$$(5)\qquad \int_{U(m)} \tilde{C}_\kappa(AUB\bar{U}')d(U) = \frac{\tilde{C}_\kappa(A)\tilde{C}_\kappa(B)}{\tilde{C}_\kappa(I_m)}$$

and

$$(6)\qquad \int_{U(n)} \mathrm{etr}(XU+\bar{U}'\bar{X}')d(U) = {}_0\tilde{F}_1(n;X\bar{X}'),$$

where $d(U)$ in (5) and (6) is the invariant measure normalized to make the total measure unity on the unitary group $U(m)$ and $U(n)$ of order m and n, respectively, and A and B are Hermitian matrices, and X is an arbitrary $n\times n$ complex matrix.

Hsu's Lemma. Let $f(Z\bar{Z}')$ be a probability density function (p.d.f.) of Z $m\times n$ $(m\le n)$, then the p.d.f. of the Hermitian matrix $R=Z\bar{Z}'$ is given by

$$(7)\qquad \frac{\pi^{mn}}{\tilde{\Gamma}_m(n)}(\det R)^{n-m} f(R).$$

The p.d.f. of the latent roots $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_m)$ of R is also given by

(8) $$\frac{\pi^{m(n+m)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)} (\det\Lambda)^{n-m} f(\Lambda) \prod_{i<j} (\lambda_i-\lambda_j)^2,$$

if $f(U\Lambda\bar{U}') = f(\Lambda)$.

3. Generalized Hermite polynomials

Hayakawa [5] defined the generalized Hermite polynomial (g.H.p.) with a real matrix argument in order to discuss the distribution problems of the latent roots of the non-central Wishart matrix. In this section, we show that they can be extended to the complex variate case and they can be treated analogously as the real variate case.

Here we define the g.H.p. $\tilde{H}_\kappa(T)$ and the generalized Laguerre polynomial (g.L.p.) $\tilde{L}_\kappa(S)$, $\gamma > -1$, which correspond to the partition κ of k, in the following way:

(9) $$\mathrm{etr}(-T\bar{T}')\tilde{H}_\kappa(T) = \frac{(-1)^k}{\pi^{mn}} \int_W \mathrm{etr}(-i(T\bar{W}'+W\bar{T}'))\mathrm{etr}(-W\bar{W}')\tilde{C}_\kappa(W\bar{W}')dW,$$

where T and W are $m\times n$ $(m\leq n)$ arbitrary complex matrices, and for $\gamma > -1$,

(10) $$\mathrm{etr}(-S)\tilde{L}_\kappa^\gamma(S) = \int_{\bar{R}'=R>0} \tilde{A}_\gamma(RS)(\det R)^\gamma \mathrm{etr}(-R)\tilde{C}_\kappa(R)dR,$$

where S and R are m×m Hermitian matrices and

$$\tilde{\Gamma}_m(\gamma+m)\tilde{A}_\gamma(RS) = {}_0\tilde{F}_1(\gamma+m, -RS), \quad [2] .$$

The next theorem gives a relation between g.H.p. and g.L.p.

<u>Theorem 1.</u>

(11) $$\tilde{H}_\kappa(T) = (-1)^k \tilde{L}_\kappa^{n-m}(T\bar{T}').$$

<u>Proof.</u> From the definition of g.H.p. and the Hsu's Lemma in the complex case, we have

$$\begin{aligned}
\operatorname{etr}(-T\bar{T}')\tilde{H}_\kappa(T) &= \frac{(-1)^k}{\pi^{mn}} \int_W \int_{U(n)} \operatorname{etr}\{-i(WU\bar{T}'+T\bar{U}'\bar{W}')\}\operatorname{etr}(-W\bar{W}')\tilde{C}_\kappa(W\bar{W}')\,dWd(U) \\
&= \frac{(-1)^k}{\pi^{mn}} \int_W {}_0\tilde{F}_1(n;\ -T\bar{T}'W\bar{W}')\operatorname{etr}(-W\bar{W}')\tilde{C}_\kappa(W\bar{W}')\,dW \\
&= (-1)^k \int_{\bar{R}'=R>0} \tilde{A}_{n-m}(T\bar{T}'R)(\det R)^{n-m} \operatorname{etr}(-R)\tilde{C}_\kappa(R)\,dR \\
&= (-1)^k \operatorname{etr}(-T\bar{T}')\tilde{L}_\kappa^{n-m}(T\bar{T}') ,
\end{aligned}$$

which completes the proof.

We can show easily the following corollaries from the definition (8) and Hsu's Lemma.

<u>Corollary 1.</u>

(12) $$\tilde{H}_\kappa(T) = \tilde{H}_\kappa(U_1T) = \tilde{H}_\kappa(TU_2),$$

where $U_1 \in U(m)$ and $U_2 \in U(n)$, respectively.

Corollary 2.

(13) $$|\tilde{H}_\kappa(T)| \le \operatorname{etr}(T\bar{T}')[n]_\kappa \tilde{C}_\kappa(I),$$

(14) $$\tilde{H}_\kappa(0) = (-1)^k[n]_\kappa \operatorname{etr}(T\bar{T}')\tilde{C}_\kappa(I),$$

Theorem 2. (Generating function of g.H.p.'s)

Let S and T be $m\times n$ $(m\le n)$ arbitrary complex matrices, then

(15) $$\int_{U(m)}\int_{U(n)} \operatorname{etr}\{-S\bar{S}'+U_1SU_2\bar{T}'+T\bar{U}_2'\bar{S}'\bar{U}_1'\}\, d(U_1)d(U_2)$$

$$= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\tilde{H}_\kappa(T)\tilde{C}_\kappa(S\bar{S}')}{k![n]_\kappa\, \tilde{C}_\kappa(I)},$$

where $U_1 \in U(m)$ and $U_2 \in U(n)$, respectively. The right hand side of (15) converges absolutely.

Proof. We show (15) by the direct method, using (5), (6) and (9).

$$\text{R.H.S.} = \operatorname{etr}(T\bar{T}')\frac{1}{\pi^{mn}}\int_W \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k\tilde{C}_\kappa(SS')}{k![n]_\kappa\,\tilde{C}_\kappa(I)}\operatorname{etr}\{-i(T\bar{W}'+W\bar{T}')\}\cdot$$

$$\cdot \operatorname{etr}(-W\bar{W}')\tilde{C}_\kappa(W\bar{W}')dW$$

$$= \frac{\operatorname{etr}(T\bar{T}')}{\pi^{mn}}\int_W\int_{U(m)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k}{k![n]_\kappa}\tilde{C}_\kappa(S\bar{S}'U_1W\bar{W}'\bar{U}_1')\cdot$$

$$\cdot \operatorname{etr}\{-i(T\bar{W}'+W\bar{T}')\}\operatorname{etr}(-W\bar{W}')dW$$

$$= \frac{\operatorname{etr}(TT')}{\pi^{mn}}\int_W\int_{U(m)}\int_{U(n)} \operatorname{etr}\{i(\bar{W}'U_1SU_2+\bar{U}_2'\bar{S}'\bar{U}_1'W)\}\cdot$$

$$\cdot \operatorname{etr}\{-i(T\bar{W}'+W\bar{T}')\}\operatorname{etr}(-W\bar{W}')dW$$

$$= \int_{U(m)} \int_{U(n)} \operatorname{etr}(-S\bar{S}'+U_1 S U_2 \bar{T}'+T\bar{U}_2' \bar{S}' \bar{U}_1') d(U_1) d(U_2) \ .$$

The absolute convergence of (15) is shown by using (13). This completes the proof.

Theorem 3. (Generating function of g.L.p.'s)

Let S and Z be $m\times m$ positive definite Hermitian matrices, then

$$(16) \qquad \det(I-Z)^{-\gamma-m} \int_{U(m)} \operatorname{etr}(-SUZ(I-Z)^{-1}\bar{U}')d(U)$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{L}_\kappa^\gamma(S)\tilde{C}_\kappa(Z)}{k!\tilde{C}_\kappa(I_m)}, \quad ||Z|| < 1,$$

where $||Z||$ means the maximum of the absolute values of the characteristic roots of Z.

Proof. Similar to Theorem 2, using (5), (7) and (10).

Remark 1: The invariance of $\tilde{H}_\kappa(T)$ with respect to $U(m)$ from left and $U(n)$ from right is obvious from Theorem 2, because $d(U_1)$ and $d(U_2)$ are invariant unitary measures on $U(m)$ and $U(n)$, respectively.

Remark 2: The relation between the generating functions of the g.H.p.'s and the g.L.p.'s when $\gamma = n-m$ is as follows. Multiply both sides of (15) by $\pi^{mn}(\det Z)^{-n}\operatorname{etr}(-S\bar{S}'Z)$ and integrate with respect to S. Then we have (16) by replacing Z with $-Z$.

Theorem 4. (Mehler's formula)

Let S and T be an $m\times n$ arbitrary complex matrix. Then

(17) $$\frac{1}{(1-u^2)^{mn}} \int_{U(m)} \int_{U(n)} \mathrm{etr}\{-\frac{u^2}{1-u^2}(S\bar{S}'+T\bar{T}') + \frac{u}{1-u^2}(U_1SU_2\bar{T}' + T\bar{U}_2'\bar{S}'\bar{U}_1')\}d(U_1)d(U_2)$$

$$= \sum_{k=0}^{\infty} \frac{u^{2k}}{k!} \sum_{\kappa} \frac{\tilde{H}_\kappa(T)\tilde{H}_\kappa(S)}{[n]_\kappa \tilde{C}_\kappa(I_m)}, \qquad |u| < 1.$$

<u>Proof</u>. The proof is exactly similar to that of Theorem 2.

<u>Corollary 4</u>.

(18) $$\sum_{\kappa} \tilde{L}_\kappa^{n-m}(T\bar{T}') = L_k^{mn-1}(\mathrm{tr}\, T\bar{T}').$$

<u>Proof</u>. If we set $S=0$ in (17), we have

(19) $$\frac{1}{(1-u^2)^{mn}} \mathrm{etr}[-\frac{u^2}{1-u^2} T\bar{T}'] = \sum_{k=0}^{\infty} \frac{u^{2k}}{k!} (-1)^k \sum_{\kappa} \tilde{H}_\kappa(T),$$

since $\tilde{H}_\kappa(0) = (-1)^k [n]_\kappa \tilde{C}_\kappa(I)$.

The left hand side of (19) is a generating function of a univariate Laguerre polynomial $L_k^{mn-1}(\mathrm{tr}T\bar{T}')$, i.e.,

$$\text{L.H.S.} = \sum_{k=0}^{\infty} \frac{u^{2k}}{k!} L_k^{mn-1}(\mathrm{tr}\, T\bar{T}').$$

Hence by comparing the coefficients of u^{2k} and by Theorem 1, we have (18), which completes the proof.

We can obtain a more general formula for the g.L.p.'s than (18).

Corollary 5.

(20) $$\sum_{\kappa} \tilde{L}_{\kappa}^{\gamma}(S) = L_k^{m(\gamma+m)-1}(\operatorname{tr} S),$$

where S is an $m\times m$ positive definite Hermitian matrix.

Proof. From the definition (10), we have for $|x| < 1$

$$\operatorname{etr}(-S) \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{\kappa} \tilde{L}_{\kappa}^{\gamma}(S) = \int_{\bar{R}'=R>0} \tilde{A}_{\gamma}(RS)(\det R)^{\gamma} \operatorname{etr}(-(1-x)R)\,dR$$

$$= \frac{1}{\tilde{\Gamma}_m(\gamma+m)} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \frac{1}{[\gamma+m]_{\kappa}} \cdot$$

$$\cdot \int_{\bar{R}'=R>0} \operatorname{etr}(-(1-x)R)(\det R)^{\gamma} \tilde{C}_{\kappa}(-RS)\,dR$$

$$= \frac{1}{(1-x)^{(\gamma+m)m}} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \tilde{C}_{\kappa}\left(\frac{-S}{1-x}\right)$$

$$= \frac{1}{(1-x)^{m(\gamma+m)}} \operatorname{etr}\left(-\frac{1}{1-x} S\right).$$

Hence

(21) $$\sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{\kappa} \tilde{L}_{\kappa}^{\gamma}(S) = \frac{1}{(1-x)^{m(\gamma+m)}} \operatorname{etr}\left(-\frac{x}{1-x} S\right)$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} L_k^{m(\gamma+m)-1}(\operatorname{tr} S), \quad |x| < 1.$$

By comparing the two sides of (21), we obtain (20).

4. JACOBIANS OF THE LATENT ROOTS AND INTEGRALS

In this section, we give some useful transformations for finding the p.d.f. of the maximum latent root of a positive definite Hermitian matrix, and of the latent roots of a non-central complex Wishart matrix with known covariance. We also give some related Beta-type integrals.

Lemma 1. Let S be an $m\times m$ positive definite random Hermitian matrix. We decompose S as follows,

$$S = U\begin{bmatrix} \lambda_1 & \\ & V \end{bmatrix}\bar{U}', \tag{22}$$

where U is an $m\times m$ unitary matrix which has only $2(m-1)$ independent variables $u^R_{21},\ldots,u^R_{m1}$, $u^I_{21},\ldots,u^I_{m1}$, the first column elements of U except u_{11}, and V is an $(m-1)\times(m-1)$ positive definite random Hermitian matrix which ranges $\lambda_1 I_{m-1} > \bar{V}' = V > 0$. Then

$$dS = \det(\lambda_1 I - V)^2 d\lambda_1 dV \prod_{\alpha=2}^{m} du^R_{\alpha 1} du^I_{\alpha 1} . \tag{23}$$

Proof. The proof of this lemma is given in Appendix I.

Note: If we decompose S further such that

$$S = U_1\begin{bmatrix} 1 & \\ & U_2 \end{bmatrix}\cdots\begin{bmatrix} I_{k-1} & \\ & U_k \end{bmatrix}\begin{bmatrix} \Lambda_k & \\ & V_k \end{bmatrix}\begin{bmatrix} I_{k-1} & \\ & \bar{U}'_k \end{bmatrix}\cdots\begin{bmatrix} 1 & \\ & \bar{U}'_2 \end{bmatrix}\bar{U}'_1 , \tag{24}$$

where U_ν is a unitary matrix of order $(m-\nu+1)$ whose first column vector contains independent variables, and V_k is an $(m-k)\times(m-k)$ Hermitian matrix which ranges $\lambda_k I > \bar{V}'_k = V_k > 0$, then

$$(25) \qquad dS = \prod_{i<j}^{k} (\lambda_i-\lambda_j)^2 \prod_{i=1}^{k} \det(\lambda_i I_{m-k}-V_k)^2 d\Lambda_k dV_k \prod_{\beta=1}^{k-1} \prod_{\alpha=\beta+1}^{m} \cdot$$

$$\cdot\, du^R_{\alpha\beta} du^I_{\alpha\beta}.$$

This can be shown, by induction, from Lemma 1.

Lemma 2: Let X be an $m\times n$ $(m\leq n)$ arbitrary complex matrix, U an $m\times m$ unitary matrix whose diagonal elements are all real values, and L an $n\times m$ $(m\leq n)$ semi-unitary matrix (i.e., $\bar{L}'L = I_m$) whose diagonal elements are all complex variables, and $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_m)$, where the λ^2_α's are all real-valued latent roots of $X\bar{X}'$. Then the Jacobian of the decomposition

$$(26) \qquad X = U\Lambda\bar{L}'$$

is given by

$$(27) \qquad dX = \frac{\pi^{m(n+m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)} (\det\Lambda)^{2(n-m)+1} \prod_{i<j} (\lambda_i^2-\lambda_j^2)^2 d\Lambda d(U) d(L),$$

where $d(L)$ is a semi-unitary invariant measure.

Proof. The proof is shown in Appendix II.

Note: If we decompose X as follows:

$$X = U_1 \begin{bmatrix} 1 & \\ & U_2 \end{bmatrix} \cdots \begin{bmatrix} I_{k-1} & \\ & U_k \end{bmatrix} \begin{bmatrix} \Lambda_k & \\ & V_k \end{bmatrix} \bar{L}',$$

then we have

$$dX = c\Big(\prod_{i=1}^{k} \lambda_i\Big)^{2(n-m)+1} \prod_{i<j}^{k} (\lambda_i^2-\lambda_j^2)^2 \prod_{i=1}^{k} \det(\lambda_i^2 I_k - V_k\bar{V}_k')^2 d\Lambda_k d(L) \cdot$$

$$\cdot \prod_{\beta=1}^{k-1} \prod_{\alpha=\beta+1}^{m} du^R_{\alpha\beta} du^I_{\alpha\beta} dV_k,$$

where

$$c = \frac{\pi^{mn}}{\tilde{\Gamma}_m(n)} .$$

Theorem 5.

(28) $$\int_{I_{m-1}>\bar{W}'=W>0} (\det W)^{\alpha-m} \det(I-W)^2 \, \tilde{C}_\kappa\begin{pmatrix}1 & \\ & W\end{pmatrix} dW$$

$$= \frac{\tilde{\Gamma}_m(\alpha)\tilde{\Gamma}_m(m)\Gamma(m)}{\pi^{m-1}\tilde{\Gamma}_m(\alpha+m)} (m\alpha+k) \cdot \frac{[\alpha]_\kappa}{[m+\alpha]_\kappa} \tilde{C}_\kappa(I_m),$$

(29) $$\int_{1>\omega_2>\dots>\omega_m>0} \left(\prod_{i=2}^m \omega_i\right)^{\alpha-m} \prod_{i=2}^m (1-\omega_i)^2 \prod_{2\le i<j\le m} (\omega_i-\omega_j)^2 \tilde{C}_\kappa\begin{bmatrix}1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_m\end{bmatrix} \cdot \prod_{i=2}^m d\omega_i$$

$$= \frac{\tilde{\Gamma}_m(\alpha)\tilde{\Gamma}_m(m)^2}{\pi^{m(m-1)}\tilde{\Gamma}_m(\alpha+m)} (m\alpha+k) \frac{[\alpha]_\kappa}{[\alpha+m]_\kappa} \tilde{C}_\kappa(I_m).$$

Proof. The following formula is well-known:

(30) $$\int_{I>\bar{S}'=S>0} (\det S)^{\alpha-m} \tilde{C}_\kappa(S) dS = \frac{\tilde{\Gamma}_m(\alpha)\tilde{\Gamma}_m(m)}{\tilde{\Gamma}_m(\alpha+m)} \frac{[\alpha]_\kappa}{[\alpha+m]_\kappa} \tilde{C}_\kappa(I_m).$$

To prove (28), we decompose S as follows:

$$S = U\begin{bmatrix}\lambda_1 & \\ & \lambda_1 W\end{bmatrix}\bar{U}' .$$

Then from Lemma 1, the integral element can be written as

$$dS = \lambda_1^{m^2-1} \det(I-W)^2 \, d\lambda_1 dW \prod_{i=2}^{m} du_{i1}^R du_{i1}^I .$$

Hence inserting these results in (30), and noting that

$$\int_{\sum_{i=2}^{m} u_{i1}^{R^2} + \sum_{i=2}^{m} u_{i1}^{I^2} \le 1} \prod_{i=2}^{m} du_{i1}^R du_{i1}^I = \frac{\pi^{m-1}}{\Gamma(m)} \quad \text{and} \quad \int_0^1 \lambda_1^{m\alpha+k-1} d\lambda_1 = \frac{1}{m\alpha+k} ,$$

we have (28). To prove (29), we decompose W further, so that

$$W = U_2 \Lambda_\omega U_2, \quad U_2 \in U(m-1), \quad \Lambda_\omega = \mathrm{diag}(\omega_2, \ldots, \omega_m).$$

Then

$$dW = \prod_{2 \le i < j \le m} (\omega_i - \omega_j)^2 d\Lambda_\omega d^*(U_2) ,$$

where $d^*(U_2)$ is an invariant measure on $U(m-1)$. Hence by integrating over $U(m-1)$, we have (29), since

$$\int_{U(m-1)} d^*(U_2) = \frac{\pi^{(m-1)(m-2)}}{\tilde{\Gamma}_{m-1}(m-1)} .$$

5. The distribution of the latent roots of a complex non-central Wishart matrix with known covariance

The probability density function of the latent roots of a complex non-central Wishart matrix was given by James [6]. However, in some cases it is not convenient to treat the distribution problems of the related statistics. We here give another formula, expressed in terms of g.H.p.'s.

Theorem 6.

Let X $m\times n$ $(m\le n)$ be distributed with p.d.f.,

$$(31)\qquad \frac{1}{\pi^{mn}(\det\Sigma)^{n}}\,\mathrm{etr}[-\Sigma^{-1}(X-M)\,\overline{(X-M)}']$$

then the p.d.f. of the latent roots of $\det(X\bar{X}'-\lambda\Sigma)=0$ is given by

$$(32)\qquad \frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}\,\mathrm{etr}(-\Sigma^{-1}MM')(\det\Lambda)^{n-m}\prod_{i<j}(\lambda_i-\lambda_j)^2\,\cdot$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{H}_{\kappa}(\Sigma^{-\frac{1}{2}}M)\tilde{C}_{\kappa}(\Lambda)}{k!\,[n]_{\kappa}\,\tilde{C}_{\kappa}(I_m)}\,,$$

where $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_m)$.

Proof. We decompose $\Sigma^{-\frac{1}{2}}X$ in (31) as follows:

$$(33)\qquad Y=\Sigma^{-\frac{1}{2}}X=U_1\Lambda^{\frac{1}{2}}\bar{L}'\,,$$

where U_1, L and Λ are same matrices as those of Lemma 2. Then

$$(34)\qquad dY=\frac{\pi^{m(m+n-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}(\det\Lambda)^{n-m}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\Lambda\, d(U_1)\, d(L).$$

Hence by inserting (33) and (34) into (31), we have the joint p.d.f. of Λ, U_1 and L;

$$(35)\qquad \frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}\,\mathrm{etr}(-\Sigma^{-1}MM')(\det\Lambda)^{n-m}\prod_{i<j}(\lambda_i-\lambda_j)^2\,\cdot$$

$$\cdot\ \mathrm{etr}(-\Lambda^{\frac{1}{2}}\bar{L}'L\Lambda^{\frac{1}{2}}+U_1\Lambda^{\frac{1}{2}}\bar{L}'\bar{M}'\Sigma^{-\frac{1}{2}}+\Sigma^{-\frac{1}{2}}ML\Lambda^{\frac{1}{2}}\bar{U}_1')\,.$$

If we set $\bar{L}'\to\bar{L}'U_2$, $U_2\in U(n)$, then $\bar{L}'U_2\overline{(\bar{L}'U_2)}'=I_m$ and the semi unitary invariant measure $d(L)$ is unchanged by the unitary transformation. Therefore, we integrate (33) with respect to U_1 and L.

$$\int_{\bar{L}'L=I_m} \int_{U(m)} \mathrm{etr}(-\Lambda^{\frac{1}{2}}\bar{L}'L\Lambda^{\frac{1}{2}}+U_1\Lambda^{\frac{1}{2}}\bar{L}'\bar{M}'\Sigma^{-\frac{1}{2}}+\Sigma^{-\frac{1}{2}}ML\Lambda^{\frac{1}{2}}\bar{U}_1')d(U_1)d(L)$$

$$= \int_{\bar{L}'L=I_m} \int_{U(m)} \int_{U(n)} \mathrm{etr}(-\Lambda^{\frac{1}{2}}\bar{L}'L\Lambda +U_1\Lambda^{\frac{1}{2}}\bar{L}'U_2\bar{M}'\Sigma^{-\frac{1}{2}}+\Sigma^{-\frac{1}{2}}M\bar{U}_2'L\Lambda^{\frac{1}{2}}\bar{U}_1') \cdot$$

$$\cdot\ d(U_1)d(U_2)d(L).$$

The integral of the R.H.S. with respect to $U(m)$ and $U(n)$ is the same form as the generating function of g.H.p.'s if we set $S = \Lambda^{\frac{1}{2}}\bar{L}'$ and $T = \Sigma^{-\frac{1}{2}}M$. Thus we have

$$\text{R.H.S.} = \int_{\bar{L}'L=I_m} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{H}_{\kappa}(\Sigma^{-\frac{1}{2}}M)\tilde{C}_{\kappa}(\Lambda)}{k![n]_{\kappa}\ \tilde{C}_{\kappa}(I_m)}\, d(L)$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{H}_{\kappa}(\Sigma^{-\frac{1}{2}}M)\tilde{C}_{\kappa}(\Lambda)}{k![n]_{\kappa}\ \tilde{C}_{\kappa}(I_m)},$$

which completes the proof.

Corollary 6. If we set $M=0$ in (32), then we have the p.d.f. of Λ in the central case from (14) [1].

Theorem 7.

Let Λ be distributed with the p.d.f. (32). Then the p.d.f. of the maximum latent root λ_1 of Λ is given by

$$(36) \qquad \frac{\tilde{\Gamma}_m(n)}{\tilde{\Gamma}_m(n+m)} \operatorname{etr}(-\Sigma^{-1}M\bar{M}')\lambda_1^{mn-1} \sum_{k=0}^{\infty} \frac{(mn+k)}{k!} \lambda_1^k \sum_{\kappa} \frac{H_\kappa(\Sigma^{-\frac{1}{2}}M)}{[n+m]_\kappa} .$$

Proof. If we set $\lambda_i = \lambda_1\omega_i$ $(i = 2,\ldots,m)$ in (32), and integrate it with respect to $\omega_2,\ldots,\omega_m$, then from (29) we have (36) immediately.

Corollary 7. The c.d.f. of λ_1 is given by

$$(37) \qquad \Pr\{\lambda_1<x\} = \frac{\tilde{\Gamma}_m(n)}{\tilde{\Gamma}_m(n+m)} \operatorname{etr}(-\Sigma^{-1}M\bar{M}')x^{mn} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{\kappa} \frac{\tilde{H}_\kappa(\Sigma^{-\frac{1}{2}}M)}{[n+m]_\kappa} .$$

Theorem 8.

Let Λ be distributed with p.d.f. (32). Then the p.d.f. of $T = \operatorname{tr}\Lambda$ is given by

$$(38) \qquad \frac{1}{\Gamma(mn)} \operatorname{etr}(-\Sigma^{-1}M\bar{M}')T^{mn-1} \sum_{k=0}^{\infty} \frac{T^k}{k!(mn)_k} \sum_{\kappa} \tilde{H}_\kappa(\Sigma^{-\frac{1}{2}}M).$$

Proof. We can obtain (38) as in the proof of Theorem 5 in [5].

Remark: (38) is of the same form as the p.d.f. of the $T = 2\chi^2_{2mn}(\delta^2)$, where $\chi^2_{2mn}(\delta^2)$ is a non-central chi-square variable of $2mn$ degrees of freedom with non-central parameter $\delta^2 = \operatorname{tr}\Sigma^{-1}M\bar{M}'$. Since from Theorem 1 and Corollary 4, the series term can be represented by

$$\sum_{k=0}^{\infty} \frac{(-T)^k L_k^{mn-1}(\operatorname{tr}\Sigma^{-1}M\bar{M}')}{k!(mn)_k} ,$$

this can also be represented using a Bessel function as

$$\Gamma(mn)(-T\delta^2)^{-\frac{1}{2}(mn-1)} \exp(-T) J_{mn-1}(2(-T\delta^2)^{\frac{1}{2}}), \quad \delta^2 = \mathrm{tr}\Sigma^{-1}M\bar{M}'.$$

However, as the Bessel function $J_{mn-1}(2(-T\delta^2)^{\frac{1}{2}})$ is written as

$$\frac{1}{\Gamma(mn)} \sum_{k=0}^{\infty} \frac{(T\delta^2)^k}{k!(mn)_k} \cdot (-T\delta^2)^{\frac{1}{2}(mn-1)},$$

the p.d.f. of T is written as

$$\frac{1}{\Gamma(mn)} \exp(-\delta^2-T) T^{mn-1} \sum_{k=0}^{\infty} \frac{(T\delta^2)^k}{k!(mn)_k}.$$

This is the p.d.f. of $2\chi^2_{2mn}(\delta^2)$.

APPENDIX 1. (PROOF OF LEMMA 1.)

Differentiating both sides of (22), we have

$$\text{(A1.1)} \qquad dS = dU\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}\bar{U}' + U\begin{bmatrix}d\lambda_1 & \\ & dV\end{bmatrix}\bar{U}' + U\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}d\bar{U}' .$$

Multiply by $\bar{U}'$ from the left and U from the right to obtain

$$\bar{U}'dSU = \bar{U}'dU\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix} + \begin{bmatrix}d\lambda_1 & \\ & dV\end{bmatrix} - \begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}\bar{U}'dU,$$

since $\bar{U}'dU = -d\bar{U}'U$. By setting $dW = \bar{U}'dSU$ and $dT = \bar{U}'dU$, we can see that dW is a Hermitian matrix whose diagonal elements are all real variates and dT is a skew Hermitian matrix whose diagonal elements are all pure imaginary. Hence we have

$$\text{(A1.2)} \qquad dW = dT\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix} + \begin{bmatrix}d\lambda_1 & \\ & dV\end{bmatrix} - \begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}dT,$$

$$\text{(A1.3)} \qquad dT = \bar{U}'dU.$$

To obtain the Jacobian of this transformation, we shall use the exterior product of the differentials. First of all, we know that $\Pi_{\alpha=1}^{m} d\omega_{\alpha\alpha}\ \Pi_{\alpha>\beta}^{m} d\omega_{\alpha\beta}^{R} d\omega_{\alpha\beta}^{I} = \Pi_{\alpha=1}^{m} dS_{\alpha\beta} \Pi_{\alpha>\beta} dS_{\alpha\beta}^{R} dS_{\alpha\beta}^{I}$, [7]. Before we calculate the exterior product of dT, we must consider the structure of U. The matrix U has only $2(m-1)$ independent elements. Hence we represent U as follows

(A1.4) $$U = \begin{bmatrix} u_{11} & \underline{a}_1^R + i\underline{a}_1^I \\ \underline{u}_1^R + i\underline{u}_1^I & U_{22}^R + iU_{22}^I \end{bmatrix} \begin{matrix} 1 \\ m-1 \end{matrix},$$
$$\begin{matrix} \quad 1 \quad & \quad m-1 \quad \end{matrix}$$

where

(A1.5) $$\begin{cases} u_{11}^2 + \underline{u}_1^{R'}\underline{u}_1^R + \underline{u}_1^{I'}\underline{u}_1^I = u_{11}^2 + \underline{a}_1^R\underline{a}_1^{R'} + \underline{a}_1^I\underline{a}_1^{I'} = 1, \\ u_{11}\underline{u}_1^{R'} + \underline{a}_1^R U_{22}^{R'} + \underline{a}_1^I U_{22}^{I'} = \underline{0}', \quad -u_{11}\underline{u}_1^{I'} - \underline{a}_1^R U_{22}^{I'} + \underline{a}_1^I U_{22}^{R'} = \underline{0}', \\ \underline{u}_1^R\underline{u}_1^{R'} + \underline{u}_1^I\underline{u}_1^{I'} + U_{22}^R U_{22}^{R'} + U_{22}^I U_{22}^{I'} = I_{m-1}, \\ -\underline{u}_1^R\underline{u}_1^{I'} + \underline{u}_1^I\underline{u}_1^{R'} - U_{22}^R U_{22}^{I'} + U_{22}^I U_{22}^{R'} = 0. \end{cases}$$

$\underline{u}_1^R$ and $\underline{u}_1^I$ are vectors of independent variables, and $\underline{a}_1^R$, $\underline{a}_1^I$, U_{22}^R and U_{22}^I are functional vectors and matrices of $\underline{u}_1^R$ and $\underline{u}_1^I$, respectively.

Let

$$dT = \begin{bmatrix} dt_{11} & \vdots & dT_2 \\ d\underline{t}_1^R + id\underline{t}_1^I & \vdots & \end{bmatrix} \begin{matrix} 1 \\ m-1 \end{matrix} = \bar{U}' \begin{bmatrix} du_{11} & \vdots & dU_2 \\ d\underline{u}_1^R + id\underline{u}_1^I & \vdots & \end{bmatrix} \begin{matrix} 1 \\ m-1 \end{matrix},$$
$$\begin{matrix} 1 & m-1 & \qquad & 1 & m-1 \end{matrix}$$

then

$$\begin{bmatrix} dt_{11} \\ d\underline{t}_1^R + id\underline{t}_1^I \end{bmatrix} = \bar{U}' \begin{bmatrix} du_{11} \\ d\underline{u}_1^R + id\underline{u}_1^I \end{bmatrix}.$$

Since dT is a skew Hermitian matrix, $dt_{11} = idt_{11}^I$. We will show later that dt_{11}^I can be represented by the linear combination of $dt_{21}^R, \ldots, dt_{m1}^R$, $dt_{21}^I, \ldots, dt_{m1}^I$. Therefore, we need only $dt_{21}^R, \ldots, dt_{m1}^I$. As the first column has a constraint such that $u_{11}^2 + \underline{u}_1^{R'}\underline{u}_1^R + \underline{u}_1^{I'}\underline{u}_1^I = 1$, the differential of the first column of dU is represented as follows;

$$\begin{bmatrix} -\frac{1}{u_{11}}(\underline{u}_1^{R'} d\underline{u}_1^R + \underline{u}_1^{I'} d\underline{u}_1^I) \\ d\underline{u}_1^R + i d\underline{u}_1^I \end{bmatrix}.$$

Hence we have

$$\text{(A1.7)} \quad \begin{bmatrix} d\underline{t}_1^R \\ d\underline{t}_1^I \end{bmatrix} = \begin{bmatrix} -\frac{1}{u_{11}} \underline{a}_1^{R'} \underline{u}_1^{R'} + U_{22}^{R'} & -\frac{1}{u_{11}} \underline{a}_1^{R'} \underline{u}_1^{I'} + U_{22}^{I'} \\ \frac{1}{u_{11}} \underline{a}_1^{I'} \underline{u}_1^{R'} - U_{22}^{I'} & \frac{1}{u_{11}} \underline{a}_1^{I'} \underline{u}_1^{I'} + U_{22}^{R'} \end{bmatrix} \begin{bmatrix} d\underline{u}_1^R \\ d\underline{u}_1^I \end{bmatrix}.$$

Thus by forming the exterior product of $dt_{21}^R, \ldots, dt_{m1}^I$, we have

$$\text{(A1.8)} \quad \prod_{\alpha=2}^{m} dt_{\alpha 1}^R \prod_{\alpha=2}^{m} dt_{\alpha 1}^I = \det \begin{bmatrix} -\frac{1}{u_{11}}\underline{a}_1^{R'} \underline{u}_1^{R'} + U_{22}^{R'} & -\frac{1}{u_{11}}\underline{a}_1^{R'} \underline{u}_1^{I'} + U_{22}^{I'} \\ \frac{1}{u_{11}}\underline{a}_1^{I'} \underline{u}_1^{R'} - U_{22}^{I'} & \frac{1}{u_{11}}\underline{a}_1^{I'} \underline{u}_1^{I'} + U_{22}^{R'} \end{bmatrix}.$$

$$\cdot \prod_{\alpha=2}^{m} du_{\alpha 1}^R \prod_{\alpha=2}^{m} du_{\alpha 1}^I.$$

We can easily rewrite the determinant (say J_1) as

$$\text{(A1.9)} \quad J_1 = \frac{1}{u_{11}} \det \begin{bmatrix} u_{11} & \underline{a}_1^R & -\underline{a}_1^I \\ \underline{u}_1^R & U_{22}^R & -U_{22}^I \\ \underline{u}_1^I & U_{22}^I & U_{22}^R \end{bmatrix}.$$

From (A1.5),

(A1.10) $$J_1^2 = \frac{1}{u_{11}^2} \det\begin{bmatrix} 1 & \underline{0}' & \underline{0}' \\ \underline{0} & I - \underline{u}_1^I\underline{u}_1^{I'} & \underline{u}_1^I\underline{u}_1^{R'} \\ \underline{0} & \underline{u}_1^R\underline{u}_1^{I'} & I - \underline{u}_1^R\underline{u}_1^{R'} \end{bmatrix}$$

$$= \frac{1}{u_{11}^2} \det\begin{bmatrix} 1 & \underline{u}_1^{I'} & -\underline{u}_1^{R'} \\ \underline{u}_1^I & I & 0' \\ -\underline{u}_1^R & 0 & I \end{bmatrix}$$

$$= \frac{1}{u_{11}^2} (1 - \underline{u}_1^{R'}\underline{u}_1^R - \underline{u}_1^{I'}\underline{u}_1^I)$$

$$= 1.$$

From this calculation, $\Pi_{\alpha=2}^m dt_{\alpha 1}^R \Pi_{\alpha=2}^m dt_{\alpha 1}^I = \Pi_{\alpha=2}^m du_{\alpha 1}^R \Pi_{\alpha=2}^m du_{\alpha 1}^I$. Hence there exists a unimodular matrix D such that

(A1.11) $$[d\underline{t}_1^{R'} \; d\underline{t}_1^{I'}] = [d\underline{u}_1^{R'} \; d\underline{u}_1^{I'}] D'.$$

dt_{11}^I of dT is represented by

$$dt_{11}^I = -\underline{u}_1^{I'} d\underline{u}_1^R + \underline{u}_1^{R'} d\underline{u}_1^I$$

$$= (-\underline{u}_1^{I'}, \; \underline{u}_1^{R'}) \begin{bmatrix} d\underline{u}_1^R \\ d\underline{u}_1^I \end{bmatrix}$$

$$= (-\underline{u}_1^{I'}, \; \underline{u}_1^{R'}) D^{-1} \begin{bmatrix} d\underline{t}_1^R \\ d\underline{t}_1^I \end{bmatrix}.$$

Therefore dt_{11}^I is a function of $d\underline{t}_1^R$ and $d\underline{t}_1^I$, which proves the previous assertion.

Secondly, we see the relation between dW and $d\lambda_1$, dV and dT. Here we must also note that in dT, dT_2 can be represented by $d\underline{t}_1^R$ and $d\underline{t}_1^I$. In fact, U_2 is a functional matrix of $\underline{u}_1^R$ and $\underline{u}_1^I$, i.e.,

$$u_{k\ell} = f_{k\ell}^R(\underline{u}_1^R,\underline{u}_1^I) + i f_{k\ell}(\underline{u}_1^R,\underline{u}_1^I), \quad (1 \le k \le m, \ 2 \le \ell \le m).$$

Therefore

$$du_{k\ell} = \sum_{r=2}^{m} \frac{\partial f_{k\ell}^R}{\partial u_{r1}^R} du_{r1}^R + \sum_{r=2}^{m} \frac{\partial f_{k\ell}^R}{\partial u_{r1}^I} du_{r1}^I$$
$$+ i\left\{ \sum_{r=2}^{m} \frac{\partial f_{k\ell}^I}{\partial u_{r1}^R} du_{r1}^R + \sum_{r=2}^{m} \frac{\partial f_{k\ell}^I}{\partial u_{r1}^I} du_{r1}^R \right\} .$$

By setting

$$F_{k\ell}^R = \left(\frac{\partial f_{k\ell}}{\partial u_{21}^R}, \ldots, \frac{\partial f_{k\ell}}{\partial u_{m1}^R}\right) \text{ and } F_{k\ell}^I = \left(\frac{\partial f_{k\ell}}{\partial u_{21}^I}, \ldots, \frac{\partial f_{k\ell}}{\partial u_{m1}^I}\right),$$

where $f_{k\ell} = f_{k\ell}^R + i f_{k\ell}^I$, we have

$$dU_2 = F^R \dot{\otimes} d\underline{u}_1^R + F^I \dot{\otimes} d\underline{u}_1^I ,$$

where $F^R = (F_{k\ell}^R)_{m\times(m-1)^2}$ and $F^I = (F_{k\ell}^I)_{m\times(m-1)^2}$. The notation $\dot{\otimes}$ means: $F^R \dot{\otimes} d\underline{u}_1^R = (F_{k\ell}^R d\underline{u}_1^R)$ and $F^I \dot{\otimes} d\underline{u}_1^I = (F_{k\ell}^I d\underline{u}_1^I)$, respectively.

Therefore we have from (A1.11),

$$dT_2 = \bar{U}' dU_2 = \bar{U}'[F^R \dot{\otimes} \{(D^{-1})_{11} d\underline{t}_1^R + (D^{-1})_{12} d\underline{t}_1^I\}$$
$$+ F^I \dot{\otimes} \{(D^{-1})_{21} d\underline{t}_1^R + (D^{-1})_{22} d\underline{t}_1^I\}] ,$$

where $(D^{-1})_{ij}$ $(i,j = 1,2)$ is a submatrix of D^{-1} corresponding to $d\underline{t}_1^R$ and $d\underline{t}_1^I$, which establishes our assertion. Hence we need only $d\underline{t}_1^R$, $d\underline{t}_1^I$, dV and $d\lambda_1$.

From (A1.2),

$$\text{(A1.12)} \quad \begin{cases} d\omega_{11} = d\lambda_1 , \\ d\underline{\omega}_1^R = (\lambda_1 I - V^R) d\underline{t}_1^R + V^I d\underline{t}_1^I , \\ d\underline{\omega}_1^I = - V^I d\underline{t}_1^I + (\lambda_1 I - V^R) d\underline{t}_1^R , \\ dW_{22} = dT_{22} V + dV - V dT_{22} , \end{cases}$$

where

$$dW = \begin{bmatrix} d\omega_{11} & d\bar{\underline{\omega}}_1' \\ d\underline{\omega}_1 & dW_{22} \end{bmatrix} \quad \text{and} \quad dT = \begin{bmatrix} dt_{11} & d\underline{t}_{12} \\ d\underline{t}_1^R + i d\underline{t}_1^I & dT_{22} \end{bmatrix}$$

Hence, forming the exterior product, we have

$$d\omega_{11} \prod_{\alpha=2}^{m} d\omega_{\alpha 1}^R \prod_{\alpha=2}^{m} d\omega_{\alpha 1}^I \prod_{\alpha>\beta\geq 2} d\omega_{\alpha\beta}^R \prod_{\alpha>\beta\geq 2} d\omega_{\alpha\beta}^I$$

$$= J_2 d\lambda_1 \prod_{\alpha=2}^{m} dt_{\alpha 1}^R \prod_{\alpha=2}^{m} dt_{\alpha 1}^I \prod_{\alpha>\beta\geq 2} dv_{\alpha\beta}^R \prod_{\alpha>\beta\geq 2} dv_{\alpha\beta}^I ,$$

$$J_2 = \det \begin{bmatrix} 1 & \underline{0}' & \underline{0}' & \underline{0}' \\ \underline{0} & \lambda_1 I - V^R & V^I & 0 \\ \underline{0} & -V^I & \lambda_1 I - V^R & 0 \\ \underline{0} & * & * & I \end{bmatrix}$$

$$= \det(\lambda_1 I - V)^2 .$$

Summarizing these results, we have the Jacobian of the transformation as $J = \det(\lambda_1 I - V)^2$, which completes the proof.

APPENDIX II. (THE PROOF OF LEMMA 2.)

Differentiating both sides of (26), we have

(A2.1) $$dX = dU\Lambda\bar{L}' + Ud\Lambda\bar{L}' + U\Lambda d\bar{L}' .$$

Let $B_{n\times(n-m)}$ be a semi-unitary matrix such that $[L:B]$ is a unitary matrix. Multiplying both sides of (A2.1) by $\bar{U}'$ from the left and $[L:B]$ from the right, we have

(A2.2) $$\bar{U}'dX[L:B] = [\bar{U}'dU\Lambda:0] + [d\Lambda:0] + [\Lambda d\bar{L}'L:\Lambda d\bar{L}'B].$$

By setting $dF = [\overset{m}{dF_1}:\overset{n-m}{dF_2}]m$, $dT = \bar{U}'dU$, $dP = -\bar{L}'dL$ and $dQ = -\bar{L}'dB$, we have

(A2.3) $$dF_1 = dT\Lambda + d\Lambda - \Lambda dP$$

(A2.4) $$dF_2 = -\Lambda dQ.$$

Here we must note that dT is a skew Hermitian matrix and dT has m^2 elements. However, dT has only m^2-m independent elements since $dT = \bar{U}'dU$. Therefore we can assume that the diagonal elements $dt^I_{\alpha\alpha}$ of dT are represented by another element of dT, i.e., $dt^I_{\alpha\alpha} = h_{\alpha\alpha}(dt^R_{21}, \ldots, dt^R_{m,m-1}, dt^I_{21}, \ldots, dt^I_{m,m-1})$. Hence we have from (A2.3)

$$df^{(1)R}_{\alpha\alpha} = d\lambda_\alpha$$

$$df^{(1)I}_{\alpha\alpha} = \lambda_\alpha\{h_{\alpha\alpha}(dt^R_{21}, \ldots, dt^R_{m,m-1}, dt^I_{21}, \ldots, dt^I_{m,m-1}) - dp^I_{\alpha\alpha}\},$$

$$df^{(1)R}_{\alpha\beta} = \lambda_\beta dt^R_{\alpha\beta} - \lambda_\alpha dp^R_{\alpha\beta}, \qquad (\alpha > \beta),$$

$$df^{(1)I}_{\alpha\beta} = \lambda_\beta dt^I_{\alpha\beta} - \lambda_\alpha dt^I_{\alpha\beta}, \qquad (\alpha > \beta),$$

$$df^{(1)R}_{\beta\alpha} = \lambda_\alpha dt^R_{\beta\alpha} - \lambda_\beta dp^R_{\beta\alpha} = -\lambda_\alpha dt^R_{\alpha\beta} + \lambda_\beta dp^I_{\alpha\beta}, \qquad (\alpha > \beta),$$

$$df^{(1)I}_{\beta\alpha} = \lambda_\alpha dt^I_{\beta\alpha} - \lambda_\beta dp^I_{\beta\alpha} = \lambda_\alpha dt^I_{\alpha\beta} - \lambda_\beta dp^I_{\alpha\beta}, \qquad (\alpha > \beta),$$

since dT and dP are skew Hermitian matrices. Hence, forming the exterior product, we have

(A2.5)
$$\prod_{\alpha>\beta} df^{(1)R}_{\alpha\beta} df^{(1)R}_{\beta\alpha} \prod_{\alpha>\beta} df^{(1)I}_{\alpha\beta} df^{(1)I}_{\beta\alpha}$$

$$= \prod_{\alpha>\beta} (\lambda^2_\alpha - \lambda^2_\beta)^2 \prod_{\alpha>\beta} dp^R_{\alpha\beta}\, dt^R_{\alpha\beta} \prod_{\alpha>\beta} dp^I_{\alpha\beta}\, dt^I_{\alpha\beta}.$$

We have from (A2.4)

$$df^{(2)R}_{\alpha\beta} = -\lambda_\alpha dq^R_{\alpha\beta}, \qquad df^{(2)I}_{\alpha\beta} = -\lambda_\alpha dq^I_{\alpha\beta}.$$

Hence

$$\prod_{\beta=1}^{n-m} \prod_{\alpha=1}^{m} df^{(2)R}_{\alpha\beta} df^{(2)I}_{\alpha\beta} = \left(\prod_{\alpha=1}^{m} \lambda_\alpha\right)^{2(n-m)} \prod_{\beta=1}^{n-m} \prod_{\alpha=1}^{m} dq^R_{\alpha\beta}\, dq^I_{\alpha\beta}.$$

Therefore we have

$$dF \overset{\text{def}}{\equiv} \prod_{\beta=1}^{n} \prod_{\alpha=1}^{m} df^R_{\alpha\beta}\, df^I_{\alpha\beta} = \left(\prod_{\alpha=1}^{m} \lambda_\alpha\right)^{2(n-m)+1} \prod_{\alpha>\beta} (\lambda^2_\alpha - \lambda^2_\beta)^2 \cdot$$

$$\cdot \prod_{\alpha=1}^{m} d\lambda_\alpha \prod_{\alpha>\beta} dt^R_{\alpha\beta}\, dt^I_{\alpha\beta} \prod_{\alpha>\beta} dp^R_{\alpha\beta}\, dp^I_{\alpha\beta} \cdot$$

$$\cdot \prod_{\alpha=1}^{m} dp^I_{\alpha\alpha} \prod_{\beta=1}^{n-m} \prod_{\alpha=1}^{m} dq^R_{\alpha\beta}\, dq^I_{\alpha\beta} .$$

Since

$$\prod_{\beta=1}^{n} \prod_{\alpha=1}^{m} dx^R_{\alpha\beta} dx^I_{\alpha\beta} = \prod_{\beta=1}^{n} \prod_{\alpha=1}^{m} df^R_{\alpha\beta} df^I_{\alpha\beta},$$

$\Pi_{\alpha>\beta} dt^R_{\alpha\beta} dt^I_{\alpha\beta}$ is an invariant measure on a unitary group, and

$$\prod_{\alpha=1}^{m} dp^I_{\alpha\alpha} \prod_{\alpha>\beta} dp^R_{\alpha\beta} dp^I_{\alpha\beta} \prod_{\beta=1}^{n-m} \prod_{\alpha=1}^{m} dq^R_{\alpha\beta} dq^I_{\alpha\beta}$$

is an invariant measure of a semi-unitary matrix group, we have finally

$$\text{(A2.6)} \qquad dX \overset{\text{def}}{\equiv} \prod_{\beta=1}^{n} \prod_{\alpha=1}^{m} dx^R_{\alpha\beta} dx^I_{\alpha\beta} = (\det\Lambda)^{2(n-m)+1} \prod_{\alpha>\beta} (\lambda^2_\alpha - \lambda^2_\beta)^2 \cdot$$

$$\cdot\, d\Lambda d^*(U) d^*(L) ,$$

(putting $d\Lambda = \Pi^m_{\alpha=1} d\lambda_\alpha$, $d^*(U) = \Pi_{\alpha>\beta} dt^R_{\alpha\beta} dt^I_{\alpha\beta}$ and $d^*(L) = \Pi^m_{\alpha=1} dp^I_{\alpha\alpha}\, \Pi_{\alpha>\beta} \cdot dp^R_{\alpha\beta} dp^I_{\alpha\beta} \Pi^{n-m}_{\beta=1} \Pi^m_{\alpha=1} dq^R_{\alpha\beta} dq^I_{\alpha\beta}$). We also note that

$$\int_{U(m)} d^*(U) = \frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(m)} \quad \text{and} \quad \int_{\bar{L}'L=I_m} d^*(L) = \frac{\pi^{mn}}{\tilde{\Gamma}_m(n)} .$$

Q.E.D.

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