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THE L_1 NORM OF THE APPROXIMATION ERROR FOR BERNSTEIN-TYPE POLYNOMIALS*

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1. *Introduction and statement of results.* This paper is concerned with the estimation of the L_1 norm of the difference between a function of bounded variation and an associated Bernstein polynomial, and with the analogous problem for a Lebesgue integrable function of bounded variation inside $(0,1)$. A real-valued function defined in the open interval $(0,1)$ is said to be of bounded variation inside $(0,1)$ if it is of bounded variation in every closed sub-interval of $(0,1)$. The class of these functions will be denoted by BV^*. To formulate some of the results we state the following lemma, which is a simple consequence of the well-known canonical representation of a function of bounded variation.

LEMMA 1. The function f is in BV^* if and only if it can be represented as $f = f_1 - f_2$, where f_1 and f_2 are nondecreasing real-valued functions on $(0,1)$. Moreover, if $f \in BV^*$, the functions f_1 and f_2

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can be so chosen that, for $0 < x < y < 1$, the total variation of f on $[x,y]$ is the sum of the total variations of f_1 and f_2 on $[x,y]$, implying

$$(1) \qquad f = f_1 - f_2, \quad d|f| = df_1 + df_2, \quad f_1, f_2 \text{ nondecreasing.}$$

If f is finite in the closed interval $[0,1]$, the associated Bernstein polynomial of order n, denoted in operator notation by $B_n f$, is defined by

$$(2) \qquad B_n f(x) = \sum_{i=0}^{n} f(i/n) p_{n,i}(x),$$

where

$$(3) \qquad p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}.$$

For f Lebesgue integrable on $(0,1)$, we shall use the modified Bernstein polynomials $P_n f(x) = \frac{d}{dx} B_{n+1} F(x)$, where $F(x) = \int_0^x f(y)dy$. Explicitly (see Lorentz [1], Chapter II),

$$(4) \qquad P_n f(x) = \sum_{i=0}^{n} (n+1) \int_{i/(n+1)}^{(i+1)/(n+1)} f(y)dy \; p_{n,i}(x).$$

For $f \in BV^*$, let

$$(5) \qquad J(f) = \int_0^1 x^{\frac{1}{2}} (1-x)^{\frac{1}{2}} \, d|f(x)|.$$

If f is represented in the form (1), we have $J(f) = J(f_1) + J(f_2)$. If f is nondecreasing,

$$(6) \qquad J(f) = \int_0^1 x^{\frac{1}{2}} (1-x)^{\frac{1}{2}} df(x) = \int_0^1 f(x)(x-\tfrac{1}{2}) x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} dx.$$

The following four theorems will be proved.

THEOREM 1. *If f is a Lebesgue integrable function of bounded variation inside $(0,1)$, then*

$$(7) \qquad \int_0^1 |P_n f(x) - f(x)|\,dx \;\le\; C_n J(f),$$

where

$$(8) \qquad C_n = 2^{\frac{1}{2}}(n+\tfrac{1}{2})^{n+\frac{1}{2}}(n+1)^{-n-1} < (2/e)^{\frac{1}{2}} n^{-\frac{1}{2}}.$$

The sign of equality in (7) holds if and only if f is constant in each of the intervals $(0,a)$ and $(a,1)$, where $a = \frac{1}{2}(n+1)^{-1}$ or $a = 1-\frac{1}{2}(n+1)^{-1}$.

THEOREM 2. *Let f be a step function with finitely many steps in every closed sub-interval of $(0,1)$, and such that the functions f_1 and f_2 in representation (1) are Lebesgue integrable. Then*

$$(9) \qquad \lim_{n\to\infty} n^{\frac{1}{2}} \int_0^1 |P_n f(x) - f(x)|\,dx = (2/\pi)^{\frac{1}{2}} J(f),$$

irrespective of whether $J(f)$ is finite or infinite.

Theorem 1 shows that the finiteness of $J(f)$ is sufficient for the L_1 norm of the approximation error to be of order $n^{-\frac{1}{2}}$. Theorem 2 implies that the latter is guaranteed only if $J(f)$ is finite when no restrictions beyond $f \in BV^*$ are imposed. It also shows that the upper bound in (7), with the numerical constant $(2/e)^{\frac{1}{2}}$ reduced to $(2/\pi)^{\frac{1}{2}}$, is asymptotically attained for every fixed step function of the specified type.

If f is nondecreasing, the condition $J(f) < \infty$ is stronger, but not much stronger than square integrability of f. Explicitly, it can be shown that if f is nondecreasing, $J(f) < \infty$ implies $\int_0^1 f^2(x)\,dx < \infty$ (but not conversely), and that $\int_0^1 f^2(x)\{\mathcal{L}og(1+|f(x)|)\}^{2+\delta}dx < \infty$ for some $\delta > 0$

implies $J(f) < \infty$. If f is nondecreasing and square integrable, we have for $n \geq 2$

$$(10) \qquad \int_0^1 |P_n f(x) - f(x)|dx \leq C(n^{-1}\log n)^{\frac{1}{2}}\left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}},$$

where C is an absolute constant. The proof of (10) is sketched at the end of Section 2.

If f is convex, (10) is true with $\log n$ removed (as can be shown by means of Jensen's inequality).

Concerning the Bernstein polynomials (2), Theorem 1 immediately implies the following. If F is the difference of two bounded, convex, absolutely continuous functions on $[0,1]$ and $J(F')$ is finite, then $var_{[0,1]}(B_nF-F) = O(n^{-\frac{1}{2}})$. (I am indebted to Professor G.G. Lorentz for this observation.) We also have the following two analogs of Theorems 1 and 2.

THEOREM 3. Let f be of bounded variation in $[0,1]$. Then

$$(11) \qquad \int_0^1 |B_n f(x) - f(x)|dx \leq C_n J(f) + (n+1)^{-1}\, var_{[0,1]}(f),$$

where C_n is given by (8).

THEOREM 4. Let f be a step function of bounded variation on $[0,1]$ with finitely many steps in every closed sub-interval of $(0,1)$. Then (9) holds true with P_n replaced by B_n.

The upper bound in (11) can not be replaced by $C\, n^{-\frac{1}{2}}J(f)$ with C an absolute constant, as the following example shows. Let $f(x) = b$ if $0 \leq x < a_n$, $f(x) = c$ if $a_n < x \leq 1$, where $a_n = o(n^{-1})$. By a simple calculation

$$\int_0^1 |B_n f - f|\,dx = |b-c|n^{-1}(1+o(1)), \qquad J(f) = |b-c|\, a_n^{\frac{1}{2}}(1+o(1)).$$

Hence $n^{\frac{1}{2}}\int_0^1 |B_n f - f|\,dx/J(f) \sim (na_n)^{-\frac{1}{2}} \to \infty$.

2. *Proof of Theorem 1.* The modified Bernstein polynomial defined by (4) may be written in the form

(12) $$P_n f(x) = \int_0^1 K_n(x,y)\, f(y)\, dy,$$

where

(13) $$K_n(x,y) = (n+1)\, p_{n,[(n+1)y]}(x)$$

and $[u]$ denotes the largest integer contained in u. We note that

(14) $$\int_0^1 K_n(x,y)\, dy = 1, \qquad \int_0^1 K_n(x,y)\, dx = 1.$$

Let

(15) $$H_n(x,u) = \int_u^1 K_n(x,y)\, dy.$$

A simple calculation shows that

(16) $$H_n(x,u) = \delta_n(u) G_{n,[(n+1)u]+1}(x) + (1-\delta_n(u)) G_{n,[(n+1)u]}(x),$$

where

(17) $$\delta_n(u) = (n+1)u - [(n+1)u],$$

(18) $$G_{n,k}(x) = \sum_{i=k}^{n} p_{n,i}(x) = n\int_0^x p_{n-1,k-1}(t)\, dt, \quad k = 1,\ldots,n\ ,$$

and $G_{n,0}(x) = 1$, $G_{n,n+1}(x) = 0$ for $0 < x < 1$.

Let $x \in (0,1)$ be a continuity point of f. We have from (12) and (14)

$$P_nf(x) - f(x) = \int_0^1 K_n(x,y)\{f(y)-f(x)\}dy$$

$$= -\int_0^x K_n(x,y) \int_y^x df(u)dy + \int_x^1 K_n(x,y) \int_x^y df(u)dy$$

$$= -\int_0^x \int_0^u K_n(x,y)\ dy\ df(u) + \int_x^1 \int_u^1 K_n(x,y)dy\ df(u)\ .$$

Since $\int_0^u K_n(x,y)dy = 1-H_n(x,u)$, we thus have

(19) $$P_nf(x) - f(x) = -\int_0^x (1-H_n(x,u))df(u) + \int_x^1 H_n(x,u)df(u).$$

Hence

$$\int_0^1 |P_nf(x) - f(x)|dx \le \int_0^1 \int_0^x (1-H_n(x,u))d|f(u)|dx + \int_0^1 \int_x^1 H_n(x,u)d|f(u)|dx$$

(20)

$$= \int_0^1 D_n(u)d|f(u)|,$$

where

(21) $$D_n(u) = \int_u^1 (1-H_n(x,u))dx + \int_0^u H_n(x,u)dx.$$

Therefore,

(22) $$\int_0^1 |P_nf(x) - f(x)|dx \le C_nJ(f),$$

where

(23) $$C_n = \sup_{0 \le u \le 1} u^{-\frac{1}{2}} (1-u)^{-\frac{1}{2}} D_n(u).$$

From (15) and (14), it is easily seen that

(24) $$D_n(u) = 2\int_0^u H_n(x,u)dx.$$

We now show that

$$(25)\qquad D_n(u) = 2u(1-u)\,p_{n,[(n+1)u]}(u).$$

For $k \le (n+1)u < k+1$ $(k = 0,1,\ldots,n)$, we have $[(n+1)u] = k$, $1-\delta_n(u) = k+1-(n+1)u$, and, by (16) and (18),

$$H_n(x,u) = G_{n,k+1}(x) + (k+1-(n+1)u)p_{n,k}(x).$$

Hence it is sufficient to show that the function

$$g(u) = \int_0^u G_{n,k+1}(x)dx + (k+1-(n+1)u)\int_0^u p_{n,k}(x)dx - u(1-u)p_{n,k}(u)$$

is identically zero.

It is easy to verify the identities

$$u(1-u)p'_{n,k}(u) = (k-nu)p_{n,k}(u),$$

$$up'_{n,k}(u) = np_{n,k}(u) - np_{n-1,k}(u)\ .$$

Hence

$$g'(u) = G_{n,k+1}(u) - (n+1)\int_0^u p_{n,k}(x)dx + (k+1-(n+1)u)\ p_{n,k}(u)$$

$$- (1-2u)\ p_{n,k}(u) - u(1-u)\ p'_{n,k}(u)$$

$$= G_{n,k+1}(u) - (n+1)\int_0^u p_{n,k}(x)dx + up_{n,k}(u),$$

$$g''(u) = np_{n-1,k}(u) - (n+1)p_{n,k}(u) + p_{n,k}(u) + up'_{n,k}(u).$$

Thus $g''(u) = 0$, and since $g(0) = g'(0) = 0$, identity (25) is proved.

For $[(n+1)u] = k$ fixed, $u^{\frac12}(1-u)^{\frac12}p_{n,k}(u) = \binom{n}{k}u^{k+\frac12}(1-u)^{n-k+\frac12}$ attains its maximum at $u = (k+\frac12)/(n+1)$. Hence, by (25),

$$(26)\qquad u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}D_n(u) = 2u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}p_{n,k}(u)$$

$$\le 2\binom{n}{k}(k+\tfrac{1}{2})^{k+\frac{1}{2}}(n-k+\tfrac{1}{2})^{n-k+\frac{1}{2}}(n+1)^{-n-1}$$

$$= c_n(k), \quad \text{say.}$$

Now

$$\frac{c_n(k+1)}{c_n(k)} = \frac{n-k}{k+1}\,\frac{(k+\frac{3}{2})^{k+\frac{3}{2}}(n-k-\frac{1}{2})^{n-k-\frac{1}{2}}}{(k+\frac{1}{2})^{k+\frac{1}{2}}(n-k+\frac{1}{2})^{n-k+\frac{1}{2}}} = \frac{F(k)}{F(n-k-1)},$$

where

$$F(k) = (k+\tfrac{3}{2})^{k+\frac{3}{2}}(k+1)^{-1}(k+\tfrac{1}{2})^{-k-\frac{1}{2}}.$$

It is readily seen that

$$\frac{d}{dk}\log F(k) = \log(k+\tfrac{3}{2}) - \log(k+\tfrac{1}{2}) - (k+1)^{-1}$$

is strictly positive for $k \ge 0$. Hence $F(k)$ is strictly increasing. Therefore

$$(27)\qquad \max_{0\le k\le n} c_n(k) = c_n(0) = c_n(n) = 2^{\frac{1}{2}}(n+\tfrac{1}{2})^{n+\frac{1}{2}}(n+1)^{-n-1}.$$

Also, the left-hand side of (26) is equal to $c_n(0)$ if and only if $u = \frac{1}{2}(n+1)^{-1}$ or $u = 1-\frac{1}{2}(n+1)^{-1}$. Thus C_n, as defined by (23), is equal to the expression in (27). By (20), equality in (22) can hold only if f takes two values and the saltus is at $\frac{1}{2}(n+1)^{-1}$ or $1-\frac{1}{2}(n+1)^{-1}$. A direct calculation shows that equality does hold in this case.

The inequality in (8) is easily verified, completing the proof.

We now indicate the proof of inequality (10). It has been shown that $D_n(u) \le (2/e)^{\frac{1}{2}}n^{-\frac{1}{2}}u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}$. Hence if f is nondecreasing,

$$\int_{\varepsilon}^{1-\varepsilon} D_n(u)\,d|f(u)| \quad \le \quad (2e)^{\frac{1}{2}}\, n^{-\frac{1}{2}} \int_{\varepsilon}^{1-\varepsilon} u^{\frac{1}{2}}\,(1-u)^{\frac{1}{2}}\, df(u) \ .$$

Integration by parts and application of Schwarz's inequality shows that the right-hand side does not exceed

$$Cn^{-\frac{1}{2}}\ (\log\frac{1-\varepsilon}{\varepsilon})^{\frac{1}{2}}\ (\int_0^1 f^2(u)du)^{\frac{1}{2}}$$

for $0 < \varepsilon \le 1/3$. If we set $\varepsilon = (n+1)^{-1}$, the remaining contribution to $\int_0^1 D_n(u)d|f(u)|$ is of smaller order of magnitude, and (10) follows from (20).

3. *Proof of Theorem 2.* For convenience of notation, the proof will be given for a step function f with finitely many steps in every interval $(0,\delta)$ with $\delta < 1$. In the general case of Theorem 2, the proof requires only trivial modifications. It is irrelevant how f is defined at its points of discontinuity, and we may assume that

$$\begin{aligned} &f(x) = b_j \quad \text{if} \quad a_{j-1} < x \le a_j, \quad j = 1,2,\dots\ , \\ &0 = a_0 < a_1 < \dots\ , \qquad \lim_{j\to\infty} a_j = 1. \end{aligned} \tag{28}$$

Let

$$\begin{aligned} &I_n(x,u) = H_n(x,u) - 1 \quad \text{if} \quad 0 < u \le x < 1, \\ &I_n(x,u) = H_n(x,u) \quad \text{if} \quad 0 < x < u < 1\ , \end{aligned} \tag{29}$$

and let m be a fixed positive integer. By (19), if $x\in(0,1)$ is a continuity point of f,

$$P_n f(x) - f(x) = \sum_{i=1}^{m} I_n(x,a_i)(b_{i+1}-b_i) + \int_{a_{m+1}-}^{1} I_n(x,u)\,df(u).$$

Hence

$$(30) \qquad \int_0^{a_m} |P_n f(x)-f(x)|\,dx = A_n + \theta R_n, \quad |\theta| \le 1,$$

where

$$(31) \qquad A_n = \int_0^{a_m} \left| \sum_{i=1}^{m} I_n(x,a_i)(b_{i+1}-b_i) \right| dx,$$

$$(32) \qquad R_n = \int_{a_{m+1}-}^{1} \int_0^{a_m} |I_n(x,u)|\,dx\,d|f(u)|.$$

From (16) and (18) we obtain by straightforward calculation

$$\int_0^1 (x-u)^2\, d_x H_n(x,u) \le 3u(1-u)n^{-1}, \quad 0 \le u \le 1.$$

Hence if $0 < x < u$,

$$H_n(x,u) \le (u-x)^{-2} \int_0^x (u-y)^2\, d_y H_n(y,u) \le 3u(1-u)(u-x)^{-2}\, n^{-1}.$$

For $u < x < 1$, we have the same upper bound for $1-H_n(x,u)$, so that

$$(33) \qquad |I_n(x,u)| \le 3u(1-u)(u-x)^{-2}\, n^{-1}, \quad 0 < x, \quad u < 1.$$

From (32) and (33), we have

$$R_n \le 3(a_{m+1}-a_m)^{-2}\, n^{-1} \int_{0+}^{1} (1-u)\,d|f(u)| .$$

The last integral is finite since f_1 and f_2 in (1) are Lebesgue integrable. Hence $R_n = O(n^{-1})$ and

$$(34) \qquad \int_0^{a_m} |P_n f(x) - f(x)|\,dx = A_n + O(n^{-1}).$$

Let $a_{j-1} < x < (a_{j-1}+a_j)/2$. Then, by (33), $I_n(x,a_i) = O(n^{-1})$ if $i \neq j-1$, uniformly in x for $i = 1,\ldots,m$. Hence

$$(35) \qquad \sum_{i=1}^{m} I_n(x,a_i)(b_{i+1}-b_i) = (H_n(x,a_{j-1})-1)(b_j-b_{j-1}) + O(n^{-1})$$

$$\text{if} \qquad a_{j-1} < x < \frac{a_{j-1}+a_j}{2}$$

for $j = 1,\ldots,m$, uniformly for $x \in (0,a_m)$. (For $j = 1$, the first term on the right is zero.)

In a similar way, it is seen that

$$(36) \qquad \sum_{i=1}^{m} I_n(x,a_i)(b_{i+1}-b_i) = H_n(x,a_j)(b_{j+1}-b_j) + O(n^{-1})$$

$$\text{if} \qquad \frac{a_{j-1}+a_j}{2} < x < a_j$$

for $j = 1,\ldots,m$, uniformly for $x \in (0,a_m)$.

It follows that

$$(37) \qquad A_n = \sum_{j=2}^{m} |b_j-b_{j-1}| \int_{a_{j-1}}^{(a_{j-1}+a_j)/2} (1-H_n(x,a_{j-1}))dx$$

$$+ \sum_{j=1}^{m} |b_{j+1}-b_j| \int_{(a_{j-1}+a_j)/2}^{a_j} H_n(x,a_j)dx + O(n^{-1}).$$

Another application of (33) shows that if the upper limits of integration in the first sum in (37) are replaced by 1 and the lower limits in the second sum by 0, then a term of order n^{-1} is added. Hence

$$(38) \qquad A_n = \sum_{j=1}^{m-1} |b_{j+1}-b_j| \left\{ \int_{a_j}^{1} (1-H_n(x,a_j))dx + \int_0^{a_j} H_n(x,a_j)dx \right\}$$

$$+ |b_{m+1}-b_m| \int_0^{a_m} H_n(x,a_m)dx + O(n^{-1}).$$

With (34), (21) and (24), we thus have

$$(39)\qquad \int_0^{a_m} |P_n f(x)-f(x)|\,dx = \sum_{j=1}^{m-1} D_n(a_j)|b_{j+1}-b_j| + \tfrac{1}{2}D_n(a_m)|b_{m+1}-b_m| + O(n^{-1}).$$

For $u\epsilon(0,1)$ fixed, we have by (25) and Stirling's formula

$$D_n(u) = (2/\pi)^{\frac{1}{2}}\, n^{-\frac{1}{2}}\, u^{\frac{1}{2}}\,(1-u)^{\frac{1}{2}} + o(n^{-\frac{1}{2}}).$$

Inserting this expression in (39) and recalling (28), we obtain

$$(40)\qquad \int_0^{a_m} |P_n f(x)-f(x)|\,dx = (2/\pi)^{\frac{1}{2}}\, n^{-\frac{1}{2}} \left\{\int_0^{a_{m-1}^+} x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} d|f(x)| + \tfrac{1}{2}\int_{x=a_m} x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} d|f(x)|\right\} + o(n^{-\frac{1}{2}}).$$

If $J(f) = \infty$, the integral on the right side of (40) may be made as large as we please by choice of m, and (9) is proved in this case.

Let $J(f) < \infty$. Given a positive ε, choose $\eta = \eta(\varepsilon)\epsilon(0,1)$ so that $\int_\eta^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} d|f(u)| < \varepsilon$. Let $f_1(x) = f(x)$ if $0 < x \le \eta$, $f_1(x) = f(\eta)$ if $\eta < x < 1$, and let $f_2(x) = f(x)-f_1(x)$. Then

$$J(f) = J(f_1) + J(f_2), \qquad J(f_2) < \varepsilon,$$

and $\int_0^1 |P_n f-f|\,dx$ differs from $\int_0^1 |P_n f_1-f_1|\,dx$ by at most $\int_0^1 |P_n f_2-f_2|\,dx$. Since f_1 has finitely many steps, (40) with $f = f_1$ implies

$$\lim_{n\to\infty} n^{\frac{1}{2}} \int_0^1 |P_n f_1 - f_1|\,dx = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} J(f_1)\ .$$

By Theorem 1,

$$n^{\frac{1}{2}} \int_0^1 |P_n f_2 - f_2| dx \le (2/e)^{\frac{1}{2}} J(f_2) < (2/e)^{\frac{1}{2}} \varepsilon.$$

Since ε is arbitrary, these facts imply that (9) holds if $J(f) < \infty$. The proof is complete.

4. *Proof of Theorems 3 and 4.* Since a function of bounded variation can be represented in the form (1), and $\int |B_n f - f| dx \le \Sigma_{i=1}^2 \int |B_n f_i - f_i| dx$, we may assume, in the proof of Theorem 3, that f is a nondecreasing function of bounded variation. By (4),

$$P_n f(x) = \sum_{i=0}^{n} c_{n,i}\, p_{n,i}(x), \quad c_{n,i} = (n+1) \int_{i/(n+1)}^{(i+1)/(n+1)} f(y) dy.$$

Since f is nondecreasing,

$$f(\tfrac{i}{n+1}) \le c_{n,i} \le f(\tfrac{i+1}{n+1}), \quad f(\tfrac{i}{n+1}) \le f(\tfrac{i}{n}) \le f(\tfrac{i+1}{n+1}), \qquad i = 0,\ldots,n.$$

Hence $|c_{n,i} - f(\frac{i}{n})| \le f(\frac{i+1}{n+1}) - f(\frac{i}{n+1})$ and therefore

$$\text{(41)} \qquad \int_0^1 |P_n f(x) - B_n f(x)| dx \le \sum_{i=0}^{n} \int_0^1 |c_{n,i} - f(\tfrac{i}{n})| p_{n,i}(x) dx$$

$$\le \sum_{i=0}^{n} \{f(\tfrac{i+1}{n+1}) - f(\tfrac{i}{n})\} (n+1)^{-1}$$

$$= \mathit{var}_{[0,1]}(f)(n+1)^{-1}.$$

Inequality (11) now follows from

$$\int_0^1 |B_n f(x) - f(x)|dx \le \int_0^1 |P_n f(x) - f(x)|dx + \int_0^1 |B_n f(x) - P_n f(x)|dx$$

and Theorem 1.

The conditions of Theorem 4 imply those of Theorem 2, and Theorem 4 follows from (9) and (41).

Reference

1. Lorentz, G.G., *Bernstein Polynomials*, University of Toronto Press, Toronto, 1953.