ON BOUNDED LENGTH CONFIDENCE INTERVAL FOR THE REGRESSION COEFFICIENT BASED ON A CLASS OF RANK STATISTICS

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SUMMARY. Bounded length (sequential) confidence intervals for the regression coefficient (in a simple linear regression model) based on a class of robust rank statistics are considered here, and their various asymptotic properties are studied. In this context, several strong convergence results on some simple and weighted empirical processes as well as on a class of rank order processes are established. Comparison with an alternative procedure based on the least squares estimators is also made.

1. INTRODUCTION

For the simple linear regression model $X_i = \beta_0+\beta c_i+e_i$, $i=1,2,\ldots$, where the c_i are known regression constants and the e_i are independent and identically distributed random variables (iidrv) with an absolutely continuous (unknown) cumulative distribution function (cdf) $F(x)$, defined on the real line $(-\infty,\infty)$, we desire to provide a robust confidence interval (of confidence coefficient $1-\alpha$, $0<\alpha<1$) for the regression coefficient β, such that the length of this confidence interval is bounded above by $2d$, for some specified $d>0$. The proposed procedure rests on the use of a class of regression rank statistics [due to Hájek (1962, 1968)] for the derivation of robust confidence intervals for β [cf. Sen (1969, section 4)], as extended here to the sequential case along the lines of Chow and Robbins (1965).

Several results, needed for this sequential extension, are derived here. First an elegant result of Jurečková (1969) on the weak convergence of a class of

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rank order processes to some appropriate linear processes is strengthened here to almost sure (a.s.) convergence under more stringent regularity conditions on the c_i and the score-functions underlying the rank statistics. This result along with a martingale (or semi-martingale) property of the regression rank statistics, deduced in section 4, guarantees the "asymptotic (as $d \to 0$) consistency" and "efficiency" [see Chow and Robbins (1965)] of the proposed sequential procedure, and also enables one to study its asymptotic relative efficiency (ARE) with respect to the procedure by Gleser (1965) and Albert (1966), which are based on the least squares estimators. In this context, several well-known rank statistics are considered and the allied ARE results are briefly presented. In particular, for the so called normal scores statistic, it is shown that the ARE is bounded below by 1, uniformly in a broad class of $\{F\}$.

2. NOTATIONS, ASSUMPTIONS AND THE MAIN THEOREM

In accordance with the model of section 1, consider a sequence $\{X_1, X_2, \ldots\}$ of independent random variables for which

$$F_i(x) = P\{X_i \leq x\} = F(x - \beta_0 - \beta c_i),\ i=1,2,\ldots, \tag{2.1}$$

where β_0 is a nuisance parameter. We intend to determine a confidence interval $I_n = \{\beta:\ \hat{\beta}_{L,n} \leq \beta \leq \hat{\beta}_{U,n}\}$ (where $\hat{\beta}_{L,n}$ and $\hat{\beta}_{U,n}$ are sample statistics) such that

$$P\{\beta \varepsilon I_n\} = 1-\alpha, \text{ the preassigned confidence coefficient,} \tag{2.2}$$

$$0 \leq \hat{\beta}_{U,n} - \hat{\beta}_{L,n} \leq 2d, \text{ for some predetermined } d(>0). \tag{2.3}$$

Since F is not known, no fixed-sample size procedure sounds valid for all F. It is therefore desired to determine sequentially a stopping variable N (a positive integer) and the corresponding $(\hat{\beta}_{L,N}, \hat{\beta}_{U,N})$, such that (2.2) and (2.3) hold. Our proposed procedure is based on the following class of regression rank statistics.

In a sample $\underset{\sim}{X}_n = (X_1,\ldots,X_n)$ of size $n(\geq 1)$, let $R_{ni} = \sum_{j=1}^{n} u(X_i - X_j)$ [where $u(t)$ is 1 or 0 according as $t \geq$ or < 0] be the rank of X_i, $1 \leq i \leq n$. Let

$$c^*_{ni} = (c_i - \bar{c}_n)/C_n,\ 1 \leq i \leq n, \text{ where } \bar{c}_n = n^{-1} \sum_1^n c_i \text{ and } C_n^2 = \sum_1^n (c_i - \bar{c}_n)^2. \tag{2.4}$$

Then, as in Hájek (1962, 1968), a regression rank statistic is defined as

$$T_n = T(\underset{\sim}{X}_n) = \sum_{i=1}^{n} c^*_{ni} J_n(R_{ni}/(n+1)), \tag{2.5}$$

where the "scores" $J_n(i/(n+1))$, $1 \leq i \leq n$, are generated by a "score-function" $\{J(u):\ 0<u<1\}$ in the following manner. We let $J_n(u) = J_n(\frac{i}{n+1})$, for $(i-1)/n < u \leq i/n$, $1 \leq i \leq n$, and define

$$J_n(\tfrac{i}{n+1}) \text{ as equal to } J(\tfrac{i}{n+1}) \text{ or } EJ(U_{ni}),\ 1 \leq i \leq n, \tag{2.6}$$

where $U_{n1} < \ldots < U_{nn}$ are the n ordered random variables in a sample of size n from the rectangular (0,1) distribution. The score-function $J(u)$ is defined as $\Psi^{-1}(u)$: $0<u<1$, where $\Psi(x)$ is an absolutely continuous cdf satisfying the two conditions

$$\text{(i)}\quad M(t) = \int_{-\infty}^{\infty} e^{tx} d\Psi(x) < \infty \text{ for all } 0<t\leq t_o, \tag{2.7}$$

$$\text{(ii)}\quad \lim_{u\to 0} \{\psi[\Psi^{-1}(u)]/u,\ \psi[\Psi^{-1}(1-u)]/(1-u)\} \geq K>0, \tag{2.8}$$

where $\psi(x) = d\Psi(x)/dx$. Thus, we have $M(t) = \int_0^1 e^{tJ(u)} du < \infty$ for all $0<t\leq t_o$, and by (2.8),

$$|J(u)| \leq K_o[-\log u(1-u)],\ |J'(u)| \leq K_o[u(1-u)]^{-1},\ 0<u<1;\ K_o<\infty. \tag{2.9}$$

These assumptions appear to be more restrictive than those in Hájek (1962, 1968) and Jurečková (1969) [where the weak convergence results are only studied], but are somewhat needed for our purpose, as we require, in the sequel, to establish certain a.s. convergence results where the exponential bounds [to follow from (2.9)] are quite useful. It may be noted that (2.7) and (2.8) hold for the entire class of normal, logistic, double-exponential, exponential, rectangular and many other cdf's. In fact, if we use $J(u) = u$ [i.e., Ψ rectangular cdf] or $J(u)$ as the inverse of the standard normal cdf, we obtain respectively the Wilcoxon and the normal scores; the corresponding T_n are termed the Wilcoxon and the normal scores regression rank statistics. For both of these, (2.7) and (2.8) hold.

Regarding $\{\underset{\sim}{c}_n\}$ [where $\underset{\sim}{c}_n = (c_1,\ldots,c_n)$], we make the following assumptions, where the first one is due to Hájek (1968):

$$\text{(i)} \quad \max_{1\leq i\leq n} |c^*_{ni}| = O(n^{-\frac{1}{2}}), \tag{2.10}$$

$$\text{(ii)} \quad \liminf_n n^{-1} C_n^2 \geq C_o^2 > 0, \tag{2.11}$$

(iii) $C_n^2 = Q(n)$ is a strictly increasing function of n, such that for every a>0,

$$\lim_{n\to\infty} Q(a_n n)/Q(n) = s(a) \text{ whenever } \lim_{n\to\infty} a_n = a, \tag{2.12}$$

where s(a) is strictly monotone (increasing) with s(1)=1. The condition (2.10) is again more stringent than the classical Noether-condition [cf. Hájek (1962, 1968)], but is satisfied in the majority of practical situations. (2.11) is less restrictive than the parallel condition: $\lim_{n\to\infty} n^{-1}C_n^2 = C_o^2 > 0$, made by Gleser (1965) and Albert (1966) in connection with least squares theory. For example, if $c_i = a+(i-1)h$, $h>0$, $i=1,2,\ldots$, (2.10) and (2.11) hold but not the Gleser-condition. Also, in this case, $Q(n) = nh^2(n^2-1)/12$, so that $s(a) = a^3$.

Finally, regarding the cdf F in (2.1), we assume that $F \in \mathfrak{F}(\Psi)$, where $\mathfrak{F}(\Psi)$ is the case of all absolutely continuous F for which the density function f(x) and its first derivative f'(x) are bounded for almost all x(a•a•x) and further

$$\lim_{x\to\pm\infty} f(x)\, J'[F(x)] \text{ are bounded.} \tag{2.13}$$

From (2.1), it follows that under H_o: $\beta=0$, implying that $X_1,\ldots, X_n$ are iidrv, $T(\underset{\sim}{X}_n)$ has a completely specified distribution generated by the n! equally likely realizations of $(R_{n1},\ldots, R_{nn})$ over the permutations of $(1,\ldots, n)$. Hence, there exists two known constants $T_n^{(1)}$ and $T_n^{(2)}$ (depending on $\underset{\sim}{c}_n$), and an α_n (known) such that

$$P\{T_n^{(1)} \leq T(\underset{\sim}{X}_n) \leq T_n^{(2)} | H_o\} = 1-\alpha_n (\to 1-\alpha \text{ as } n\to\infty) \tag{2.14}$$

[For small n, α_n may not be equal to α]. If we let

$$A_n^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} [J_n(\tfrac{i}{n+1})-\bar{J}_n]^2;\ \bar{J}_n = \frac{1}{n}\sum_{i=1}^{n} J_n(\tfrac{i}{n+1}), \tag{2.15}$$

then from the classical Wald-Wolfowitz-Noether-Hoeffding-Hájek permutational central limit theorem, (cf. [8, p. 160]), we have

$$\lim_{n\to\infty} \{T_n^{(2)}/A_n\} = \lim_{n\to\infty} \{-T_n^{(1)}/A_n\} = \tau_{\alpha/2}, \tag{2.16}$$

where $\Phi(\tau_\varepsilon) = 1-\varepsilon$ and $\Phi(x)$ is the standard normal cdf.

It follows from Sen (1969, Section 6) that $T(\underset{\sim}{X}_n - a\cdot\underset{\sim}{c}_n)$ is $\downarrow$ in a: $-\infty<a<\infty$. Hence, if we let

$$\hat{\beta}_{L,n} = \text{Sup}\{a:\ T(\underset{\sim}{X}_n - a\cdot\underset{\sim}{c}_n) > T_n^{(2)}\},\ \hat{\beta}_{U,n} = \inf\{a:\ T(\underset{\sim}{X}_n - a\cdot\underset{\sim}{c}_n) < T_n^{(1)}\}, \tag{2.17}$$

it follows as in Sen (1969, Section 4) that

$$P\{\hat{\beta}_{L,n} \leq \beta \leq \hat{\beta}_{U,n} | \beta\} = 1-\alpha_n (\to 1-\alpha \text{ as } n\to\infty). \tag{2.18}$$

We are now in a position to define our sequential procedure. For every $d>0$, let $N(d)$, the stopping variable, be the smallest positive integer [$\geq n_o$, an initial sample size (≥ 3)] for which $\hat{\beta}_{U,N(d)}-\hat{\beta}_{L,N(d)}\leq 2d$. Then our proposed confidence interval for β is $I_{N(d)} = \{\beta:\ \hat{\beta}_{L,N(d)}\leq\beta<\hat{\beta}_{U,N(d)}\}$ and is based on the stopping variable $N(d)$. We justify the proposed procedure on the ground of its robustness (for outliers or gross errors etc) and its asymptotic properties considered in the following theorem, sketched in the same fashion as in Chow and Robbins (1965).

Theorem 2.1. Under the assumptions made above, $N(d)$ is a non-increasing function of $d(>0)$, it is finite a.s., $EN(d) <\infty$ for all $d>0$, $\lim_{d\to 0} N(d) = \infty$ a.s., and $\lim_{d\to 0} EN(d) = \infty$. Further,

$$\lim_{d\to 0} N(d)/Q^{-1}(\nu(d)) = 1 \text{ a.s.,} \tag{2.19}$$

$$\lim_{d\to 0} P\{\beta\varepsilon I_{N(d)}\} = 1-\alpha \text{ for all } F\varepsilon\ \mathfrak{F}(\Psi), \tag{2.20}$$

$$\lim_{d\to 0} [EN(d)]/Q^{-1}(\nu(d)) = 1, \tag{2.21}$$

where

$$\nu(d) = [\{A\tau_{\alpha/2}\}|\{dB(F)\}]^2,\ B(F) = \int_{-\infty}^{\infty} (d/dx)J[F(x)]dF(x), \tag{2.22}$$

$$A^2 = \int_0^1 J^2(u)du-\mu^2 \text{ and } \mu = \int_0^1 J(u)du. \tag{2.23}$$

The proof of the theorem is postponed to section 5; certain other results needed in this context and having importance of their own are derived in the next two sections.

3. ASYMPTOTIC BEHAVIOUR OF SOME EMPIRICAL PROCESSES

It has been shown by Jurečková (1969) that under H_o: $\beta=0$, for all real and finite b, denoting $W(\underset{\sim}{X}_n,b) = T(\underset{\sim}{X}_n)-T(\underset{\sim}{X}_n-b\underset{\sim}{c}^*_n)-bB(F)$,

$$|W(\underset{\sim}{X}_n, b)| \xrightarrow{p} 0 \text{ as } n\to\infty; \tag{3.1}$$

our primary concern is to strengthen this statement to almost sure convergence, specifying the order of convergence as well as extending the domain of b to $|b| \leq C(\log n)^k$, $k\geq 1$, $0<C<\infty$. We define now

$$H_n(x;b) = \frac{1}{n+1}\sum_1^n u(x+bc^*_{ni}-X_i), \quad S^*_n(x;b) = \sum_1^n c^*_{ni}u(x+bc^*_{ni}-X_i), \tag{3.2}$$

for $-\infty<x<\infty$ and $-\infty<b<\infty$. Then, by (2.5) and (3.2), we have

$$T(\underset{\sim}{X}_n - b\underset{\sim}{c}^*_n) = \int_{-\infty}^{\infty} J_n[H_n(x;b)]dS^*_n(x;b). \tag{3.3}$$

It will be convenient for us to study first the asymptotic behaviour of the two processes in (3.2).

Let $\{Y_1, Y_2, \ldots\}$ be a sequence of iidrv having the rectangular (0,1) distribution. For every t: $0<t<1$, define

$$d_{ni}(t;b) = F(F^{-1}(t)+bc^*_{ni})-t, \quad 1\leq i\leq n, \quad -\infty<b<\infty. \tag{3.4}$$

It follows that for every i: $1\leq i\leq n$ and b: $-\infty<b<\infty$,

(i) $0<t+d_{ni}(t;b)\leq 1$; $t+d_{ni}(t;b)$ is $\uparrow$ in t: $0<t<1$, (3.5)

(ii) $d_{ni}(t;b)$ and bc^*_{ni} have the same sign, (3.6)

(iii) $\bar{d}_n(t;b) = n^{-1}\sum_1^n d_{ni}(t,b) = b^2[O(n^{-1})] \;\forall\, F\in\mathfrak{F}(\Psi)$. (3.7)

Consider then the two stochastic processes

$$G^*_n(t;b) = \sum_1^n c^*_{ni}u(t+d_{ni}(t;b)-Y_i) - \sum_1^n c^*_{ni}d_{ni}(t;b), \quad -\infty<b<\infty, \quad 0<t<1; \tag{3.8}$$

$$L_n^*(t;b) = (n+1)^{-\frac{1}{2}} \sum_1^n u(t+d_{ni}(t;b)-Y_i), \quad -\infty<b<\infty, \quad 0<t<1. \tag{3.9}$$

Simple computations yield that

$$\zeta_n(t;b) = \sum_{i=1}^n c_{ni}^* d_{ni}(t;b) = bf(F^{-1}(t)) + b^2[O(n^{-\frac{1}{2}})]; \tag{3.10}$$

$$G_n^*(t;b) = S_n^*(F^{-1}(t);b) - bf(F^{-1}(t)) + b^2[O(n^{-\frac{1}{2}})]; \tag{3.11}$$

$$L_n^*(t;b) = (n+1)^{\frac{1}{2}} H_n(F^{-1}(t);b), \quad 0<t<1, \quad -\infty<b<\infty. \tag{3.12}$$

Theorem 3.1. For every $h(>0)$, there exists two positive constants (K_1,K_2) and an n^* (which all may depend on h), such that for $n\geq n^*$,

$$P\{\sup_{0<t<1} \sup_{b\in I_n^*} |G_n^*(t;b)-G_n^*(t;0)| \geq K_1 n^{-\delta}(\log n)^k\} \leq K_2 n^{-h}, \tag{3.13}$$

$$P\{\sup_{0<t<1} \sup_{b\in I_n^*} |L_n^*(t;b)-L_n^*(t;0)| \geq K_1 n^{-\delta}(\log n)^k\} \leq K_2 n^{-h}, \tag{3.14}$$

where $\delta>0$, $k\geq 1$ and $I_n^* = \{b: |b| \leq C(\log n)^k, 0<C<\infty\}$.

Proof. Let $a_n = C(\log n)^k$, $r_n = [n^{\delta_1}]$ and $\eta_{r,n} = ra_n/r_n$, $r=0,\pm1,\ldots,\pm r_n$, where $0<\delta<\delta_1<\frac{1}{4}$ and $[s]$ denotes the integral part of $s(>0)$. Then, $\eta_{r+1,n}-\eta_{r,n} = O(n^{-\delta_1}(\log n)^k)$, for all $r(=-r_n,\ldots,r_n-1)$. Also, let $s_n = [n^{\frac{1}{2}+\delta_1}]$ and $\xi_{s,n} = s/s_n$, $s=0,1,\ldots,s_n$. Now, for $b\in[\eta_{r,n},\eta_{r+1,n}]$, we have

$$\begin{aligned} c_{ni}^*[u(t+d_{ni}(t;\eta_{r,n})-Y_i)-u(t-Y_i)] &\leq c_{ni}^*[u(t+d_{ni}(t;b)-Y_i)-u(t-Y_i)] \\ &\leq c_{ni}^*[u(t+d_{ni}(t;\eta_{r+1,n})-Y_i)-u(t-Y_i)], \quad 1\leq i\leq n, \end{aligned} \tag{3.15}$$

$$\zeta_n(t;\eta_{r,n}) \leq \zeta_n(t;b) \leq \zeta_n(t;\eta_{r+1,n}) \tag{3.16}$$

$$\zeta_n(t;\eta_{r+1,n}) - \zeta_n(t;\eta_{r,n}) = O(n^{-\delta_1}(\log n)^k), \tag{3.17}$$

for all $r = -r_n,\ldots,r_n-1$, $0<t<1$. Hence, for $b\epsilon[\eta_{r,n},\eta_{r+1,n}]$

$$|G_n^*(t;b)-G_n^*(t;0)| \le \max_{r,r+1} |G_n^*(t;\eta_{j,n})-G_n^*(t;0)|$$

$$+ O(n^{-\delta_1}(\log n)^k), \ -r_n\le r\le r_n-1. \tag{3.18}$$

We denote by $S_1^{(n)} = \{i: c_{ni}^* \ge 0,\ 1\le i\le n\}$, $S_2^{(n)} = \{i: c_{ni}^* < 0,\ 1\le i\le n\}$,
$U_{b,n}^{(s)} = \sum_{i\epsilon S_2^{(n)}} c_{ni}^*[u(\xi_{s,n}+d_{ni}(\xi_{s,n};b)-Y_i)-u(\xi_{s+1,n}+d_{ni}(\xi_{s+1,n};b)-Y_i)]$ and
$V_{b,n}^{(s)} = \sum_{i\epsilon S_1^{(n)}} c_{ni}^*[\xi_{s+1,n}+d_{ni}(\xi_{s+1,n};b)-\xi_{s,n}-d_{ni}(\xi_{s,n};b)]$, for $s=0,1,\ldots,s_n-1$,
$b\epsilon I_n$. Then, we get after some simple algebraic manipulations that

$$\sum_{i=1}^n c_{ni}^*[u(\xi_{s,n}+d_{ni}(\xi_{s,n};b)-Y_i) - (\xi_{s,n}+d_{ni}(\xi_{s,n};b))] - V_{b,n}^{(s)}-U_{b,n}^{(s)}$$

$$\le \sum_{i=1}^n c_{ni}^*[u(t+d_{ni}(t;b)-Y_i) - (t+d_{ni}(t;b))]$$

$$\le \sum_{i=1}^n c_{ni}^*[u(\xi_{s+1,n}+d_{ni}(\xi_{s+1,n};b)-Y_i) - (\xi_{s+1,n}+d_{ni}(\xi_{s+1,n};b))]$$

$$+ V_{b,n}^{(s)} + U_{b,n}^{(s)}.$$

Thus,

$$|\sum_{i=1}^n c_{ni}^*[u(t+d_{ni}(t;b)-Y_i) - (t+d_{ni}(t;b))]|$$

$$\le \max_{j=s,s+1}|\sum_{i=1}^n c_{ni}^*[u(\xi_{j,n}+d_{ni}(\xi_{j,n};b)-Y_i)-(\xi_{j,n}+d_{ni}(\xi_{j,n};b))]|$$

$$+ V_{b,n}^{(s)} + U_{b,n}^{(s)}. \tag{3.19}$$

Hence, by (3.18) and (3.19), we have,

$$\sup_{0<t<1} \sup_{|b|\leq a_n} |G^*_n(t;b)-G^*_n(t;0)|$$

$$\leq \max_{0\leq s\leq s_n} \max_{|r|\leq r_n} |G^*_n(\xi_{s,n};\eta_{r,n})-G^*_n(\xi_{s,n};0)|$$

$$+ \max_{0\leq s\leq s_n} \max_{-r_n\leq r\leq r_{n-1}} V^{(s)}_{\eta_{r,n},n} + \max_{0\leq s\leq s_n} \max_{-r_n\leq r\leq r_{n-1}} U^{(s)}_{\eta_{r,n},n}$$

$$+ O(n^{-\delta_1}(\log n)^k). \tag{3.20}$$

Hence, it suffices to prove that the first three terms on the right hand side of (3.20) are bounded above by $O(n^{-h}(\log n)^k)$ with probability $\geq 1-O(n^{-h})$ for every fixed $h>0$.

Consider any fixed r (say ≥ 0) and any fixed s. Then, $G^*_n(\xi_{s,n};\eta_{r,n})-G^*_n(\xi_{s,n};0) = \sum_{i=1}^n(Z_{ni}-EZ_{ni})$, where, $Z_{ni} = c^*_{ni}[u(\xi_{s,n}+d_{ni}(\xi_{s,n};\eta_{r,n})-Y_i) - u(\xi_{s,n}-Y_i)]$, $i=1,2,\ldots,n$ are independent and Z_{ni} can assume the values 0 and $|c^*_{ni}|$ respectively with probability $1-d_{ni}(\xi_{s,n};\eta_{r,n})$ and $d_{ni}(\xi_{s,n};\eta_{r,n})$ respectively. Hence, for every $g>0$, and $h>0$,

$$P\{\textstyle\sum_{i=1}^n(Z_{ni}-EZ_{ni})>g\} \leq \exp\{-h[g+\textstyle\sum_{i=1}^n EZ_{ni}]\}$$

$$\Pi_{i=1}^n E\{\exp(h\, Z_{ni})\}$$

$$\leq \exp\{-h[g+\zeta_n(\xi_{s,n},\eta_{r,n})]\}\ \Pi_{i=1}^n\{1+(\exp[h|c^*_{ni}|]-1)d_{ni}(\xi_{s,n};\eta_{r,n})\}$$

$$= \chi_n(g,h),\ \text{(say)}. \tag{3.21}$$

Upon taking now $g = g_n = K_1' n^{-\delta}(\log n)^k$, $h = h_n = n^{\delta_1}$, where $0<\delta<\delta_1<\frac{1}{4}$, $k\geq 1$, $0<K_1'<\infty$, we obtain after some simplifications that

$$\log \chi_n(g_n,h_n) = -K_1' n^{\delta_1-\delta}(\log n)^k[1+O(n^{-(\frac{1}{2}-\delta-\delta_1)}(\log n)^k] \tag{3.22}$$

uniformly in $s=0,1,\ldots,s_n$ and $|r|\leq r_n$. Thus, for n adequately large, we have, uniformly in r and s,

$$P\{G_n^*(\xi_{s,n};\eta_{r,n}) - G_n^*(\xi_{s,n};0) > K_1' n^{-\delta}(\log n)^k\}$$
$$\leq \exp\{-K_1' n^{\delta_1-\delta}(\log n)^k\}[1+o(1)], \tag{3.23}$$

and a similar bound is easily obtained for the left hand tail. Thus, for large n, $0\leq s\leq s_n$, $|r|\leq r_n$,

$$P\{|G_n^*(\xi_{s,n};\eta_{r,n}) - G_n^*(\xi_{s,n};0)| > K_1' n^{-\delta}(\log n)^k\}$$
$$\leq 2\exp\{-K_1' n^{\delta_1-\delta}(\log n)^k[1+o(1)]\}. \tag{3.24}$$

Also, it follows obviously that

$$V_{\eta_{r_n},n}^{(s)} = O(n^{-\delta_1}) \text{ uniformly in } |r|\leq r_n,\ 0\leq s\leq s_n. \tag{3.25}$$

By similar techniques as in (3.21), we get after putting $h=n^{\delta_1}$ that for fixed r and s,

$$P\{U_{\eta_{r_n},n}^{(s)} > g_n\} \leq \exp[-K_1' n^{\delta_1-\delta}(\log n)^k(1+O(n^{\delta-\delta_1}(\log n)^{-k}))]. \tag{3.26}$$

From (3.24)-(3.26), we obtain (by use of the Bonferroni inequality) that for large n, the left hand side of (3.20) is bounded above by $K_1' n^{-\delta}(\log n)^k$ with probability greater than or equal to

$$1-2(s_n+1)2r_n \exp\{-K_1' n^{\delta_1-\delta}(\log n)^k(1+o(1))\}. \tag{3.27}$$

Since, $s_n = [n^{\frac{1}{2}+\delta_1}]$, $r_n = n^{\delta_1}$ and $\delta_1>\delta$, K_1' can always be so selected that for $n\geq n^*$, (3.27) is bounded above by $4\exp\{-h(\log n)^k\} \leq 4n^{-h}$ (since, $k\geq 1$). This completes the proof of (3.13).

It can be shown by the same technique as in above that

$$\sup_{0<t<1}\ \sup_{|b|\leq a_n} |L_n^*(t;b)-L_n^*(t;0)|$$

$$\leq \max_{0\leq s\leq s_n}\ \max_{|r|\leq r_n} |L_n^*(\xi_{s,n};\eta_{r,n})-L_n^*(\xi_{s,n};0)|$$

$$+ \max_{0\leq s\leq s_{n-1}} [L_n^*(\xi_{s+1,n};0)-L_n^*(\xi_{s,n};0)]$$

$$+ \max_{0\leq s\leq s_n}\ \max_{|r|\leq r_n} |W_{r,n}^{(s)}| , \tag{3.28}$$

where $W_{r,n}^{(s)} = (n+1)^{-\frac{1}{2}} \sum_{i\varepsilon S_2^{(n)}} [u(\xi_{s,n}+d_{ni}(\xi_{s,n};\eta_{r,n})-Y_i)-u(\xi_{s,n}+d_{ni}(\xi_{s,n};\eta_{r+1,n})-Y_i)]$.

Using (3.9) we obtain that for every $g_n>0$, $h_n>0$,

$$\log P\{L_n^*(\xi_{s,n};\eta_{r,n})-L_n^*(\xi_{s,n};0)>g_n\}$$

$$\leq -(n+1)^{\frac{1}{2}}h_n g_n + \{\sum_{i\varepsilon S_1^{(n)}} + \sum_{i\varepsilon S_2^{(n)}} \log E\exp[h_n\{u(\xi_{s,n}+d_{ni}(\xi_{s,n};\eta_{r,n}) -Y_i) - u(\xi_{s,n}-Y_i)\}]$$

$$= -(n+1)^{\frac{1}{2}}h_n g_n + \sum_{i\varepsilon S_1^{(n)}} \log[1+(e^{h_n}-1)d_{ni}(\xi_{s;n};\eta_{r,n})]$$

$$+ \sum_{i\varepsilon S_2^{(n)}} \log[1-(1-e^{-h_n})(-d_{ni}(\xi_{s,n};\eta_{r,n}))]. \tag{3.29}$$

On using the inequality that for $|x|<1$, $\log(1\pm|x|) \le \pm|x|$, and on taking $h_n = n^{-\delta_1}$, $g_n = K_1 n^{-\delta}(\log n)^k$, $(0<\delta<\delta_1<\frac{1}{4})$, we get for adequately large n that,

$$\log P\{L_n^*(\xi_{s,n};\eta_{r,n})-L_n^*(\xi_{s,n};0) > K_1 n^{-\delta}(\log n)^k\}$$

$$\le -K_1 n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k + n^{-\delta_1}\{\sum_{i=1}^{n} d_{ni}(\xi_{s,n};\eta_{r,n})\}$$

$$+ O(n^{-2\delta_1})\{\textstyle\sum_{i=1}^{n}|d_{ni}(\xi_{s,n};\eta_{r,n})|\}$$

$$= -K_1 n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k + n^{-\delta_1}[n\{O(n^{-1})(\log n)^{2k}\}]$$

$$+ O(n^{-2\delta_1})\cdot O(n^{\frac{1}{2}}(\log n)^k), \text{ by (2.10), (3.4) and (3.7).} \tag{3.30}$$

Since $\delta<\delta_1$, if we let $\delta_1\le\frac{1}{4}$, the right hand side of (3.30) is

$$-K_1 n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k[1+O(n^{-\frac{1}{2}+\delta}(\log n)^k)+O(n^{-(\delta_1-\delta)})], \tag{3.31}$$

and hence, for n sufficiently large,

$$P\{L_n^*(\xi_{s,n};\eta_{r,n})-L_n^*(\xi_{s,n};0) > K_1 n^{-\delta}(\log n)^k\}$$

$$\le \exp\{-K_1' n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k\},\ 0<K_1'<\infty, \tag{3.32}$$

and the same bound holds for the left hand tail. Hence, by the Bonferroni inequality, for large n,

$$P\{\max_{0\le s\le s_n}\max_{|r|\le r_n}|L_n^*(\xi_{s,n};\eta_{r,n})-L_n^*(\xi_{s,n};0)| > Kn^{-\delta}(\log n)^k\}$$

$$\le 4r_n(s_n+1)\exp\{-K'n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k\}$$

$$\le 4n^{-h},\ \text{for } k\ge 1 \text{ and } n\ge n^*, \tag{3.33}$$

where h is any (fixed) positive number.

Since, $L_n^*(\xi_{s+1,n};0)-L_n^*(\xi_{s,n};0) = (n+1)^{-\frac{1}{2}}\sum_1^n [u(\xi_{s+1,n}-Y_i)-u(\xi_{s,n}-Y_i)]$ involves iid (0,1) valued random variables, proceeding as in (3.21)-(3.24) and using the Bonferroni inequality, we obtain

$$P\{\max_{0\le s\le s_n-1}|L_n^*(\xi_{s+1,n};0)-L_n^*(\xi_{s,n};0)| > K_2n^{-\delta}(\log n)^k\}$$

$$\le 2s_n\ \exp[-K_2n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k\{1+O(n^{\delta-\delta_1})\}]$$

$$\le 2n^{-h}\ \text{for any fixed } h>0 \text{ by proper choice of } K_2 \text{ (since } k\ge 1). \tag{3.34}$$

Similarly it can be shown that

$$P\{|W_{r,n}^{(s)}| > K_3n^{-\delta}(\log n)^k\}$$

$$\le 2\ \exp[-K_3n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k[1+O(n^{\delta-\delta_1})]], \tag{3.35}$$

uniformly in $|r|\le r_n$ and $0\le s\le s_n$. As before, on using the Bonferroni inequality, we get

$$P\{\max_{0\le s\le s_n}\max_{|r|\le r_n}|W_{r,n}^{(s)}| > K_3n^{-\delta}(\log n)^k\}$$

$$\le 4r_n(s_n+1)\exp\{-K_3n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k(1+o(1))\}$$

$$\le 4n^{-h},\ \text{for any fixed } h(>0),\ \text{as } k\ge 1. \tag{3.36}$$

(3.14) then follows from (3.28), (3.33), (3.34) and (3.36). Q.E.D.

Theorem 3.1 will be utilized to prove the following basic result which strengthens (3.1) to a.s. convergence.

Theorem 3.2. Under the assumptions of Section 2, for every $s(>0)$, there exists positive constants $(c_s^{(1)}, c_s^{(2)})$ and a sample size n_s, such that $\beta=0$ and all $n\geq n_s$

$$P\{\sup_{b\in I_n^*}|W(\underset{\sim}{X}_n,b)| > c_s^{(1)}n^{-\delta}(\log n)^{k+1}\} < c_s^{(2)}n^{-s}.$$

Hence, $W(\underset{\sim}{X}_n,b)\to 0$ a.s., as $n\to\infty$.

Proof. Note that if we let $J_n(\frac{i}{n+1}) = EJ(U_{ni})$, $1\leq i\leq n$ [cf. (2.6)], we have

$$\begin{aligned} &|\textstyle\sum_1^n c_{ni}^*[J_n(\frac{R_{ni}}{n+1}) - J(\frac{R_{ni}}{n+1})]| \\ &\leq \max_{1\leq i\leq n}|c_{ni}^*|\textstyle\sum_{i=1}^n|J_n(i/(n+1))-J(i/(n+1))| \\ &= O(n^{-\frac{1}{2}})\cdot o(n^{\frac{1}{2}}) = o(1), \end{aligned} \tag{3.37}$$

by (2.10) and theorem 2 of Chernoff and Savage (1958). Hence, for our purpose, it suffices to work with $J_n(\frac{i}{n+1}) = J(\frac{i}{n+1})$, $1\leq i\leq n$. If we define $R_{ni}^{(b)} = \sum_{j=1}^n u(X_i-bc_{ni}^*-[X_j-bc_{nj}^*])$, $1\leq i\leq n$, $-\infty<b<\infty$, we have

$$\begin{aligned} T(\underset{\sim}{X}_n)-T(\underset{\sim}{X}_n-b\underset{\sim}{c}_n^*) &= \int_{-\infty}^{\infty} J[H_n(x,0)]dS_n^*(x;0) \\ &\quad - \int_{-\infty}^{\infty} J[H_n(x;b)]dS_n^*(x;b) \\ &= I_{n1}(b) + I_{n2}(b), \end{aligned} \tag{3.38}$$

where

$$I_{n1}(b) = \int_{-\infty}^{\infty} J[H_n(x;0)]d[S_n^*(x;0)-S_n^*(x;b)], \tag{3.39}$$

$$I_{n2}(b) = \int_{-\infty}^{\infty} \{J[H_n(x;0)]-J[H_n(x;b)]\}dS_n^*(x;b). \tag{3.40}$$

In (2.1), without any loss of generality, we may let $\beta_0=0$ and assume that $0<F(0)<1$. We select $x_n^{(1)}$ and $x_n^{(2)}$ such that

$$F(x_n^{(1)}) = 1-F(x_n^{(2)}) = 4K_1 n^{-(\frac{1}{2}+\delta)}(\log n)^k, \tag{3.41}$$

where K_1 is defined in (3.14). Then, for n sufficiently large, say for $n \geq n_o$, we have $x_n^{(1)}<0<x_n^{(2)}$, and

$$I_{n2}(b) = \int_{-\infty}^{x_n^{(1)}} + \int_{x_n^{(1)}}^{0} + \int_{0}^{x_n^{(2)}} + \int_{x_n^{(2)}}^{\infty} \{J[H_n(x;0)]-J[H_n(x;b)]\}dS_n^*(x,b)$$

$$= I_{n2}^{(1)}(b) + I_{n2}^{(2)}(b) + I_{n2}^{(3)}(b) + I_{n2}^{(4)}(b), \text{ say.} \tag{3.42}$$

Since, by (3.2), $|S_n^*(x;b)| \leq \{(n+1) \max_{1\leq i\leq n}|c_{ni}^*|\}H_n(x;b)$, we obtain by some standard computations that

$$|I_{n2}^{(3)}(b)| \leq (n+1) \max_{1\leq i\leq n}|c_{ni}^*| \int_0^{x_n^{(2)}} |H_n(x;0)-H_n(x;b)||J'[H_n(x;b,\theta_n)]|dH_n(x,b), \tag{3.43}$$

where $H_n(x;b,\theta_n) = \theta_n H_n(x;0) + (1-\theta_n)H_n(x;b)$, $0<\theta_n<1$. On using (3.12) and (3.14), we obtain for large n

$$\sup_{-\infty<x<\infty}\ \sup_{b\in I_n^*} |H_n(x;b)-H_n(x;0)| = O(n^{-\frac{1}{2}-\delta}(\log n)^k), \tag{3.44}$$

with probability $\geq 1-O(n^{-s})$, for every (fixed) $s(>0)$. Hence, on using (3.44), (2.9) and (2.10), we obtain that

$$|I_{n2}^{(3)}(b)| \leq O(n^{-\delta}(\log n)^k) \int_0^{x_n^{(2)}} \{H_n(x;b,\theta_n)[1-H_n(x;b,\theta_n)]\}^{-1} dH_n(x,b), \tag{3.45}$$

with probability $\geq 1-O(n^{-s})$. Let us define $\bar{F}_n(x;b) = (n+1)^{-1}\sum_{i=1}^n F(x+bc_{ni}^*)$, so that $\bar{F}_n(x;0) = (n/(n+1))F(x)$. Note that by (2.10)-(2.13)

$$\sup_{b\in I_n^*} |\bar{F}_n(x;b)-\bar{F}_n(x;0)| = O(n^{-1}(\log n)^{2k}). \tag{3.46}$$

Since $H_n(x;0) = (n+1)^{-1}\sum_{i=1}^n u(x-X_i)$ involves average of iid (bounded valued) rv's, using the results of Hoeffding (1963), we obtain that for large n,

$$P\{|H_n(x_n^{(2)};0)-\bar{F}_n(x_n^{(2)};0)| > K_1 n^{-3/4-\delta/2}(\log n)^k\} < K_2 n^{-s}, \tag{3.47}$$

where K_1 and K_2 are the same as in (3.14). Using then (3.12), (3.14), (3.41), it follows that for all $x \leq x_n^{(2)}$,

$$1-H_n(x;0) \geq (1/3)[1-H_n(x;b)],\ \forall\ b\in I_n^*, \tag{3.48}$$

with probability $\geq 1-O(n^{-s})$. Again, by using the same results of Hoeffding (1963), we have $P\{|L_n^*(F(0);0)-\sqrt{n}\,F(0)|>K_1\sqrt{\log n}\} \leq K_2 n^{-s}$, for every $s(>0)$, and hence, by (3.12), (3.14) and the fact that $F(0)>0$, we obtain that for all $x \geq 0$, as $n\to\infty$,

$$H_n(x;b) \geq F(0)-K_1 n^{-\frac{1}{2}}(\log n)^k[1+O(n^{-\delta})] \geq g_o > 0,\ \forall\ b\in I_n^*, \tag{3.49}$$

with probability $\geq 1-O(n^{-s})$. From the definition of $H_n(x;b,\theta_n)$ and (3.48) - (3.49), it follows that for all $0 \leq x \leq x_n^{(2)}$,

$$H_n(x;b,\theta_n)[1-H_n(x;b,\theta_n)] \geq C[1-H_n(x;b)] \ , \tag{3.50}$$

(where C>0), with probability $\geq 1-O(n^{-s})$. From (3.45) and (3.50), for large n, with probability $\geq 1-O(n^{-s})$,

$$\begin{aligned}\sup_{b\varepsilon I_n^*}|I_{n2}^{(3)}(b)| &\leq O(n^{-\delta}(\log n)^k)\{\sup_{b\varepsilon I_n^*}\int_0^{x_n^{(2)}} \{1-H_n(x;b)\}^{-1}dH_n(x;b)\} \\ &\leq O(n^{-\delta}(\log n)^k)\{\sup_{b\varepsilon I_n^*}\int_{-\infty}^{\infty} \{1-H_n(x;b)\}^{-1}dH_n(x;b)\} \\ &= O(n^{-\delta}(\log n)^k)\{\sup_{b\varepsilon I_n^*}(n+1)^{-1}\sum_{i=1}^{n}(1-\frac{i}{n+1})^{-1}\} \\ &= O(n^{-\delta}(\log n)^k)\{1+\tfrac{1}{2}+\ldots+\tfrac{1}{n}\} = O(n^{-\delta}(\log n)^{k+1}).\end{aligned} \tag{3.51}$$

It follows similarly that for large n, with probability $\geq 1-O(n^{-s})$, $\sup_{b\varepsilon I_n^*}|I_{n2}^{(2)}(b)| = O(n^{-\delta}(\log n)^{k+1})$. Again

$$\begin{aligned}|I_{n2}^{(4)}(b)| &\leq \{(n+1)\max_{1\leq i\leq n}|c_{ni}^*|\}\{\int_{x_n^{(2)}}^{\infty} |J[H_n(x;b)]|dH_n(x;b) \\ &+ \int_{x_n^{(2)}}^{\infty} |J[H_n(x;0)]|dH_n(x;b)\} = I_{n2}^{(4)}(b)_1+I_{n2}^{(4)}(b)_2, \text{ say.}\end{aligned} \tag{3.52}$$

On using (2.9), (2.10), (3.41), (3.47) and (3.48), we obtain that for large n, with probability $\geq 1-O(n^{-s})$,

$$\sup_{b\varepsilon I_n^*}|I_{n2}^{(4)}(b)_1| \leq O(n^{\frac{1}{2}})\{\sup_{b\varepsilon I_n^*}\int_{H_n(x_n^{(2)};b)}^{1}\{-\log u(1-u)\}du\}$$

$$\leq O(n^{\frac{1}{2}})\{K\int_{1-O(n^{-\frac{1}{2}-\delta}(\log n)^k)}^{1}[-\log(1-u)]du\}$$

$$= O(n^{\frac{1}{2}})\cdot O(n^{-\frac{1}{2}-\delta}(\log n)^{k+1}) = O(n^{-\delta}(\log n)^{k+1}). \quad (3.53)$$

Also, $I_{n2}^{(4)}(b)_2 = O(n^{\frac{1}{2}})\int_{x_n^{(2)}}^{\infty}|J[H_n(x;0)]|dH_n(x;0)+O(n^{\frac{1}{2}})\int_{x_n^{(2)}}^{\infty}|J[H_n(x;0)]|\cdot d[H_n(x;b) - H_n(x;0)]$. By (3.53), the first term is $O(n^{-\delta}(\log n)^{k+1})$; with probability $\geq 1-O(n^{-s})$ (note that it does not depend on $b\varepsilon I_n^*$), while integrating by parts and using the same trick as in the earlier cases, it follows that the second term is also $O(n^{-\delta}(\log n)^{k+1})$ for all $b\varepsilon I_n^*$, with probability $\geq 1-O(n^{-s})$. Thus, as $n\to\infty$,

$$\sup_{b\varepsilon I_n^*} I_{n2}^{(4)}(b) = O(n^{-\delta}(\log n)^{k+1}) \text{ with probability } \geq 1-O(n^{-s}). \quad (3.54)$$

Similarly, it follows that for large n, with probability $\geq 1-O(n^{-s})$, $\sup_{b\varepsilon I_n^*}|I_{n2}^{(1)}(b)| = O(n^{-\delta}(\log n)^{k+1})$. Consequently, for large n,

$$\sup_{b\varepsilon I_n^*}|I_{n2}(b)| = O(n^{-\delta}(\log n)^{k+1}), \text{ with probability } \geq 1-O(n^{-s}). \quad (3.55)$$

In (3.39), we now write $I_{n1}(b) = I_{n1}^{(1)}(b) + I_{n1}^{(2)}(b)$, where

$$I_{n1}^{(1)}(b) = \int_{-\infty}^{\infty} J[F(x)]d[S_n^*(x;0)-S_n^*(x;b)], \quad (3.56)$$

$$I_{n1}^{(2)}(b) = \int_{-\infty}^{\infty}\{J[H_n(x;0)]-J[F(x)]\}d[S_n^*(x;0)-S_n^*(x;b)]. \quad (3.57)$$

By (2.10), (3.2) and (3.57), we have

$$\sup_{b\varepsilon I_n^*}|I_{n1}^{(2)}(b)| = \sup_{b\varepsilon I_n^*}\Big|\sum_{i=1}^{n} c_{ni}^*\{J[H_n(X_i;0)]-J[F(X_i)]\}\{u(-X_i)-u(bc_{ni}^*-X_i)\}\Big|$$

$$\le \{\max_{1\le i\le n}|c_{ni}^*|\}\{\sup_{x\varepsilon I_n^*}|J[H_n(Cn^{-\frac{1}{2}}x;0)]-J[F(Cn^{-\frac{1}{2}}x)]|\}(n+1)\cdot$$

$$\cdot\{|H_n(Cn^{-\frac{1}{2}}(\log n)^k;0) - H_n(Cn^{-\frac{1}{2}}(\log n)^k;0)|\}, \qquad (3.58)$$

where C(>0) is finite. On using the well-known result that

$$P\{\sup_{-\infty<x<\infty} n^{\frac{1}{2}}|H_n(x;0)-F(x)| > C_1\sqrt{\log n}\} \le C_2 n^{-s}, \qquad (3.59)$$

where C_1, C_2 are finite (positive) constants, and that in the neighbourhood of 0 (where $0<F(0)<1$), $|J'(F(x))|$ is bounded [by (2.9)], and further that $F(Cn^{-\frac{1}{2}}(\log n)^k) - F(-Cn^{-\frac{1}{2}}(\log n)^k) = O(n^{-\frac{1}{2}}(\log n)^k)$, we obtain from (3.58) that

$$\sup_{b\varepsilon I_n^*}|I_{n1}^{(2)}(b)| \le [O(n^{-\frac{1}{2}})][O(n^{-\frac{1}{2}}(\log n)\ ][O(n)][O(n^{-\frac{1}{2}}(\log n)^k]$$

$$= O(n^{-\frac{1}{2}}(\log n)^{k+\frac{1}{2}}), \text{ with probability } \ge 1-O(n^{-s}). \qquad (3.60)$$

Further, on denoting by $c_n^* = \max_{1\le i\le n}|c_{ni}^*|$, we have

$$I_{n1}^{(1)}(b) = \int_{X_{(1)}-|b|c_n^*}^{X_{(n)}+|b|c_n^*} J[F(x)]d[S_n^*(x;0)-S_n^*(x;b)]$$

$$= \int_{X_{(1)}-|b|c_n^*}^{X_{(n)}+|b|c_n^*} [S_n^*(x;b)-S_n^*(x;0)]J'[F(x)]dF(x), \qquad (3.61)$$

where $X_{(1)} = \min_{1\leq i\leq n} X_i$ and $X_{(n)} = \max_{1\leq i\leq n} X_i$. Hence, by (2.22) and (3.61),

$$I_{n1}^{(1)}(b)-bB(F) = \int_{X_{(1)}-|b|c_n^*}^{X_{(n)}+|b|c_n^*} [S_n^*(x;b)-S_n^*(x;0)-bf(x)]J'[F(x)]dF(x)$$

$$-\{\int_{-\infty}^{X_{(1)}-|b|c_n^*} + \int_{X_{(n)}+|b|c_n^*}^{\infty} bf(x)J'[F(x)]dF(x)\}. \tag{3.62}$$

Now, $P\{F(X_{(1)}) \geq cn^{-\frac{1}{2}}\} = [1-cn^{-\frac{1}{2}}]^n \leq 0(n^{-s})$, for every (fixed) $s(>0)$, and similarly, $P\{1-F(X_{(n)}) \geq cn^{-\frac{1}{2}}\} \leq 0(n^{-s})$. Hence, by (2.13) the last term on the right hand side of (3.62) is bounded (for all $b\varepsilon I_n^*$) above by $0(n^{-\frac{1}{2}}(\log n)^{k+1})$, with probability $\geq 1-0(n^{-s})$. Also, by (2.9), (3.11) and (3.13), the first term on the right hand side of (3.62) is bounded (for all $b\varepsilon I_n^*$) above by

$$\{K_1 n^{-\delta}(\log n)^k[1+0(n^{-\frac{1}{2}})]\}K \int_{X_{(1)}-Cn^{-\frac{1}{2}}(\log n)^k}^{X_{(n)}+Cn^{-\frac{1}{2}}(\log n)^k} \{F(x)[1-F(x)]\}^{-1}dF(x) \tag{3.63}$$

$$\leq 0(n^{-\delta}(\log n)^k) \quad \{-\log F(X_{(1)}-Cn^{-\frac{1}{2}}(\log n)^k)-\log\{1-F(X_{(n)}+Cn^{-\frac{1}{2}}(\log n)^k)\}\},$$

with probability $\geq 1-0(n^{-s})$, where $c>0$. In the same way as in after (3.62), it can be shown that $F(X_{(1)}-Cn^{-\frac{1}{2}}(\log n)^k) \geq 0(n^{-k})$ with probability $\geq 1-0(n^{-s})$, where h is some fixed positive number, and a similar statement holds for $1-F(X_{(n)}+Cn^{-\frac{1}{2}}(\log n)^k))$. Hence, with probability $\geq 1-0(n^{-s})$, the right hand side of (3.63) is bounded above by

$$0(n^{-\delta}(\log n)^k)\cdot 0(h \log n) = 0(n^{-\delta}(\log n)^{k+1}). \tag{3.64}$$

The proof of the theorem is then completed by (3.55), (3.62), (3.63), (3.64) and the definition of $W(\underset{\sim}{X}_n, b)$, given at the begining of this section. Q.E.D.

4. A MARTINGALE PROPERTY OF $T(\underset{\sim}{X}_n)$.

Let $\mathcal{F}_n$ denote the σ-field generated by $\underset{\sim}{R}_n=(R_{n1},\ldots,R_{nn})$; note that $\mathcal{F}_n$ is $\uparrow$ in n. Also, let $T^*_n=C_nT_n$ and assume that H_o: $\beta=0$ holds. Then, the following theorem holds; the result is of fundamental use in proving the "uniform continuity in probability" (to be explained in section 5) of $T(\underset{\sim}{X}_n)$ with respect to C_n^{-1}.

Theorem 4.1. If $J_n(u) = EJ(U_{ni})$, $(i-1)/n<u\leq i/n$, $1\leq i\leq n$, then under H_o: $\beta=0$, $\{T^*_n, \mathcal{F}_n\}$ forms a martingale sequence.

Proof. We have $T^*_n = \sum_{i=1}^n(c_i-\bar{c}_n)J_n(R_{ni}/(n+1))$, and hence,

$$E[T^*_{n+1}|\mathcal{F}_n] = (c_{n+1}-\bar{c}_{n+1})E[J_{n+1}(R_{n+1,n+1}/(n+2))|\mathcal{F}_n]$$

$$+ \sum_{i=1}^n(c_i-\bar{c}_{n+1})E\{J_{n+1}(R_{n+1i}/(n+2))|\mathcal{F}_n\}. \tag{4.1}$$

Since, $E\{J_{n+1}(R_{n+1\ n+1}/(n+2))|\mathcal{F}_n\} = (n+1)^{-1}\sum_{i=1}^{n+1}J_{n+1}(i/(n+2))= \int_0^1 J(u)du = \mu$, and also for any $1\leq i\leq n$, $E[J_{n+1}(R_{n+1i}/(n+2))|\mathcal{F}_n] = ((n+1)^{-1}R_{ni})J_{n+1}((R_{ni}+1)/(n+2)) + (n+1)^{-1}(n+1-R_{ni})J_{n+1}(R_{ni}/(n+2))$, and finally, $[i/(n+1)]J_{n+1}((i+1)/(n+2)) + [(n-i+1)/(n+1)]J_{n+1}(i/(n+2)) = i\binom{n}{i}\int_0^1 J(u)u^i(1-u)^{n-i}du + (n-i+1)\binom{n}{i-1}\int_0^1 J(u)u^{i-1}(1-u)^{n-i+1}du = n\binom{n-1}{i-1}\int_0^1 J(u)u^{i-1}(i-u)^{n-i}du = J_n(i/(n+1))$, for $1\leq i\leq n$, we have from (4.1),

$$E[T^*_{n+1}|\mathcal{F}_n] = (c_{n+1}-\bar{c}_{n+1})\mu+\sum_{i=1}^n(c_i-\bar{c}_{n+1})J_n(R_{ni}/(n+1))$$

$$= (c_{n+1}-\bar{c}_{n+1})\mu + \sum_{i=1}^n(c_i-\bar{c}_n)J_n(R_{ni}/(n+1)) + (\bar{c}_n-\bar{c}_{n+1})n\mu$$

$$= \sum_{i=1}^n(c_i-\bar{c}_n)J_n(R_{ni}/(n+1)) = T^*_n, \text{ for all } n\geq 1. \tag{4.2}$$

Hence, the theorem.

Remark. Note that the above martingale property does not necessarily hold when we let $J_n(i/(n+1)) = J(i/(n+1))$, $1\leq i\leq n$ [unless $J(u) = u$: $0<u<1$]. Also, when $\beta\neq 0$, so that given the ranks of $X_1,\ldots,X_n$, X_{n+1} does not have a constant probability $(1/(n+1))$ of having the rank $i(1\leq i\leq n+1)$, the above margingale property does not hold, in general.

5. THE PROOF OF THE MAIN THEOREM

The proof is accomplished in several steps. First, we prove the following lemma.

Lemma 5.1. For $\forall s(>0)$, $\exists$ positive constants $K_s^{(1)}$, $K_s^{(2)}$ and a sample size n_s, such that for $n\geq n_s$,

$$P\{[C_n(\hat{\beta}_{U,n}-\beta)-A\tau_{\alpha/2}/B(F)] > K_s^{(1)}(\log n)^2\} < K_s^{(2)}n^{-s}, \tag{5.1}$$

$$P\{[C_n(\hat{\beta}_{L,n}-\beta)+A\tau_{\alpha/2}/B(F)] < -K_s^{(1)}(\log n)^2\} < K_s^{(2)}n^{-s}. \tag{5.2}$$

Proof. We shall only consider the proof of (5.1) as the proof of (5.2) follows on the same line. By virtue of the translation-invariance of the estimates $\hat{\beta}_{U,n}$ and $\hat{\beta}_{L,n}$, we may, without any loss of generality, assume that $\beta=0$. Then, by (2.7), we have

$$P\{C_n\hat{\beta}_{U,n}-A\tau_{\alpha/2}/B(F) > K_s^{(1)}(\log n)^2\}$$

$$= P\{T(\underset{\sim}{X}_n-C_n^{-1}[A\tau_{\alpha/2}/B(F) + K_s^{(1)}(\log n)^2]\underset{\sim}{c}_n) > T_n^{(1)}|\beta=0\}$$

$$= P_{\beta=0}\{\hat{T}_n-\tilde{T}_n>T_n^{(1)}-\tilde{T}_n\}, \tag{5.3}$$

where we write $\tilde{T}_n = T(\underset{\sim}{X}_n - b_o \underset{\sim}{c}_n^*)$, $b_o = A\tau_{\alpha/2}/B(F)$ and $\hat{T}_n = T(\underset{\sim}{X}_n - [b_o + K_s^{(1)}(\log n)^2]\underset{\sim}{c}_n^*)$. By theorem 3.2, with probability $\geq 1-O(n^{-s})$,

$$\hat{T}_n - \tilde{T}_n = -K_s^{(1)}(\log n)^2 B(F) + O(n^{-\delta}(\log n)^3),\ \delta>0. \tag{5.4}$$

Also, using the fact that $\lim_{n\to\infty} T_n^{(1)} = -A\tau_{\alpha/2}$ and writing $S_n(x;b_o) = \sum_{i=1}^n c_{ni}^* F(x-b_o c_{ni}^*)$, we obtain by some standard computations that

$$\int_{-\infty}^{\infty} J[F(x)]dS_n(x;b_o) = -b_o B(F) + O(n^{-\frac{1}{2}}) = T_n^{(1)} + O(n^{-\frac{1}{2}}). \tag{5.5}$$

Hence, it suffices to show that with probability $\geq 1-O(n^{-s})$,

$$\left|\int_{-\infty}^{\infty} J[F(x)]dS_n(x;b_o) - \tilde{T}_n\right| < K_s^{(1)}(\log n)^{3/2}. \tag{5.6}$$

The left hand side of (5.6) can be written as

$$\left|\int_{-\infty}^{\infty} \{J[F(x)]-J[H_n(x;b_o)]\}dS_n(x;b_o) + \int_{-\infty}^{\infty} J[H_n(x;b_o)]d[S_n(x;b)-S_n^*(x;b)]\right|$$

$$\leq \left|\int_{-\infty}^{\infty} \{J[F(x)]-J[H_n(x;b_o)]\}dS_n(x;b_o)\right|$$

$$+ \left|\int_{-\infty}^{\infty} [S_n^*(x;b_o)-S_n(x;b_o)]J'[H_n(x;b_o)]dH_n(x;b_o)\right| \tag{5.7}$$

It can be shown by using theorem 2 of Hoeffding (1963) that $\sup_x |S_n^*(x;b_o)-S_n(x;b)| = O((\log n)^{\frac{1}{2}})$, with probability $\geq 1-O(n^{-s})$. Also, $\int_{-\infty}^{\infty} |J'[H_n(x;b_o)]|dH_n(x;b_o) \leq (n+1)^{-1}\sum_{i=1}^n K(i(n+1-i)/(n+1)^2)^{-1} = O(\log n)$ [by (2.9)]. Hence, the second term on the right hand side of (5.7) is $O((\log n)^{3/2})$ with probability $\geq 1-O(n^{-s})$. Finally, precisely on the same line as in (3.42)-(3.55), it follows that the first term on the right hand side of (5.7) is also $O((\log n)^{3/2})$, with probability $\geq 1-O(n^{-s})$. Hence the proof is complete.

A direct consequence of theorem 3.2 and lemma 5.1 is the following.

Lemma 5.2. For every $s(>0)$, there exists positive constants $(K_s^{(1)}, K_s^{(2)})$ and a sample size n_s, such that for $n \geq n_s$,

$$P\{|C_n(\hat{\beta}_{U,n}-\hat{\beta}_{L,n})-2A\tau_{\alpha/2}/B(F)| \geq K_s^{(1)} n^{-\delta}(\log n)^2\} \leq K_s^{(2)} n^{-s}. \tag{5.8}$$

Also, we have the following lemma whose proof follows along the lines of Sen (1969, Section 3).

Lemma 5.3. For every real $x(-\infty<x<\infty)$,

$$\lim_{n\to\infty} P\{C_n(\hat{\beta}_{U,n}-\beta)B(F)/A-\tau_{\alpha/2} \leq x\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}t^2)dt. \tag{5.9}$$

Finally, for the "uniform continuity in probability" (for definition, see Anscombe (1952)) of $\{\hat{\beta}_{U,n}\}$ with respect to $\{C_n^{-1}\}$, we have the following:

Lemma 5.4. For every positive ε and η, there exists a $\delta(>0)$, such that as $n\to\infty$,

$$P\{\sup_{|n'-n|<\delta n} |C_n(\hat{\beta}_{U,n'}-\hat{\beta}_{U,n})|>\eta\} < \varepsilon, \tag{5.10}$$

and a similar statement holds for $\{\hat{\beta}_{L,n}\}$.

Proof. By theorem 3.2, lemma 5.1 (where we let $s=1+h$, $h>0$), and (2.17), we have, with probability $\geq 1-O(n^{-h})$, for all $|n'-n|<\delta n$,

$$\begin{aligned} C_n(\hat{\beta}_{U,n'}-\hat{\beta}_{U,n}) &= C_{n'}(\hat{\beta}_{U,n'}-\beta)[C_n/C_{n'}]-C_n(\hat{\beta}_{U,n}-\beta) \\ &= -[C_n/C_{n'}][1/B(F)][T(\underset{\sim}{X}_{n'}-\hat{\beta}_{U,n'}\underset{\sim}{c}_{n'})-T(\underset{\sim}{X}_{n'}-\beta\underset{\sim}{c}_{n'})+o(1)] \\ &\quad + [1/B(F)][T(\underset{\sim}{X}_n-\hat{\beta}_{U,n}\underset{\sim}{c}_n)-T(\underset{\sim}{X}_n-\beta\underset{\sim}{c}_n)+o(1)] \end{aligned}$$

$$= -[A/B(F)]\{\tau_{\alpha/2}[1-C_n/C_{n'}]\}+(C_n/C_{n'})[T(\underset{\sim}{X}_{n'}-\beta\underset{\sim}{c}_{n'})-T(\underset{\sim}{X}_n-\beta\underset{\sim}{c}_n)]/B(F)$$

$$- T(\underset{\sim}{X}_n-\beta\underset{\sim}{c}_n)[1-C_n/C_{n'}]/B(F)+o(1). \tag{5.11}$$

By (2.12), $\sup_{|n'-n|<\delta n}|1-C_n/C_{n'}|<\delta'$, where $\delta'(>0)$ depends on δ, and can be made arbitrarily small when δ is made so. Also, the asymptotic normality of $T(\underset{\sim}{X}_n-\beta\underset{\sim}{c}_n)/A$ (with zero mean and unit variance) implies that $|T(\underset{\sim}{X}_n-\beta\underset{\sim}{c}_n)|$ is bounded, in probability, as $n\to\infty$. Hence, it suffices to show that as $n\to\infty$

$$P\{\sup_{|n'-n|<\delta n}|T(\underset{\sim}{X}_n-\beta\underset{\sim}{c}_n)-T(\underset{\sim}{X}_{n'}-\beta\underset{\sim}{c}_{n'})|>\eta\} < \varepsilon. \tag{5.12}$$

Since $|T(\underset{\sim}{X}_n-\beta\underset{\sim}{c}_n)-T(\underset{\sim}{X}_{n'}-\beta\underset{\sim}{c}_{n'})|\leq|(C_{n'}-C_n)/C_{n'}||T_n| + C_{n'}^{-1}|T^*_{n'}-T^*_n|$, (where T^*_n is defined in section 4), it seems sufficient to show that

$$P_{\beta=0}\{\sup_{|n'-n|<\delta n}|T^*_{n'}-T^*_n|>\eta C_n\} < \varepsilon, \tag{5.13}$$

and in view of the martingale property of T^*_n, see theorem 4.1, from the Kolmogorov inequality (cf. Loève (1963, p. 386)) for martingales, we get,

$$P_{\beta=0}\{\sup_{|n'-n|<\delta n}|T^*_{n'}-T^*_n|>\eta C_n\}$$

$$\leq (\eta C_n)^{-2}E(T^{*2}_{n+[\delta n]}-T^{*2}_{n-[\delta n]})$$

$$= (\eta C_n)^{-2}[C^2_{n+[\delta n]}A^2_{n+[\delta n]}-C^2_{n-[\delta n]}A^2_{n-[\delta n]}]$$

The rest of the proof follows from (2.12) and the fact that $A_n^2 = A^2+o(1)$, for large n.

We now return to the proof of theorem 2.1. By (2.17) lemma 5.2 and the definition of N(d), it follows that for all d>0, N(d) is finite a.s. and is $\downarrow$ in d. We also note that

$$E[N(d)] = \sum_{n=0}^{\infty} nP\{N(d)=n\} = \sum_{n\geq 0} P\{N(d)>n\}. \tag{5.14}$$

Hence, in order to show that $E[N(d)]<\infty$, it suffices to show that for large n,

$$P\{N(d)>n\} = O(n^{-1-\eta}), \text{ where } \eta>0. \tag{5.15}$$

By definition, the left hand side of (5.15) is equal to

$$P\{\hat{\beta}_{U,k}-\hat{\beta}_{L,k}>2d \text{ for all } k\leq n\} \leq P\{\hat{\beta}_{U,n}-\hat{\beta}_{L,n}>2d\}$$

$$= P\{C_n(\hat{\beta}_{U,n}-\hat{\beta}_{L,n})>2dC_n\} = O(n^{-s}), \tag{5.16}$$

(where we can let $s>1$), as by (5.8), $C_n(\hat{\beta}_{U,n}-\hat{\beta}_{L,n}) = 2A\tau_{\alpha/2}/B(F) + O(n^{-\delta}(\log n)^2)$, where as by (2.11), $2dC_n \geq 2dC_o n^{\frac{1}{2}}$. Hence, for every $d>0$, $E[N(d)]<\infty$. Using the fact that $\hat{\beta}_{U,n}-\hat{\beta}_{L,n}>0$ a.s. for each n, we have $\lim_{d\to 0} N(d) = \infty$ and finally by the monotone convergence theorem $\lim_{d\to 0} E[N(d)] = \infty$.

Now (2.19) follows directly from (5.8), (2.12) and (2.22); (2.20) also follows from theorem 1 of Anscombe (1952) after using our lemmas 5.3 and 5.4. To prove (2.21), we let for each $d(>0)$,

$$n_i(d) = [Q^{-1}(\nu(d))(1+(-1)^i\varepsilon) + \frac{1+(-1)^i}{2}], \tag{5.17}$$

[x] being the largest integer contained in x, and write,

$$E[N(d)]/Q^{-1}(\nu(d)) = [Q^{-1}(\nu(d))]^{-1}\{\Sigma_1+\Sigma_2+\Sigma_3\ nP\{N(d)=n\}\},$$

where the summations Σ_1, Σ_2 and Σ_3 extend over $n<n_1(d)$, $n_1(d)\leq n\leq n_2(d)$ and $n>n_2(d)$, respectively. Since $Q(\ )$ is $\uparrow$, $\lim_{d\to 0} \nu(d) = \infty$ and (2.19) holds, for every $\varepsilon>0$, there exists a $d_\varepsilon(>0)$, such that for all $0<d\leq d_\varepsilon$, $P\{n_1(d)\leq N(d)\leq n_2(d)\} \geq P\{|N(d)/Q^{-1}(\nu(d))-1|<\varepsilon\} \geq 1-\eta$, where $\eta(>0)$ is a (preassigned) small number. Hence, for $d\leq d_\varepsilon$,

$$\{Q^{-1}(\nu(d))\}^{-1}\Sigma_1 nP\{N(d)=n\} \le (1-\varepsilon)P\{N(d)<n_1(d)\} \le \eta(1-\varepsilon). \quad (5.18)$$

Also, by the same technique as in (5.16), we have

$$\{Q^{-1}(\nu(d))\}^{-1}\Sigma_3 nP\{N(d)=n\} = \{Q^{-1}(\nu(d))\}^{-1}n_2(d)P\{N(d) = n_2(d)\}$$

$$+ \{Q^{-1}(\nu(d))\}^{-1}\Sigma_3 P\{N(d)>n\}$$

$$= \{Q^{-1}(\nu(d))\}^{-1}\{O(n_2(d))O[(n_2(d))^{-s}] + O[(n_2(d))^{-s+1}]\}$$

$$= O[(n_2(d))^{-s+1}] \to 0 \text{ as } n\to\infty, \quad (5.19)$$

as in (5.8) we can let $s>1$ and $Q^{-1}(\nu(d))\to\infty$ as $d\to 0$. Finally,

$$|\{Q^{-1}(\nu(d))\}^{-1}\Sigma_2 nP\{N(d)=n\}-1| \le \varepsilon\Sigma_2 P\{N(d)=n\}+\eta\le\varepsilon+\eta. \quad (5.20)$$

Thus, the proof of (2.21) follows from (5.18)-(5.20).

Remark: Using (5.12), the classical Wald-Wolfowitz-Noether-Hájek theorem [cf. Hájek and Šidák (1967, p. 160)] on the asymptotic normality of $T_n(\beta)$ for fixed but large n, and (2.19), it readily follows from theorem 1 (p. 601) of Anscombe (1952) that $T_{N(d)}(\beta)$ has asymptotically (as $d\to 0$) a normal distribution with zero mean and variance A^2 defined in (2.23). Also, on using theorem 3.2, it follows that for every $b\varepsilon I_{N(d)}$, $T_{N(d)}(\beta+b) + bB(F)$ has asymptotically the same normal distribution. These results give simple proofs of the asymptotic normality of regression rank statistics based on random sample sizes both in the null (b=0) and the non-null ($b\neq 0$) situations. The results trivially extend to any stopping variable N_r, indexed by a sequence $\{r\}$ and a sequence of positive integers $\{n_r\}$, such that $n_r\to\infty$ and $N_r/n_r \xrightarrow{p} 1$ as $r\to\infty$.

6. ASYMPTOTIC RELATIVE EFFICIENCY

For any two procedures A and B for determining (sequentially) bounded length confidence intervals for β (with the same bound 2d), let $P_A(d)$ and $P_B(d)$ be the coverage probabilities and $N_A(d)$ and $N_B(d)$ the stopping variables corresponding to the respective procedures. Then, the ARE of the procedure A with respect to the procedure B is given by

$$e_{A,B} = \lim_{d\to 0}[EN_B(d)/EN_A(d)], \tag{6.1}$$

provided $\lim_{d\to 0}P_A(d) = \lim_{d\to 0}P_B(d)$ and either of the limit exists.

Gleser (1965) considered the case when $n^{-1}C_n^2 \to C>0$ as $n\to\infty$. However, one can easily extend his results when (2.10) and (2.11) hold. Thus, if G denotes his procedure, theorem 2.1 also holds with the change that $\nu(d)$ has to be replaced by $\nu_G(d) = \sigma^2\tau^2_{\frac{1}{2}\alpha}/d^2$, σ^2 being the variance of the distribution of F(x) in (2.1). Writing R for the proposed procedure and using (2.21), we have now,

$$e_{R,G} = \lim_{d\to 0}\{Q^{-1}(\nu_G(d))/Q^{-1}(\nu(d))\}. \tag{6.2}$$

By definition, $\nu_G(d)/\nu(d) = \sigma^2B^2(F)/A^2$ is independent of d. Write $e=s(e^*)$. Then, $e^*=s^{-1}(e)$ is monotonic in e with $e^*=1$ when $e=1$. Write $\phi_d=\nu_G^*(d)/\nu^*(d)$, where $\nu_G^*(d) = Q^{-1}(\nu_G(d))$, $\nu^*(d) = Q^{-1}(\nu(d))$. Both $\nu_G^*(d)$ and $\nu^*(d)$ tend to ∞ as $d\to 0$. Thus,

$$s(e^*)=e=\nu_G(d)/\nu(d)=\lim_{d\to 0}\{\nu_G(d)/\nu(d)\}$$
$$= \lim_{d\to 0}\{Q(\phi_d\nu^*(d))/Q(\nu^*(d))\}. \tag{6.3}$$

Using (2.12) and proving by contradiction, we have,

$$\lim_{d\to 0}\phi(d) = e^* = s^{-1}(e) = s^{-1}(\sigma^2 B^2(F)/A^2) \tag{6.4}$$

The expression $\sigma^2B^2(F)/A^2$ is the Pitman-efficiency of a general rank order test with respect to Student's t test. In the particular case, when $J(u) = \Phi^{-1}(u)$, Φ being the distribution function (d.f.) of a standard normal variable, (normal score) it is well-known (cf. Chernoff and Savage (1958)) that $\sigma^2B^2(F)/A^2 \geq 1$ for all d.f. F with a density f and a finite second moment, equality being attained when and only when F is normal $(0,\sigma^2)$ d.f. From monotonicity, it follows now from (4.4) that in this case $e^*\geq 1$, equality being attained if and only if F is normal. Also, when $J(u)=u$ (Wilcoxon score), $e^* = s^{-1}(12\sigma^2(\int_{-\infty}^{\infty} f^2(x)dx)^2) \geq s^{-1}(0.864)$ (cf. Hájek and Šidák (1967, p. 280)). In the case of equispaced regression line, $t_i = a+(i-1)$, $i=1,2,\ldots$, $C_n^2 = n(n^2-1)/12$, $e^* = \{12\sigma^2(\int_{-\infty}^{\infty} f^2(x)dx)^2\}^{1/3}$. For normal F, this reduces to $(0.955)^{1/3} \approx .985$, while the infimumn is given by $(0.864)^{1/3} \approx .953$.

In the special case when c_i is either 0 of 1, β is the difference in the location parameters of the two distributions $F(x-\alpha')$ and $F(x-\alpha'-\beta)$. This is the classical two-sample problem. If at the n^{th} stage, m_n of the c_i are 1 and rest zero, $C_n^2 = m_n(n-m_n)/4 \leq n/4$, for all $n\geq 1$. Looking at the definition of C_n^2, (2.19) and (2.21), we may observe that an optimum choice of m_n is $[\frac{1}{2}n]$, the integral part of $\frac{1}{2}n$. Thus, among all designs for obtaining a bounded length confidence interval for β, in this problem, and optimum design (which minimizes EN(d) for small d) consists in taking every alternate observation for the two distributions. Here, $n^{-1}C_n^2\to\frac{1}{4}$ and the ARE reduces to $\sigma^2B^2(F)/A^2$ various bounds for which have been discussed earlier.

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