This research was partially supported by the National Science Foundation under grant No. GU-2059 and the U.S. Air Force Office of Scientific Research under grant No. AFOSR-68-1415.

A CONDITION EQUIVALENT TO FERROMAGNETISM FOR A GENERALIZED ISING MODEL

by

DOUGLAS G. KELLY
Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 668

FEBRUARY 1970

A CONDITION EQUIVALENT TO FERROMAGNETISM FOR A GENERALIZED ISING MODEL

DOUGLAS G. KELLY

ABSTRACT

In the real algebra $R(G)$ over a certain finite group, the operator exp is defined. A condition is stated on exp J which is necessary and sufficient for J in $R(G)$ to be nonnegative (where J is viewed as a function $G \to R$). Physically, this amounts to a condition on the correlations of a generalized system of Ising spins which is necessary and sufficient for the ferromagnetism of the system.

KEY WORDS AND PHRASES

Ising ferromagnet, correlations, real group algebra.

A CONDITION EQUIVALENT TO FERROMAGNETISM FOR A GENERALIZED ISING MODEL

DOUGLAS G. KELLY

INTRODUCTION. Let $N = \{1,\ldots,n\}$ and let $\{\sigma_1,\ldots,\sigma_n\}$ be a generalized system of Ising spins with Hamiltonian

$$H = - \sum_{A \subseteq N} J(A)\sigma^A \tag{1}$$

where, for any subset A of N,

$$\sigma^A = \prod_{i \in A} \sigma_i \tag{2}$$

and $J(A) \in R$ (the real numbers) is the many-body potential. The system is called *ferromagnetic* if $J(A) \geq 0$ for each nonempty $A \subseteq N$. In reference [2], hereafter referred to as (KS), the following problem is stated: Find conditions on the correlations $\pi(A) = \langle \sigma^A \rangle$ which are necessary and sufficient for ferromagnetism. In that paper, a condition equivalent to ferromagnetism is given in terms of the Fourier transform of the function π; Theorem 1 below restates this condition. In this note, we translate that into a condition on π itself, stated in terms of a sum of products $\pi(B_1)\ldots\pi(B_k)$, each product bearing a coefficient equal to the permanent of a certain matrix. (The result is given as Theorem 2 below.)

This research was partially supported by the National Science Foundation under grant No. GU-2059 and the U.S. Air Force Office of Scientific Research under grant No. AFOSR-68-1415.

ABSTRACT FORMULATION. In (KS) it is shown that the numbers $J(A)$ and $\pi(A) = \langle\sigma^A\rangle$ are naturally associated with a certain real group algebra, as follows.

Let G denote the group $(2^N, \Delta)$ of subsets of N under symmetric difference. (The identity is the empty set $\emptyset$, and each member of G is its own inverse.) The real group algebra $\mathcal{R}(G)$ will be viewed here as the vector space of all functions $G \to \mathcal{R}$, with multiplication given by convolution:

$$(f*g)(A) = \sum_{B \in G} f(B)g(A \Delta B). \tag{3}$$

The multiplicative identity is the function δ for which $\delta(\emptyset) = 1$ and $\delta(A) = 0$ when $A \neq \emptyset$.

Define the operator exp on $\mathcal{R}(G)$ by

$$\exp f = \delta + f + \frac{f*f}{2!} + \frac{f*f*f}{3!} + \dots \ . \tag{4}$$

(It was shown in Section 11 of (KS) that exp is well-defined; that is, that the series in (4) converges when applied to any $A \in G$.)

Now if the "many-body potential" $J(A)$ is viewed as a member J of $\mathcal{R}(G)$, then it happens that

$$\langle\sigma^A\rangle = \frac{(\exp J)(A)}{(\exp J)(\emptyset)} \tag{5}$$

((KS), equation (11.13)); moreover, judicious choice of $J(\emptyset)$ (whose value does not affect the physical nature of the system) will guarantee $(\exp J)(\emptyset) = 1$.

Thus the problem stated in the introduction is reduced to the following: Given $J \in \mathcal{R}(G)$, find conditions on $\exp J$ which are necessary and sufficient for $J(A)$ to be nonnegative for each nonempty $A \in G$.

STATEMENT OF THE RESULT. For any two subsets A and B of N, define

$$\sigma_B^A = \sigma_A^B = (-1)^{|A\cap B|}, \tag{6}$$

where $|R|$ denotes the cardinality of R. The identity

$$\sigma_C^{A\Delta B} = \sigma_C^A \sigma_C^B \tag{7}$$

is easily checked. Define, for subsets A and B of N,

$$A^+ = \{C\subseteq N:\ \sigma_A^C = 1\},$$

$$A^- = \{C\subseteq N:\ \sigma_A^C = -1\},$$

$$A^+B^+ = \{C\subseteq N:\ \sigma_A^C = 1 \text{ and } \sigma_B^C = 1\},$$

and similarly for A^+B^-, A^-B^+, A^-B^-.

LEMMA 1. (i) If $A \neq \emptyset$, then $|A^+| = |A^-| = 2^{n-1}$.
(ii) If A and B are unequal, nonempty subsets of N, then each of the four families A^+B^+, A^+B^-, A^-B^+, A^-B^- has 2^{n-2} members.

(This lemma appears in Section 3 of (KS).) Note that A^+ and A^+B^+ are subgroups of G and each of the other families is a coset. Thus we have

LEMMA 2. If $B \in A^-$, then $A^- = \{E\Delta B:\ E\in A^+\}$.

The characters of G are the functions σ_A: $B \mapsto \sigma_A^B$; and so the Fourier transform of $f \in \mathcal{R}(G)$ is given by

$$\hat{f}(A) = \sum_{B\in G} \sigma_B^A\, f(B). \tag{8}$$

Now let $J \in \mathcal{R}(G)$ and define, for $A \in G$,

$$\pi(A) = (\exp J)(A). \tag{9}$$

From Theorem 12.1 of (KS), from the Proof of 12.1, and from the statement following the Proof, can be gleaned

THEOREM 1. For $R \neq \emptyset$, $J(R) \geq 0$ if and only if

$$\prod_{E \in R^+} \hat{\pi}(E) \geq \prod_{F \in R^-} \hat{\pi}(F). \tag{10}$$

To interpret (10) in terms of the function π, we adopt the following notations:

Fix $R \in G$, $R \neq \emptyset$; let $R^+ = \{E_1,\ldots,E_k\}$. By Lemma 1, $k = 2^{n-1}$; let $R^- = \{F_1,\ldots,F_k\}$. For any unordered k-tuple $\mu = \{B_1,\ldots,B_k\}$ of (not necessarily distinct) members of G, define

$$\Delta\mu = B_1 \Delta \cdots \Delta B_k \tag{11}$$

and let $\mu!$ be the number of permutations ψ of $\{1,\ldots,k\}$ for which $\{B_{\psi 1},\ldots,B_{\psi k}\} = \{B_1,\ldots,B_k\}$. For example, if $k = 8$ and $\mu = \{A,A,A,B,B,C,D,E\}$, then $\Delta\mu = A\Delta C\Delta D\Delta E$ and $\mu! = 3!2!1!1!1!$. Also define the $k \times k$ matrix of ± 1's

$$M_{\mu,R} = \left|\sigma_{B_i}^{E_j}\right|. \tag{12}$$

THEOREM 2. $J(R) \geq 0$ if and only if

$$\sum_{\Delta\mu = R} \frac{1}{\mu!} \operatorname{per}(M_{\mu,R})\pi(B_1) \cdots \pi(B_k) \geq 0, \tag{13}$$

the sum extending over unordered k-tuples $\mu = \{B_1,\ldots,B_k\}$ of members of G satisfying $\Delta\mu = R$.

Proof of Theorem 2. With the notation we have established, (10) is equivalent to $E_R - O_R \geq 0$, where

$$\left.\begin{aligned} E_R &= \prod_{i=1}^{k} \sum_{B_i \in G} \sigma_{B_i}^{E_i} \pi(B_i) \\ &= \sum_{B_1 \in G} \cdots \sum_{B_k \in G} \sigma_{B_1}^{E_1} \cdots \sigma_{B_k}^{E_k} \pi(B_1) \cdots \pi(B_k). \end{aligned}\right\} \tag{14}$$

Similarly

$$O_R = \sum_{B_1 \in G} \cdots \sum_{B_k \in G} \sigma_{B_1}^{F_1} \cdots \sigma_{B_k}^{F_k} \pi(B_1) \cdots \pi(B_k). \tag{15}$$

Now (14) can be rewritten

$$E_R = \sum_{\mu} A_\mu \pi(B_1) \cdots \pi(B_k), \tag{16}$$

the sum extending over all unordered k-tuples $\mu = \{B_1,\ldots,B_k\}$, where

$$A_\mu = \sum_{\phi} \sigma_{\phi B_1}^{E_1} \cdots \sigma_{\phi B_k}^{E_k}. \tag{17}$$

Here ϕ ranges over all distinguishable permutations of $\{B_1,\ldots,B_k\}$. Similarly,

$$O_R = \sum_{\mu} B_\mu \, \pi(B_1) \cdots \pi(B_k) \tag{18}$$

where

$$B_\mu = \sum_{\phi} \sigma_{\phi B_1}^{F_1} \cdots \sigma_{\phi B_k}^{F_k}. \tag{19}$$

Now Lemma 2 says that for any $i = 1,\ldots,k$,

$$\{F_1,\ldots,F_k\} = \{E_1\Delta F_i,\ldots,E_k\Delta F_i\} \tag{20}$$

Thus for each $i = 1,\ldots,k$,

$$\left.\begin{aligned} B_\mu &= \sum_\phi \sigma_{\phi B_1}^{E_1\Delta F_i} \cdots \sigma_{\phi B_k}^{E_k\Delta F_i} \\ &= \sum_\phi \sigma_{\Delta\mu}^{F_i}\, \sigma_{\phi B_1}^{E_1} \cdots \sigma_{\phi B_k}^{E_k} = \sigma_{\Delta\mu}^{F_i}\, A_\mu . \end{aligned}\right\} \tag{21}$$

That is, we have

$$B_\mu = \sigma_{\Delta\mu}^{F_1} A_\mu = \sigma_{\Delta\mu}^{F_2} A_\mu = \cdots = \sigma_{\Delta\mu}^{F_k} A_\mu . \tag{22}$$

From (22) it follows that

(i) If $\Delta\mu = \emptyset$, then $B_\mu = A_\mu$;

(ii) If $\Delta\mu = R$, then $B_\mu = -A_\mu$.

If $\Delta\mu = A$ where $A \neq \emptyset$ and $A \neq R$, then Lemma 1 implies that exactly half of the numbers $\sigma_A^{F_1},\ldots,\sigma_A^{F_k}$ are $+1$ and the other half are -1; so

(iii) If $\Delta\mu \neq \emptyset$ or R, then $B_\mu = A_\mu = 0$.

Combining (16), (18), and (i) - (iii) above, we see that $E_R - 0_R \geq 0$ if and only if

$$\sum_{\Delta\mu = R} A_\mu \pi(B_1) \cdots \pi(B_k) \geq 0. \tag{23}$$

Finally we note that if instead of (17) we form, for a fixed μ,

$$\sum_p \sigma_{B_{p1}}^{E_1} \cdots \sigma_{B_{pk}}^{E_k} , \tag{24}$$

the sum extending over all permutations of $\{1,\ldots,k\}$ rather than over distinguishable permutations of $\{B_1,\ldots,B_k\}$, we obtain $\mu!A\mu$. But (24) is exactly $\text{per}(M_{\mu,R})$; thus Theorem 2 is proved.

Remarks. For the case $n = 2$, condition (13) reduces to the Griffiths inequalities (see [1] and (KS)). That is, if we denote $\{1\}$ by A and $\{2\}$ by B, then Theorem 2 gives

$$J(A) \geq 0 \quad \text{iff} \quad \pi(\emptyset)\pi(A) \geq \pi(B)\pi(A\Delta B) \ ;$$

$$J(B) \geq 0 \quad \text{iff} \quad \pi(\emptyset)\pi(B) \geq \pi(A)\pi(A\Delta B) \ ;$$

$$J(N) \geq 0 \quad \text{iff} \quad \pi(\emptyset)\pi(N) \geq \pi(A)\pi(B).$$

For $n \geq 3$, of course, (13) is not only unlike the Griffiths inequalities, it is also quite unwieldy. The evaluation of permanents of matrices of ± 1's is a subject that seems to have received scant attention.

References

1. R. B. Griffiths: Correlations in Ising Ferromagnets. I and II. *J. Math. Phys.* 8 (1967) 478-489.

2. D. G. Kelly and S. Sherman: General Griffiths' Inequalities on Correlations in Ising Ferromagnets. *J. Math. Phys.* 9 (1968) 466-484.