* The research described in this report was supported by the National Science Foundation under Grant GU-2059 and the U.S.A.F. under Contract AFOSR-68-1415.

A MORE PRECISE ESTIMATE OF THE COST OF NOT KNOWING THE VARIANCE WHEN MAKING A FIXED-WIDTH CONFIDENCE INTERVAL FOR THE MEAN

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Institute of Statistics Mimeo Series No. 677

April 1970

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1. Introduction.

Let $X_1, X_2, \ldots$ be a sequence of independent normal random variables with unknown mean, μ, and unknown variance, σ^2. In order to obtain a confidence interval for μ of fixed width d $(d > 0)$ and prescribed coverage probability α $(0 < \alpha < 1)$, a procedure originally suggested by Starr has been developed by Gordon Simons [3]. (For further references, see [1], [4], [5].) This procedure is based on the following sequential stopping rule:

(1) $$N = \text{smallest index } n \geq n_0 \geq 3 \text{ for which } n \geq \frac{a^2 S_n^2}{d^2}$$

where

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du \qquad \alpha = 2\Phi(a) - 1.$$

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Simons has proved that there exists a finite integer k such that if after nominally stopping according to (1), k more observations are taken, then

$$\Pr\{|\bar{X}_{N+k}-\mu| < d\} \geq \alpha \quad \text{for all} \quad \mu, \sigma^2, \quad \text{and} \quad d.$$

This note is concerned with an investigation of the value of k.

2. Method.

In order to evaluate k, the probability distribution of N must be obtained. However, the problem of finding this distribution is relatively intractable, so we modify the stopping rule N in the following way:

(2) N^* = smallest odd integer $n = 2m+1 \geq 2m_0+1 \geq 3$ for which $S_{2m+1} \leq \frac{(2m+1)a^2}{d^2}$.

Note that $N^* \geq N$. Hence the corresponding finite integer k^* such that

(3) $\Pr\{|X_{N^*+k^*}-\mu| < d\} \geq \alpha$ for all μ, σ^2, and d,

will be no larger than k (c.f. (26) of [3]). That is, if the value of k^* for (2) is found, it may underestimate the value of k for (1).

Rewriting (2), we get

(4) N^* = smallest integer $n = 2m+1 \geq 2m_0+1 \geq 3$ for which

$$\frac{1}{2m} \sum_{i=1}^{2m+1} (X_i-\bar{X}_n)^2 \leq \frac{(2m+1)d^2}{a^2}$$

At this point, we follow Robbins [2]. The joint distribution of the sequence of random variables $\{S_n^2\}$ $(n = 2,3,\ldots)$ is the same as that of the sequence $\{\sigma^2(y_1^2+\ldots+y_{n-1}^2)/(n-1)\}$, where the y_i are independent normal $(0,1)$ random variables. Thus (4) is equivalent to

(5) $N^* =$ smallest integer $n = 2m+1 \geq 2m_0+1 \geq 3$ for which

$$\frac{\sigma^2}{2m}\sum_{i=1}^{2m} y_i^2 \leq \frac{(2m+1)d^2}{a^2}, \quad \text{where } y_i \sim N(0,1)$$

for $i = 1,\ldots,2m$. And by noting that if y_1 and y_2 are independent $N(0,1)$, then $z_1 = \frac{1}{2}(y_1^2+y_2^2) \sim e^{-z}$, we get in analogy with (5)

(6) $N^* =$ smallest integer $n = 2m+1 \geq 2m_0+1 \geq 3$ for which

$$\sum_{i=1}^{m} z_i \leq \frac{(2m+1)mr^2}{a^2}, \quad \text{where } z_i \sim e^{-z} \text{ for } i = 1,\ldots,m,$$

$r = d/\sigma$.

Or, the probability that $N^* = 2m+1$ is the same as the probability that m is the smallest integer for which (6) is satisfied. Robbins has accordingly constructed an iterative scheme for determining $P\{N^*=2m+1\}$, making it possible to compute

$$P(r,x) = \sum_{m=m_0}^{M} \Phi(r\sqrt{2m+1+x})P\{N^*=2m+1\}.$$

(With M large, $2P(d/\sigma,k^*)-1$ closely approximates the probability in (3). An adequate truncation point for our range of computations was $M=[C]+10$, $C = a^2/r^2$; i.e., $P\{N^* \geq 2[C]+21\} \leq .0001$.)

3. Discussion.

For fixed value of r, we let

$$h(r) = \text{smallest integer } i \text{ such that } P(r,i) \geq \alpha$$

and

$$k(r) = \inf\{x: P(r,x) \geq \alpha\}.$$

Note that $h(r)-1 < k(r) \leq h(r)$. Linear interpolation was used to approximate $k(r)$ by computing the coverage probability at $h(r)-1$ and $h(r)$. Graphs Ia and Ib show a plot of C vs $k(r)$, where we obtained $k(r)$ for $r = .2, .5\ (.05)$ and $r = 0.5, 1.0\ (.10)$.

Although it seems extremely difficult to prove the unimodality of $k(r)$, our computations certainly support this character. If this is true, then the appropriate number of additional observations, k^*, for the cases considered is listed in the following table.

m_0	$\alpha = 0.95$	$\alpha = 0.98$
2	8	10
3	6	8
4	6	7
5	5	6

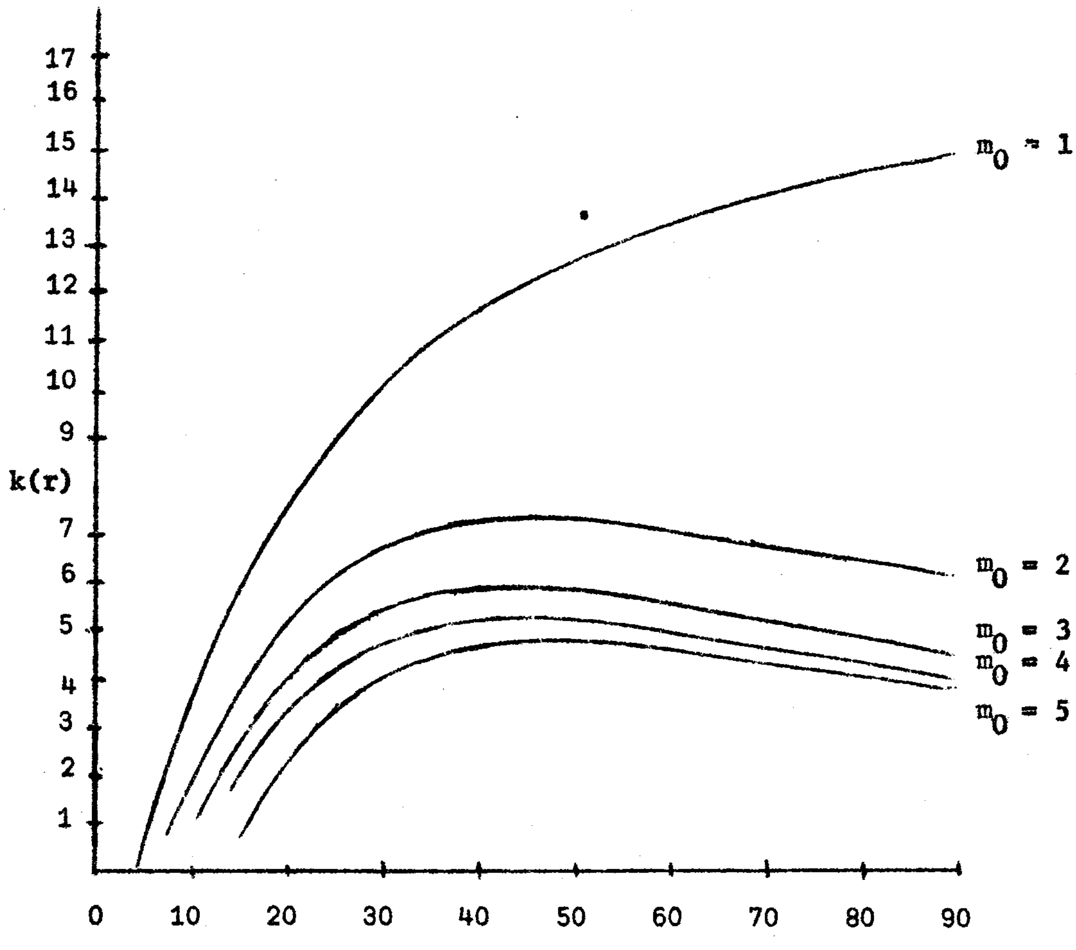
The mode for $m_0 = 1$ occurs at $C = \infty$ or at some point outside our computable range.

If σ^2 were known, then the confidence interval $[\bar{X}_n-d, \bar{X}_n+d]$ for μ of prescribed coverage probability α is assured provided $n \geq C$. Hence, in order to measure the cost due to ignorance of σ^2, we plotted C vs $EN^* + k^* - C$. Graphs IIa and IIb show the plots for the cases we considered. Due to the discreteness of k^*, in graph IIa the plot for $m_0=4$ lies entirely above the plot for $m_0=3$. It is interesting to note that the plots in set II remarkably support one's intuition.

Ia

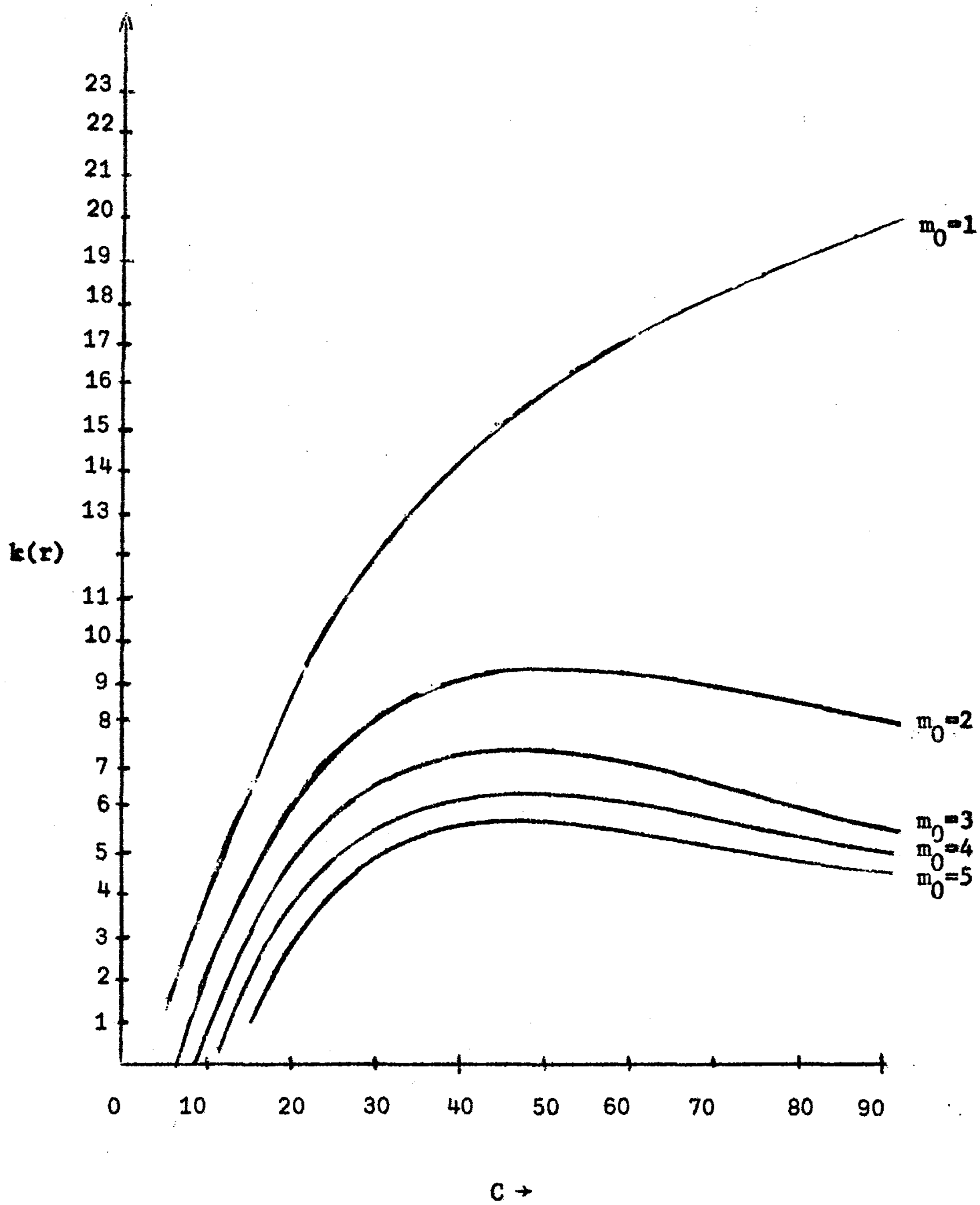
$\alpha = .95$

C versus k(r)



Ib α = .98

C versus k(r)



Tables of values obtained for graphs Ia and Ib.

$\alpha = .95$

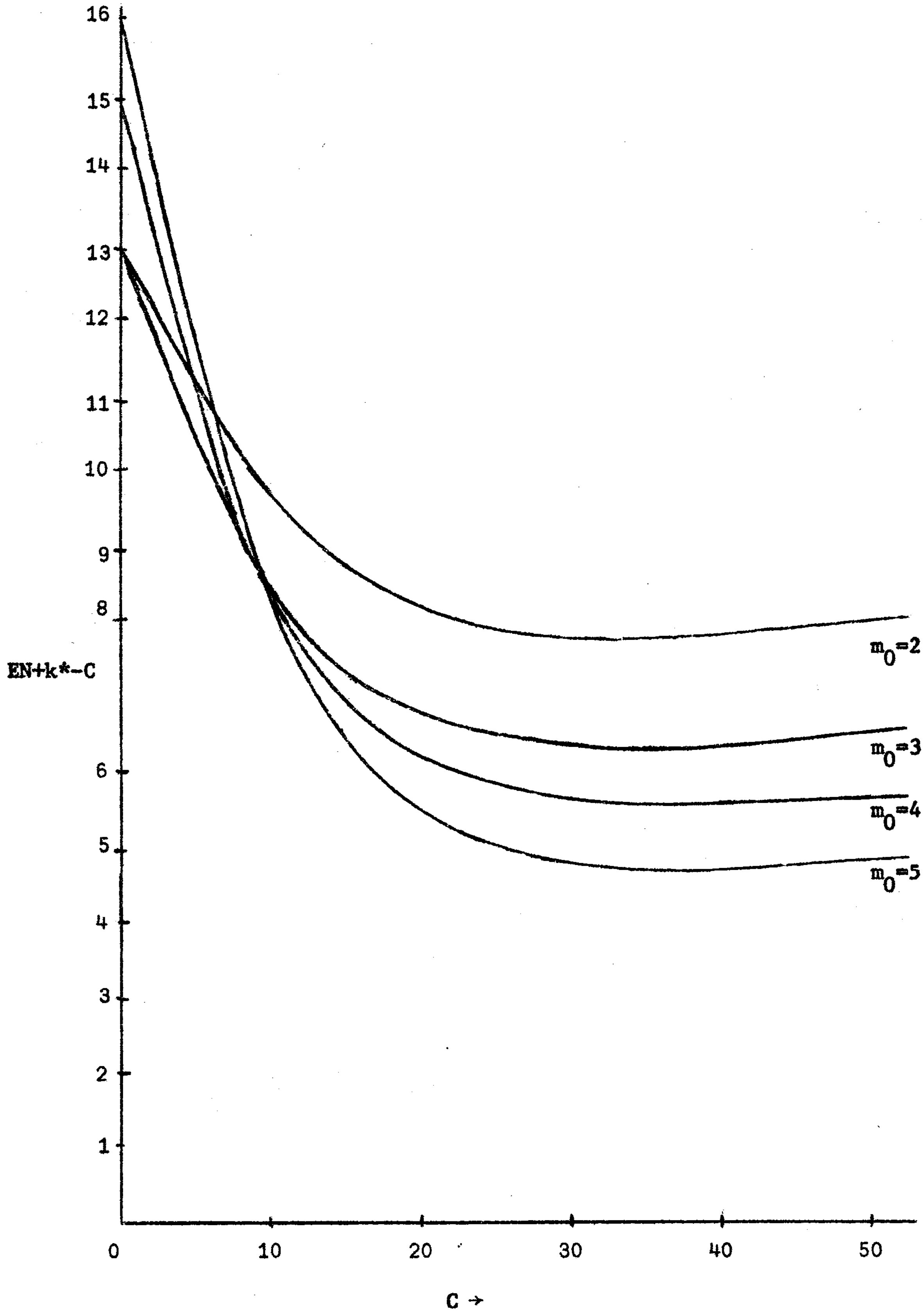
C	3.84	4.74	6.00	7.84	10.67	15.37	18.97	24.01	31.36	42.68	61.47	96.04
$m_0=1$	0.12	0.72	1.48	2.50	3.91	5.95	7.26	8.78	10.43	12.87		
$m_0=2$	—	—	—	0.90	2.23	3.96	4.97	6.02	6.92	7.39	7.02	5.93
$m_0=3$	—	—	—	—	1.00	2.80	3.79	4.79	5.58	5.87	5.35	4.26
$m_0=4$	—	—	—	—	—	1.83	2.92	3.97	4.82	5.15	4.73	3.82
$m_0=5$	—	—	—	—	—	0.84	2.12	3.33	4.29	4.74	4.43	3.66

C	75.88	120.44	170.74
$m_0=1$	14.31	15.93	16.84

$\alpha = .98$

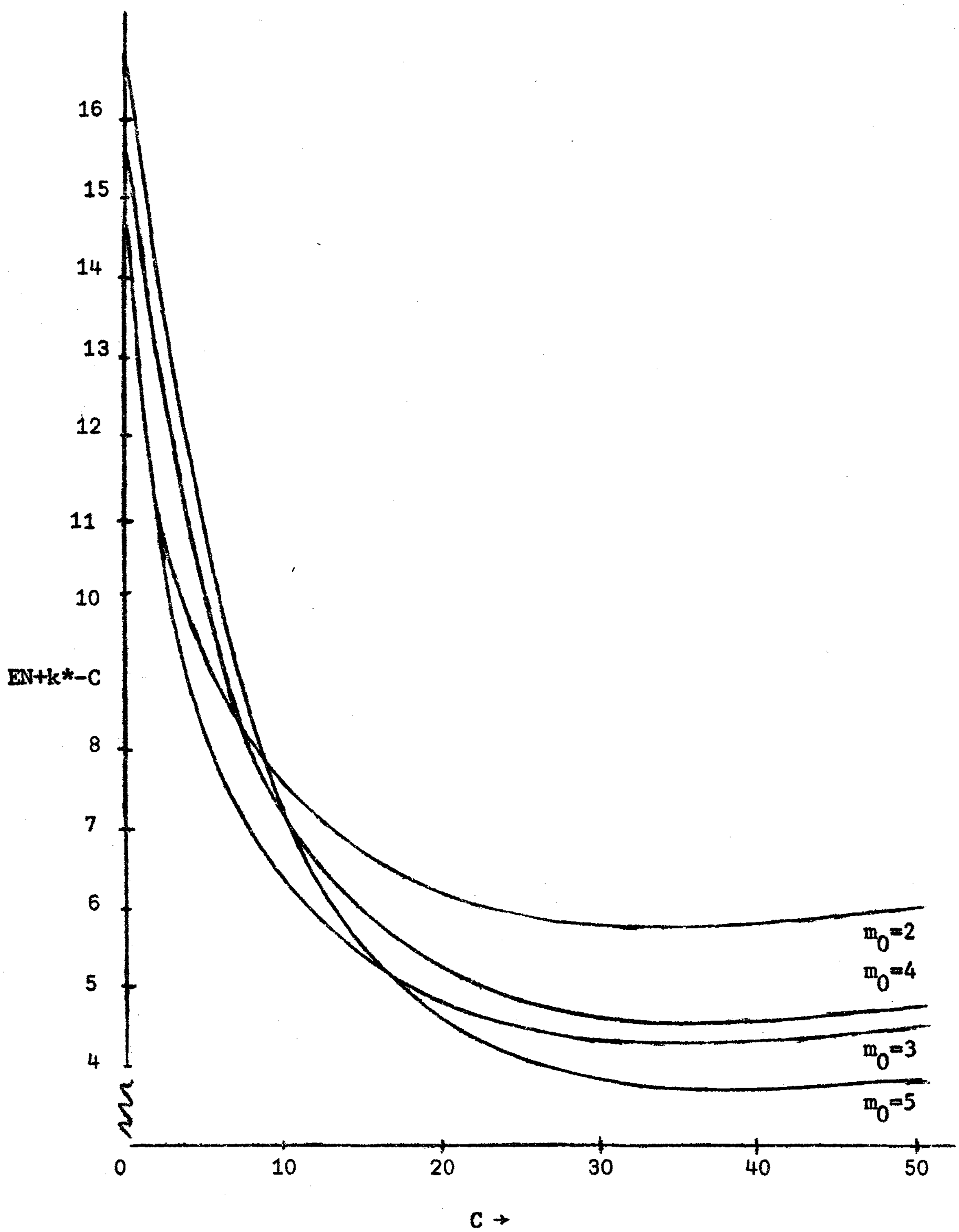
C	5.43	6.70	8.48	11.08	15.08	21.71	26.81	33.93	44.32	60.32	86.86	135.72
$m_0=1$	1.33	2.10	3.17	4.63	6.62	9.34	11.05	12.95	15.03	17.21	19.57	22.50
$m_0=2$	—	0.50	1.50	2.80	4.50	6.60	7.73	8.70	9.31	9.19	8.25	6.87
$m_0=3$	—	—	0.09	1.50	3.20	5.21	6.22	7.00	7.34	6.88	5.69	4.45
$m_0=4$	—	—	—	0.23	2.10	4.24	5.29	6.07	6.42	5.99	4.95	3.94
$m_0=5$	—	—	—	—	1.03	3.41	4.57	5.45	5.86	5.56	4.65	3.81

IIa $\alpha = .95$


16
15
14
13
12
11
10
9
8
EN+k*-C
6
5
4
3
2
1
0
10
20
30
40
50
C →
m_0=2
m_0=3
m_0=4
m_0=5

IIb

$\alpha = .98$



Tables of values obtained for graphs IIa and IIb

$\alpha = .95$

C	0	5.43	6.70	8.48	11.08	15.08	21.71	26.81	33.93	44.32	60.32	86.86	135.72
$m_0=2$	13	11.13	10.65	10.08	9.44	8.73	8.07	7.85	7.77	7.89	8.21	8.63	8.96
$m_0=3$	13	10.31	9.67	8.95	8.18	7.38	6.67	6.42	6.32	6.41	6.68	7.02	7.22
$m_0=4$	15	10.83	9.94	8.96	7.94	6.92	6.06	5.76	5.60	5.63	5.85	6.13	6.33
$m_0=5$	16	11.63	10.53	9.24	7.85	6.52	5.41	5.02	4.80	4.77	4.93	5.17	5.35

$\alpha = .98$

C	0	3.84	4.74	6.00	7.84	10.67	15.37	18.97	24.01	31.36	42.68	61.47	96.04
$m_0=2$	15	9.87	9.43	8.90	8.27	7.53	6.69	6.29	5.95	5.78	5.86	6.23	6.73
$m_0=3$	15	9.37	8.74	8.01	7.19	6.29	5.34	4.90	4.54	4.34	4.39	4.70	5.10
$m_0=4$	16	11.19	10.39	9.40	8.28	7.07	5.88	5.34	4.90	4.64	4.62	4.87	5.19
$m_0=5$	17	12.16	11.28	10.12	8.67	7.04	5.46	4.76	4.20	3.85	3.76	3.95	4.23

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