This research was supported in part by the National Science Foundation under Grant No. GU-2059 and the Sakko-kai Foundation.

ON THE DISTRIBUTION OF A TRACE OF A MULTIVARIATE QUADRATIC FORM IN THE MULTIVARIATE NORMAL SAMPLES

by

Takesi Hayakawa
University of North Carolina at Chapel Hill
Institute of Statistical Mathematics, Tokyo

Institute of Statistics Mimeo Series No. 682

Department of Statistics
University of North Carolina at Chapel Hill

APRIL 1970

On the Distribution of a Trace of a Multivariate Quadratic Form in the Multivariate Normal Samples

Takesi Hayakawa
Department of Statistics
University of North Carolina
Chapel Hill, North Carolina 27514

Abstract

This paper considers the derivation of the p.d.f. of the trace of a non-central multivariate quadratic form using a polynomial $P_\kappa(T,A)$ defined by the author and compares with the results of Kotz et al and Ruben. The complex case is also discussed.

Key words

Non-central multivariate quadratic form
generating function
Kronecker product
representation
Hermitian matrix.

Classification number: 40

This research was supported in part by the National Science Foundation under Grant No. GU-2059 and the Sakko-kai Foundation.

ON THE DISTRIBUTION OF A TRACE OF A MULTIVARIATE QUADRATIC FORM IN THE MULTIVARIATE NORMAL SAMPLES

Takesi Hayakawa
University of North Carolina at Chapel Hill
Institute of Statistical Mathematics, Tokyo

1. INTRODUCTION.

Recently the probability density function (p.d.f.) of latent roots of a multivariate non-central quadratic form and of a trace of it were obtained by the use of a polynomial $P_\kappa(T,A)$ introduced by Hayakawa [3]. However, it only shows that if we use the polynomial $P_\kappa(T,A)$, we can represent the p.d.f.'s in terms of a power series representation. In this paper, we discuss another type of representation, that is, Γ-type representation of the p.d.f. and give a more concrete form than Hayakawa [3]. By using this representation, we can compare with the results of Ruben [8] and Kotz et al [7]. We also give a p.d.f. for the case of the complex variables.

2. NOTATIONS AND SOME USEFUL RESULTS.

Let T and U be $m\times n$ $(m\leq n)$ real arbitrary matrices each of rank m, and let A be an $n\times n$ positive definite symmetric (p.d.s.) matrix. Hayakawa [3] defined a new polynomial $P_\kappa(T,A)$ as follows;

$$(1)\quad etr(-TT')P_\kappa(T,A) = \frac{(-1)^k}{\pi^{mn/2}}\int_U etr(-2iTU')etr(-UU')C_\kappa(UAU')dU,$$

This research was supported in part by the National Science Foundation under Grant No. GU-2059 and the Sakko-kai Foundation.

where κ is a partition of k into not more than m parts, i.e., $\kappa = (k_1, k_2, \ldots, k_m)$, $k = k_1 + \ldots + k_m$, $k_1 \geq k_2 \geq \ldots \geq k_m \geq 0$, and $C_\kappa(UAU')$ is a zonal polynomial of UAU' corresponding to a partition κ of k, James [5].

$P_\kappa(T,A)$ has the following properties.

$$(2) \qquad P_\kappa(0,A) = (\tfrac{n}{2})_\kappa C_\kappa(A)/C_\kappa(I_n),$$

$$(3) \qquad P_\kappa(T,I_n) = H_\kappa(T),$$

$$(4) \qquad |P_\kappa(T,A)| \leq etr(TT')(\tfrac{n}{2})_\kappa C_\kappa(A)/C_\kappa(I_n),$$

where

$$(a)_\kappa = \prod_{\alpha=1}^{m} (a - \frac{\alpha-1}{2})_{k_\alpha}, \quad (a)_n = a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},$$

and $H_\kappa(T)$ is a generalized Hermite polynomial of matrix argument T.

The generating function of $P_\kappa(T,A)$ is given by

$$(5) \qquad \int_{O(m)} \int_{O(n)} etr(-UH_2AH_2'U' + 2H_1UH_2A^{\frac{1}{2}}T')\, d(H_1)\, d(H_2)$$

$$= \sum_{k=0}^{\infty} \sum_\kappa \frac{P_\kappa(T,A)C_\kappa(UU')}{k!(\frac{n}{2})_\kappa C_\kappa(I_m)}$$

and the right hand side (R.H.S.) of (5) converges absolutely with respect to U. $d(H_1)$ and $d(H_2)$ are the orthogonal invariant measures on the orthogonal groups $O(m)$ and $O(n)$, respectively. A detailed discussion of $H_\kappa(T)$ and $P_\kappa(T,A)$ may be found in Hayakawa [3].

We give here some useful lemmas which will be applied to the representation of the p.d.f. of a trace of a non-central quadratic form.

Lemma 1.

(6)
$$\sum_{\kappa} P_{\kappa}(T,A) = (-1)^k[A_k + \frac{1}{2}\sum_{\ell=0}^{k} A_{k-\ell}A_{\ell}$$
$$+ \frac{1}{3!}\sum_{\ell_1=0}^{k}\sum_{\ell_2=0}^{\ell_1} A_{k-\ell_1} A_{\ell_1-\ell_2} A_{\ell_2} + \cdots$$
$$\cdots + \frac{1}{k!}\sum_{\ell_1=0}^{k}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_{k-1}=0}^{\ell_{k-2}} A_{k-\ell_1} A_{\ell_1-\ell_2}\cdots$$
$$\cdots A_{\ell_{k-2}-\ell_{k-1}} A_{\ell_{k-1}}] ,$$

where

$$A_{\ell} = \frac{m}{2\ell} \mathrm{tr} A^{\ell} - \mathrm{tr} T A^{\ell} T', \qquad \ell = 1,2,\ldots,k$$

and

$$A_0 = 0, \quad \text{for convenience.}$$

Proof: From the definition of $P_{\kappa}(T,A)$, we construct the generating function of $(-1)^k \sum_{\kappa} P_{\kappa}(T,A)$.

(7)
$$\sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \sum_{\kappa} P_{\kappa}(T,A)$$
$$= \frac{\mathrm{etr}(TT')}{\pi^{\frac{1}{2}mn}} \int_U \mathrm{etr}(-2iTU')\mathrm{etr}(-UU')\mathrm{etr}(xUAU')\, dU$$
$$= \det(I-xA)^{-(m/2)}\, \mathrm{etr}(TT').$$

$$\cdot \frac{1}{\pi^{mn/2}} \int_U etr(-UU'-2iT(I-xA)^{-\frac{1}{2}}U')\, dU$$

$$= det(I-xA)^{-(m/2)}\, etr(T(I-(I-xA)^{-1})T'),$$

$$||xA|| < 1,$$

where $||A||$ means the maximum value of the absolute value of the characteristic roots of A.

We expand the R.H.S. of (7) with respect to x by noting that $||xA|| < 1$ using

$$log\ det(I-xA) = xtrA + \frac{x^2}{2} trA^2 + \ldots + \frac{x^k}{k} trA^k + \ldots$$

and

$$(I-xA)^{-1} = I + xA + x^2A^2 + \ldots + x^kA^k + \ldots \quad .$$

Hence

$$\text{R.H.S.} = exp[-\frac{m}{2} log\ det(I-xA)]\ etr(-xTA(I-xA)^{-1}T')$$

$$= exp[xtr(\frac{m}{2}A-TAT') + x^2tr(\frac{m}{4}A^2-TA^2T') + \ldots$$

$$\ldots + x^ktr(\frac{m}{2k}A^k-TA^kT') + \ldots] \ .$$

Here we set

$$A_\ell = tr(\frac{m}{2\ell}A^\ell-TA^\ell T'), \quad \ell = 1,2,\ldots$$

and

$$A_0 = 0, \quad \text{for convenience.}$$

We can obtain the value of $(-1)^k \Sigma_\kappa P_\kappa(T,A)$ by comparing the coefficients of x^k on the two sides of (7). By differentiating the left hand side with respect to x and by setting $x = 0$, we have $(-1)^k \Sigma_\kappa P_\kappa(T,A)$. Let us divide the R.H.S. into two parts such that

$$\begin{aligned} \text{R.H.S.} &= \exp(g(x)) \exp(f(x)) \\ &= \{1 + g(x) + \frac{1}{2!} g^2(x) + \dots + \frac{1}{k!} g^k(x) + \dots\} \\ &\quad \cdot \{1 + f(x) + \frac{1}{2!} f^2(x) + \dots\} , \end{aligned}$$

where

$$g(x) = \sum_{j=0}^{k} A_j x^j, \quad f(x) = \sum_{j=k+1}^{\infty} A_j x^j .$$

The degrees of x in the second factor are all greater than or equal to $k+1$, and so there is no contribution to the coefficient of x^k in the R.H.S. Hence we need only consider the first factor. We have

$$g'(0) = A_1, \quad g''(0) = 2A_2, \dots, g^{(k)}(0) = k!A_k ,$$

and

$$g(0) = 0 \ (\equiv A_0, \text{ by definition}).$$

Now we differentiate k times the power series of $g(x)$ and set $x = 0$.

$$g^{(k)}(0) = k!A_k ,$$

$$(g^2(x))^{(k)}\Big|_{x=0} = \sum_{\ell=0}^{k} \binom{k}{\ell} g^{(\ell)}(x) g^{(k-\ell)}(x)\Big|_{x=0} = k! \sum_{\ell=0}^{k} A_{k-\ell} A_\ell ,$$

$$(g^3(x))^{(k)}\Big|_{x=0} = \sum_{\ell_1=0}^{k} \binom{k}{\ell_1} (g^2(x))^{(\ell_1)} g^{(k-\ell_1)}(x)\Big|_{x=0}$$

$$= \sum_{\ell_1=0}^{k} \sum_{\ell_2=0}^{\ell_1} \binom{k}{\ell_1}\binom{\ell_1}{\ell_2} g^{(\ell_2)}(x) g^{(\ell_1-\ell_2)}(x) g^{(k-\ell_1)}(x)\Big|_{x=0}$$

$$= k! \sum_{\ell_1=0}^{k} \sum_{\ell_2=0}^{\ell_1} A_{k-\ell_1} A_{\ell_1-\ell_2} A_{\ell_2}.$$

In the same way we have

$$(g^k(x))^{(k)}\Big|_{x=0} = k! \sum_{\ell_1=0}^{k} \sum_{\ell_2=0}^{\ell_1} \cdots \sum_{\ell_{k-1}=0}^{\ell_{k-2}} A_{k-\ell_1} A_{\ell_1-\ell_2} \cdots A_{\ell_{k-2}-\ell_{k-1}} A_{\ell_{k-1}},$$

and

$$(g^p(x))^{(k)}\Big|_{x=0} = 0, \quad \text{for} \quad p \geq k+1.$$

Hence combining the results, we have

$$\left\{exp(g(x))\right\}^{(k)}\Big|_{x=0} = k![A_k + \frac{1}{2}\sum_{\ell=0}^{k} A_{k-\ell}A_\ell + \frac{1}{3!}\sum_{\ell_1=0}^{k}\sum_{\ell_2=0}^{\ell_1} A_{k-\ell_1}A_{\ell_1-\ell_2}A_{\ell_2}$$

$$+ \ldots + \frac{1}{k!}\sum_{\ell_1=0}^{k_1} \cdots \sum_{\ell_{k-1}=0}^{\ell_{k-2}} A_{k-\ell_1}A_{\ell_1-\ell_2}\cdots A_{\ell_{k-2}-\ell_{k-1}}A_{\ell_{k-1}}],$$

which completes the proof.

EXAMPLES:

$k = 1$: $P_1(T,A) = (-1)A_1 = -\frac{m}{2}\, tr A + tr TAT'$,

$k = 2$: $\sum_{(2)} P_{(2)}(T,A) = 2![A_2 + \frac{1}{2} A_1]$

$$= 2![tr(\frac{m}{4} A^2 - TA^2T') + \frac{1}{2}\{tr(\frac{m}{2} A - TAT')\}^2],$$

$$k = 3: \quad \sum_{(3)} P_{(3)}(T,A) = -3![A_3 + A_1A_2 + \frac{1}{3!} A_1^3]$$

$$= -3![tr(\frac{m}{6} A^3 - TA^3T') + tr(\frac{m}{2} A - TAT') \cdot$$

$$\cdot\, tr(\frac{m}{4} A^2 - TA^2T')$$

$$+ \frac{1}{3!} \{tr(\frac{m}{2} A - TAT')\}^3],$$

etc.

REMARK. If we set $A = I_n$, then we have immediately from Hayakawa [4, (18)]

$$\sum_{\kappa} P_{\kappa}(T,I_n) = \sum_{\kappa} H_{\kappa}(T) = (-1)^k L_k^{(mn/2)-1}(trTT') .$$

3. THE P.D.F. OF $tr\, \Sigma^{-1}XAX'$.

Let X be an $m\times n$ $(m \le n)$ matrix whose density function is given by

$$(8) \qquad \frac{1}{(2\pi)^{mn/2}(det\Sigma)^{n/2}(detB)^{m/2}}\, etr[-\frac{1}{2}\Sigma^{-1}(X-M)B^{-1}(X-M)'] ,$$

Where Σ is an $m\times m$ p.d.s. matrix, B is an $n\times n$ p.d.s. matrix and M is an $m\times n$ $(m \le n)$ matrix such that $E(X) = M$ and rank $M = m$. Let A be an $n\times n$ p.d.s. matrix.

LEMMA 2. *(Power series representation.)* Let X be distributed with p.d.f. (8), then the p.d.f. of the latent roots $\Lambda = diag(\lambda_1,\ldots,\lambda_m)$ of $\Sigma^{-\frac{1}{2}}XAX'\Sigma^{-\frac{1}{2}}$ is given by

(9) $$\frac{\pi^{(m^2/2)} etr(-\frac{1}{2}MB^{-1}M'\Sigma^{-1})}{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})(det 2AB)^{(m/2)}} (det\Lambda)^{\frac{1}{2}(n-m-1)} \prod_{i<j} (\lambda_i-\lambda_j)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{P\ (\frac{1}{\sqrt{2}}\ \Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}}, C^{-1})C_\kappa(\frac{1}{2}\Lambda)}{k!(\frac{n}{2})_\kappa C_\kappa(I_m)},$$

where $\Gamma_m(a) = \pi^{\frac{1}{4}m(m-1)} \prod_{\alpha=1}^{m} \Gamma(a-\frac{\alpha-1}{2})$ and $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

PROOF: See Hayakawa [3].

Next we give another type (say, Γ-type) expression for this p.d.f.

THEOREM 1. *(Γ-type representation.)* Let X be distributed with p. d.f. (8), then the p.d.f. of the latent roots $\Lambda = diag(\lambda_1,\ldots,\lambda_m)$ of $\Sigma^{-\frac{1}{2}}XAX'\Sigma^{-\frac{1}{2}}$ is given by, for $||AB|| < p$,

(10) $$\frac{\pi^{(m^2/2)} etr(-\frac{1}{2}\Sigma^{-1}MB^{-1}M')}{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})(det 2AB)^{m/2}}\, etr(-\frac{1}{2p}\Lambda)(det\Lambda)^{\frac{1}{2}(n-m-1)} \prod_{i<j} (\lambda_i-\lambda_j)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{P_\kappa(\frac{1}{\sqrt{2}}\ \Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1}-\frac{I}{p})^{-\frac{1}{2}}, C^{-1}-\frac{I}{p})C_\kappa(\frac{1}{2}\Lambda)}{k!(\frac{n}{2})_\kappa C_\kappa(I_m)}$$

where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

The power series converges absolutely for $\Lambda > 0$.

PROOF: We decompose $Y = \Sigma^{-\frac{1}{2}}XA^{\frac{1}{2}}$ as

$$Y = \Sigma^{-\frac{1}{2}}XA^{\frac{1}{2}} = H_1\Lambda^{\frac{1}{2}}L,$$

where H_1 is an orthogonal matrix of order m whose elements of the first column are positive and $\Lambda^{\frac{1}{2}} = diag(\lambda_1^{\frac{1}{2}},\ldots,\lambda_m^{\frac{1}{2}})$ and $\lambda_1,\ldots,\lambda_m$ are latent roots of $YY' = \Sigma^{-\frac{1}{2}}XAX'\Sigma^{-\frac{1}{2}}$, and L is an $m\times n$ Stiefel matrix such that $LL' = I_m$. By inserting this decomposition into (8), we have the joint p.d.f. of Λ, H_1 and L:

$$(11)\qquad \frac{\pi^{(m^2/2}etr(-\frac{1}{2}\Sigma^{-1}MB^{-1}M')}{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})(det2AB)^{(m/2)}}\, etr(-\frac{1}{2}\Lambda^{\frac{1}{2}}LA^{-\frac{1}{2}}B^{-1}A^{-\frac{1}{2}}L'\Lambda^{\frac{1}{2}}$$

$$+ H_1\Lambda^{\frac{1}{2}}LA^{-\frac{1}{2}}B^{-1}M'\Sigma^{-\frac{1}{2}})(det\Lambda)^{\frac{1}{2}(n-m-1)} \prod_{i<j}(\lambda_i-\lambda_j)d\Lambda d(H_1)d(L)\ .$$

If we set $L \to LH_2$, $H_2\epsilon O(n)$, then $LH_2(LH_2)' = I_m$ and $d(L)$ remains invariant with respect to H_2. Then the integral with respect to $O(m)$ and $O(n)$ is the same form as (5) with $U = \frac{1}{\sqrt{2}}\Lambda^{\frac{1}{2}}L$ and $T = \frac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1} - \frac{I}{p})^{\frac{1}{2}}$, where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$. Hence .

$$\int_{LL'=I_m} d(L)etr(-\frac{1}{2p}\Lambda)\frac{1}{2^m}\int_{O(m)}\int_{O(n)} etr[-\frac{1}{2}\Lambda^{\frac{1}{2}}LH_2(C^{-1} - \frac{I}{p})H_2'L'\Lambda^{\frac{1}{2}}$$

$$+ H_1\Lambda^{\frac{1}{2}}LH_2(C^{-1} - \frac{I}{p})^{\frac{1}{2}}(C^{-1} - \frac{I}{p})^{-\frac{1}{2}}A^{-\frac{1}{2}}B^{-1}M'\Sigma^{-\frac{1}{2}})\ d(H_1)\ d(H_2)$$

$$= etr(-\frac{1}{2p}\Lambda)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\frac{1}{2}\Lambda)}{k!(\frac{n}{2})_\kappa C_\kappa(I_m)} P_\kappa(\frac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1} - \frac{I}{p})^{-\frac{1}{2}}, C^{-1} - \frac{I}{p})\ .$$

The proof of the absolutely convergence is easily achieved by using (4), which completes the proof.

NOTE. Since

$$\int_{\lambda_1>\dots>\lambda_m>0} etr(-\frac{1}{2p}\Lambda)(det\Lambda)^{\frac{1}{2}(n-m-1)}\prod_{i<j}(\lambda_i-\lambda_j)C_\kappa(\frac{1}{2}\Lambda)d\Lambda$$

$$= \frac{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})}{\pi^{m^2/2}} (2p)^{mn/2} (\tfrac{n}{2})_\kappa\, p^k\, C_\kappa(I) ,$$

we have the following relations.

$$\text{(12)} \qquad \sum_{k=0}^{\infty} \sum_{\kappa} \frac{p^k}{k!} P_\kappa(\tfrac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1} - \tfrac{I}{p})^{-\frac{1}{2}},\ C^{-1} - \tfrac{I}{p})$$

$$= etr(-\tfrac{1}{2}\Sigma^{-1}MB^{-1}M')(det AB/p)^{mn/2}.$$

This formula can be obtained from (7) directly if we replace x with $-p$, T with $\frac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1} - \frac{I}{p})^{-\frac{1}{2}}$, and A with $C^{-1} - \frac{I}{p}$, respectively.

Hayakawa [3] obtained the p.d.f. of $tr\Sigma^{-\frac{1}{2}}XAX'\Sigma^{-\frac{1}{2}}$ as a power series representation. We give this as Lemma 3 for comparison with Theorem 2.

LEMMA 3. *(Power Series representation.)* Let Λ be distributed with p.d.f. (9), then the p.d.f. of $T = tr\Lambda$ is given by

(13)

$$\frac{etr(-\frac{1}{2}\Sigma^{-1}MB^{-1}M')}{\Gamma(\frac{mn}{2})(det 2AB)^{m/2}} T^{(mn/2)-1} \sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\tfrac{T}{2})^k \sum_{\kappa} P_\kappa(\tfrac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}}, C^{-1}),$$

where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

PROOF: See [3].

THEOREM 2. *(Γ-type representation.)* Let Λ be distributed with p.d.f. (10), then the p.d.f. of $T = tr\Lambda$ is given by

$$(14)\qquad \frac{etr(-\frac{1}{2}\Sigma^{-1}MB^{-1}M')}{\Gamma(\frac{mn}{2})(det2AB)^{m/2}}\, exp(-\frac{T}{2p})\cdot$$

$$\cdot T^{(mn/2)-1}\sum_{k=0}^{\infty}\frac{1}{k!(\frac{mn}{2})_k}(\frac{T}{2})^k\sum_{\kappa}P_{\kappa}(\frac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1}-\frac{I}{p})^{-\frac{1}{2}},$$

$$C^{-1}-\frac{I}{p}),$$

where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

The series converges absolutely for $T > 0$.

PROOF: By applying a Fourier transform to $T = tr\Lambda$ and inverting it, we obtain (14) easily. Q.E.D.

4. THE RELATION WITH A UNIVARIATE QUADRATIC FORM.

We can derive the p.d.f. of $tr\Sigma^{-1}XAX'$ by another way. We denote X and M as

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad M = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix},$$

where $x_\alpha = (x_{\alpha 1},\ldots,x_{\alpha n})$ and $\mu_\alpha = (\mu_{\alpha 1},\ldots,\mu_{\alpha n})$, $\alpha = 1,2,\ldots,m$. Let $x = (x_1,x_2,\ldots,x_m)$ and $\mu = (\mu_1,\mu_2,\ldots,\mu_m)$, then x is distributed with mean μ and a covariance matrix $\Sigma\otimes B$, where $\otimes$ denotes a Kronecker product of Σ and B. On the other hand, $trXAX' = \Sigma_{\alpha=1}^{n}x_\alpha Ax_\alpha' = x[I_m\otimes A]x'$. Hence the problem is reduced to the one of the univariate non-central quadratic form. To compare with the results of Kotz et al [7] and Ruben [8], we can assume that A is a diagonal matrix whose diagonal elements are $a_1,a_2,\ldots,a_n$ and $a_1 \geq a_2 \geq \ldots \geq a_n > 0$.

The p.d.f. of $\mathit{tr}XAX'$ is derived as follows.

$$(15)\qquad \frac{1}{(2\pi)^{\frac{mn}{2}}(\det\Sigma)^{\frac{n}{2}}(\det B)^{\frac{m}{2}}}\int_{T=x(I_m\otimes A)x'} \exp[-\tfrac{1}{2}(x-\mu)(\Sigma^{-1}\otimes B^{-1})(x-\mu)']\, dX$$

$$= \frac{\exp[-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu']}{(2\pi)^{\frac{mn}{2}}(\det\Sigma)^{\frac{n}{2}}(\det AB)^{\frac{m}{2}}}\int_{T=xx'} \exp[-\tfrac{1}{2}(\Sigma^{-1}\otimes C^{-1})x' + x(\Sigma^{-1}\otimes A^{-\frac{1}{2}}B^{-1})\mu']\, dX$$

where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

From the above integral, we derive two types of the representation.

THEOREM 3. *(Power series representation.)* Let X be distributed with p.d.f. (8), then the p.d.f. of $T = \mathit{tr}XAX'$ is given by (16)

$$\frac{\exp[-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu']}{2^{\frac{mn}{2}}\Gamma(\frac{mn}{2})(\det\Sigma)^{\frac{n}{2}}(\det AB)^{\frac{m}{2}}}\, T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty}\frac{1}{k!(\frac{mn}{2})_k}\left(\frac{T}{2}\right)^k P_k\left(\frac{1}{\sqrt{2}}\mu(\Sigma^{-\frac{1}{2}}\otimes B^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}}),\ \Sigma^{-1}\otimes C^{-1}\right),$$

where $P_k(\cdot,\cdot)$ is a polynomial for $m = 1$ in the definition of $P_K(T,A)$ and $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

PROOF: As for Lemma 2.

COROLLARY. The p.d.f. of $T = \mathit{tr}\Sigma^{-1}XAX'$ is given by

$$(17)\qquad \frac{\exp[-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu']}{2^{\frac{mn}{2}}\Gamma(\frac{mn}{2})(\det AB)^{\frac{m}{2}}}\, T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\frac{T}{2})^k P_k(\frac{1}{\sqrt{2}}\mu(\Sigma^{-\frac{1}{2}}\otimes B^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}}), I \otimes C^{-1}).$$

We can easily show that (17) is the same form as (13).

Here we compare with the results of Kotz et al [7]. Kotz et al showed the following Lemma.

LEMMA. *(Kotz et al.)* Let $x = (x_1,\ldots,x_n)$ be normally distributed with mean 0 and covariance matrix I_n and A be a diagonal matrix, i.e. $diag(a_1,\ldots,a_n)$, $a_1 \geq a_2 \geq \ldots \geq a_n > 0$, and $b = (b_1,\ldots,b_n)$, then the p.d.f. of $T = (x+b)A(x+b)'$ is given by

$$(18) \qquad \sum_{k=0}^{\infty} \alpha_k^p (-1)^k (\frac{T}{2})^{(n/2)+k-1} \frac{1}{2\Gamma(\frac{n}{2}+k)} .$$

The α_k^p's are determined by

$$(19) \qquad \sum_{k=0}^{\infty} \alpha_k^p \theta^k = (detA)^{-\frac{1}{2}} exp[-\frac{1}{2}\sum_{i=1}^{n} b_i^2/(1-\theta/a_i)] \prod_{i=1}^{n} (1-\theta/a_i)^{-\frac{1}{2}}$$

where the recurrence relation

$$(20) \qquad \alpha_0^p = (detA)^{-\frac{1}{2}} exp(-\frac{1}{2}\sum_{i=1}^{n} b_i^2)$$

$$\alpha_k^p = \sum_{r=0}^{k-1} b_{k-r} \alpha_r^p/k, \quad k \geq 1,$$

with $b_k^p = \frac{1}{2}\sum_{i=1}^{n}(1-kb_i^2)(a_i)^{-k}$ can be obtained.
(α_0^p and b_k^p of Kotz et al should be changed to above form.)

To compare with Theorem 3 and Lemma (Kotz et al), we set $\Sigma = I_m$ and $B = I_n$ in (16) and we have

$$(16)' \quad \frac{exp[-\frac{1}{2}\mu\mu']}{\Gamma(\frac{mn}{2})(det2A)^{\frac{m}{2}}} T^{(mn/2)-1} \sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\frac{T}{2})^k P_k(\frac{1}{\sqrt{2}}\mu, I\otimes A^{-1})$$

$$= \frac{etr(-\frac{1}{2}MM')}{\Gamma(\frac{mn}{2})(det2A)^{\frac{m}{2}}} T^{(mn/2)-1} \sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\frac{T}{2})^k \sum_{\kappa} P_{\kappa}(\frac{1}{\sqrt{2}}M, A^{-1}).$$

Now replacing A by $I_m\otimes A$ and b by μ in Lemma (Kotz et al), we have the following form.

$$(18)' \quad \sum_{k=0}^{\infty} \alpha_k^p (-1)^k (\frac{T}{2})^{(mn/2)+k-1} \frac{1}{2\Gamma(\frac{mn}{2}+k)},$$

$$(19)' \quad \sum_{k=0}^{\infty} \alpha_k^p \theta^k = (detA)^{-(m/2)} exp[-\frac{1}{2}\sum_{j=1}^{n} \frac{1}{1-\theta/a_j} \sum_{i=1}^{m} \mu_{ij}^2] \prod_{j=1}^{n} (1-\theta/a_j)^{-(m/2)}$$

$$(20)' \quad \alpha_0^p = (detA)^{-(m/2)} etr(-\frac{1}{2}MM')$$

$$(21)' \quad b_k^p = \sum_{j=1}^{n} (\frac{1}{a_j})^k \sum_{i=1}^{m} (1-k\mu_{ij}^2).$$

Therefore, by comparing with (16)' and (18)', we have the following relation.

$$(22) \quad \alpha_k^p = \frac{(-1)^k}{k!} \frac{etr(-\frac{1}{2}MM')}{(detA)^{m/2}} \sum_{\kappa} P_{\kappa}(\frac{1}{\sqrt{2}}M, A^{-1}),$$

where $(-1)^k\Sigma_{\kappa}P_{\kappa}(\frac{1}{\sqrt{2}}M, A^{-1})$ is given by Lemma 1. Hence (22) gives an explicit form for α_k^p not involving a recurrence relation. We can also easily check that if we insert (22) into the left hand side of (19)', then we have (7).

Next we compare with the Γ-type representation.

THEOREM 4. *(Γ-type representation.)* Let X be distributed with the p.d.f. (8), then the p.d.f. of $T = tr XAX'$ is given by

$$\frac{exp(-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu')}{\Gamma(\frac{mn}{2})(det\Sigma)^{\frac{n}{2}}(det2AB)^{\frac{m}{2}}}\ exp(-\frac{1}{2p}T)\ T^{(mn/2)-1} \tag{23}$$

$$\sum_{k=0}^{\infty}\frac{1}{k!(\frac{mn}{2})_k}(\frac{T}{2})^k P_k(\frac{1}{\sqrt{2}}\mu(\Sigma^{-1}\otimes B^{-1}A^{-\frac{1}{2}})D^{-\frac{1}{2}}, D),$$

where $D = \Sigma^{-1}\otimes C^{-1} - I_m\otimes I_n/p$.

PROOF: From (15) and the proof of Theorem 3, we can easily show (23).

COROLLARY. The p.d.f. of the $T = tr\Sigma^{-1}XAX'$ is given by

$$\frac{exp[-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu']}{\Gamma(\frac{mn}{2})(det2AB)^{\frac{m}{2}}}\ exp(-\frac{1}{2p}T)\ T^{(mn/2)-1} \tag{24}$$

$$\sum_{k=0}^{\infty}\frac{1}{k!(\frac{mn}{2})_k}(\frac{T}{2})^k P_k(\frac{1}{\sqrt{2}}\mu(\Sigma^{-\frac{1}{2}}\otimes B^{-1}A^{-\frac{1}{2}})(I\otimes(C^{-1}-I/p))^{\frac{1}{2}}\ ,\ \ I\otimes(C^{-1}-I/p)).$$

We can show easily that (24) is the same form as (14).

Ruben [8] gave a Γ-type representation of a quadratic form which is obtained in the following lemma.

LEMMA. *(Ruben)* Under the same condition of Lemma (Kotz et al), the p.d.f. of $T = (x+b)A(x+b)'$ is given by

$$\sum_{k=0}^{\infty}\alpha_k^c\ \frac{e^{-(T/2p)}T^{(n/2)+k-1}}{2^{(n/2)+k}\ \Gamma(\frac{n}{2}+k)}\ (\frac{1}{p})^{(n/2)+k}\ . \tag{25}$$

The α_k^c are determined by

$$(26) \qquad \sum_{k=0}^{\infty} \alpha_k^c \theta^k = (detA/p)^{-\frac{1}{2}} exp[-\frac{1}{2}\sum_{k=1}^{n} b_k^2 \frac{1-\theta}{1-(1-p/a_k)\theta}] \prod_{j=1}^{n} \frac{1}{\{1-(1-p/a_j)\theta\}^{\frac{1}{2}}} .$$

Hence the recurrence relation

$$\alpha_0^c = (detA/p)^{-\frac{1}{2}} exp[-\frac{1}{2} \sum_{i=1}^{n} b_i^2]$$

$$\alpha_k^c = \sum_{r=0}^{k-1} b_{k-r}^c \alpha_r^c/2k, \quad k \geq 1,$$

with

$$b_k^c = \sum_{j=1}^{n} (1 - \frac{p}{a_j})^k + kp \sum_{j=1}^{n} \frac{b_j^2}{a_j} (1 - \frac{p}{a_j})^{k-1}$$

can be obtained.

To compare with Theorem 4 and Lemma (Ruben), we set $\Sigma = I_m$ and $B = I_n$ in (23) and we replace A with $I_m \otimes A$ and b with μ in (26). Then we have

$$(23)' \qquad \frac{exp(-\frac{1}{2}\mu\mu')}{2^{\frac{mn}{2}} \Gamma(\frac{mn}{2}) (detA)^{\frac{m}{2}}} exp(-\frac{T}{2p}) T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\frac{T}{2})^k P_k(\frac{1}{\sqrt{2}}\mu(I\otimes A^{-\frac{1}{2}}(A^{-1} - \frac{I}{p})^{\frac{1}{2}}, (I\otimes A^{-1}-I/p)$$

$$= \frac{etr(-\frac{1}{2}MM')}{2^{\frac{mn}{2}} \Gamma(\frac{mn}{2}) (detA)^{\frac{m}{2}}} exp(-\frac{T}{2p}) T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\frac{T}{2})^k \sum_{\kappa} P_\kappa(\frac{1}{\sqrt{2}}MA^{-\frac{1}{2}}(A^{-1}-I/p)^{\frac{1}{2}}, A^{-1} - I/p)$$

and

$$(25)' \qquad \sum_{k=0}^{\infty} \alpha_k^c \frac{e^{-(T/2p)} T^{(mn/2)+k-1}}{2^{(mn/2)+k} \Gamma(\frac{mn}{2}+k)} (\frac{1}{p})^{(mn/2)+k}$$

$$(26)' \qquad \sum_{k=0}^{\infty} \alpha_k^c \theta^k = (\det A/p)^{-(m/2)} \prod_{j=1}^{n} \{1-(1-P/a_j)\theta\}^{-(m/2)}$$

$$\exp[-\frac{1}{2} \sum_{j=1}^{n} \frac{1-\theta}{1-(1-P/a_j)\theta} \sum_{i=1}^{m} \mu_{ij}^2] .$$

Therefore, by comparing with (23)' and (25)', we have the following relation.

$$(27) \qquad \alpha_k^c = \frac{1}{k!} (\det A/p)^{-(m/2)} \mathrm{etr}(-\frac{1}{2}MM') \sum_{\kappa} P_{\kappa}(\frac{1}{\sqrt{2}}MA^{-\frac{1}{2}}(A^{-1}-I/p)^{\frac{1}{2}}, A^{-1}-I/p)$$

and $\Sigma_{\kappa} P_{\kappa}(\frac{1}{\sqrt{2}}MA^{-\frac{1}{2}}(A^{-1}-I/p)^{\frac{1}{2}}, A^{-1}-I/p)$ is given by Lemma 1. Hence (27) gives an explicit form for α_k^c, not involving a recurrence relation. We can easily check that if we insert (27) into the left hand side of (26)', then we obtain (7) by changing T and A into the appropriate variables.

From (27), we have the relation

$$(28) \qquad \frac{E(Q^k H_{2k}(L/Q^{\frac{1}{2}}))}{2^k(2k-1)!!} = \sum_{\kappa} P_{\kappa}(\frac{1}{\sqrt{2}} MA^{-\frac{1}{2}}(A^{-1}-I/p)^{\frac{1}{2}}, A^{-1}-I/p),$$

where

$$L = \sum_{j=1}^{n} \frac{1}{\sqrt{a_j}} \sum_{i=1}^{m} \mu_{ij} x_{ij}, \quad Q = \sum_{i=1}^{m} \sum_{j=1}^{n} (\frac{1}{a_j} - \frac{1}{p}) x_{ij}^2,$$

x_{ij} are independent normal variables with zero mean and unit variances, and $H_{2k}(y)$ is an Hermite polynomial of order $2k$, and $(2k-1)!! = 1\cdot 3 \ldots (2k-1)$.

NOTE. If we set $M = 0$ and $B = I_n$ in (16), then we have the p.d.f. of a central case, given in Hayakawa [2] by using the zonal polynomials.

5. THE COMPLEX MULTIVARIATE QUADRATIC FORM.

In this section, we shall state the above results for the complex Gaussian distribution studied by Goodman [1], James [5] and Khatri [6].

Let T and U be $m \times n$ $(m \le n)$ complex arbitrary matrices whose rank are m, respectively, and A be an $n \times n$ positive definite Hermitian matrix. We define $\tilde{P}_\kappa(T,A)$ as follows:

$$\text{(29)} \qquad etr(-T\bar{T}')\tilde{P}_\kappa(T,A) = \frac{(-1)^k}{\pi^{mn}} \int_U etr(-i(T\bar{U}'+U\bar{T}'))etr(-U\bar{U}')\tilde{C}_\kappa(UA\bar{U}')dU,$$

where $\tilde{C}_\kappa(UA\bar{U}')$ is a zonal polynomial of a Hermitian matrix $UA\bar{U}'$. Then we can show that

$$\text{(30)} \qquad \tilde{P}_\kappa(0,A) = [n]_\kappa \tilde{C}_\kappa(A)/\tilde{C}_\kappa(I_n),$$

$$\text{(31)} \qquad \tilde{P}_\kappa(T,I_n) = \tilde{H}_\kappa(T),$$

$$\text{(32)} \qquad |\tilde{P}_\kappa(T,A)| \le etr(T\bar{T}')[n]_\kappa \tilde{C}_\kappa(A)/\tilde{C}_\kappa(I_n),$$

where

$$[n]_\kappa = \prod_{\alpha=1}^{m} (a-\alpha+1)_{k_\alpha}$$

and $\tilde{H}_\kappa(T)$ is a generalized complex Hermite polynomial of a matrix argument T.

The generating function of $\tilde{P}_\kappa(T,A)$ is given by

$$(33)\qquad \int_{U(m)}\int_{U(n)} etr(-SU_2A\bar{U}_2'\bar{S}'+U_1SU_2A^{\frac{1}{2}}\bar{T}'+TA^{\frac{1}{2}}\bar{U}_2'\bar{S}'\bar{U}_1')d(U_1)d(U_2)$$

$$= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\tilde{P}_\kappa(T,A)\tilde{C}_\kappa(S\bar{S}')}{k![n]_\kappa \tilde{C}_\kappa(I_m)} .$$

The R.H.S. of (33) converges absolutely with respect to S, where S is an $m\times n$ $(m \le n)$ complex arbitrary matrix, U_1 and U_2 are unitary matrix of order m and n, respectively, and $d(U_1)$ and $d(U_2)$ are the unitary invariant measures over the unitary groups $U(m)$ and $U(n)$, respectively.

A detailed discussion of $\tilde{H}_\kappa(T)$ may be found in Hayakawa [4].

Lemma 2.

$$(34)\quad \sum_{\kappa}\tilde{P}_\kappa(T,A) = (-1)^k k!\,[\tilde{A}_k+\frac{1}{2}\sum_{\ell=0}^{k}\tilde{A}_{k-\ell}\tilde{A}_\ell+\frac{1}{3!}\sum_{\ell_1=0}^{k}\sum_{\ell_2=0}^{\ell_1}\tilde{A}_{k-\ell_1}\tilde{A}_{\ell_1-\ell_2}\tilde{A}_{\ell_2}$$

$$+\dots+\frac{1}{k!}\sum_{\ell_1=0}^{k}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_{n-1}=0}^{\ell_{k-2}}\tilde{A}_{k-\ell_1}\cdots\tilde{A}_{\ell_{k-2}-\ell_{k-1}}\tilde{A}_{\ell_{k-1}}],$$

where

$$\tilde{A}_\ell = \frac{m}{k}\,tr A^\ell - tr TA^\ell\bar{T}', \quad \ell = 1,2,\dots,k$$

and

$$\tilde{A}_0 = 0, \quad \text{for convenience.}$$

Proof: Similar to Lemma 1.

Let X be an $m\times n$ $(m\le n)$ complex matrix whose denisty function is given by

$$(35) \qquad \frac{1}{\pi^{mn}(\det\Sigma)^n(\det B)^m}\, etr[-\Sigma^{-1}(X-M)B^{-1}(\bar{X}'-\bar{M}')],$$

where Σ is an $m\times m$ p.d. Hermite matrix, M is an $m\times n$ complex matrix whose rank is m, and B is an $n\times n$ p.d. Hermitian matrix. Let A be an $n\times n$ p.d. Hermitian matrix. We denote X and M as

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad M = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix},$$

where $x_\alpha = (x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha n})$ and $\mu_\alpha = (\mu_{\alpha 1}, \ldots, \mu_{\alpha n})$, $\alpha = 1,2,\ldots,m$. By setting $x = (x_1, x_2, \ldots, x_m)$ and $\mu = (\mu_1\ \mu_2\ ..\mu_m)$, we rewrite $trXA\bar{X}'$ $= x(I_m \otimes A)\bar{x}'$ and x has an mn dimensional complex Gaussian distribution with mean μ and covariance $\Sigma\otimes B$. Then by applying the same method as in the real variate case, we have the following theorem.

THEOREM 5. *(Power series representation.)* Let X be distributed with the p.d.f. (35), then the p.d.f. of $T = trXA\bar{X}'$ is given by

$$(36) \qquad \frac{exp(-\mu(\Sigma^{-1}\otimes B^{-1})\bar{\mu}')}{\Gamma(mn)(\det\Sigma)^n(\det AB)^m}\, T^{mn-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!(mn)_k}\, T^k\, \tilde{P}_\kappa(\mu(\Sigma^{-\frac{1}{2}}\otimes B^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}},\ \Sigma^{-1}\otimes C^{-1}),$$

where $\tilde{P}_k(\cdot,\cdot)$ is a polynomial for $m = 1$ in the definition of $\tilde{P}_\kappa(T,A)$ and $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

THEOREM 6. *(Γ-type representation.)* Let X be distributed with the p.s.f. (35), then the p.d.f. of $T = trXA\bar{X}'$ is given by

(37) $$\frac{exp(-\mu(\Sigma^{-1}\otimes B^{-1})\mu')}{\Gamma(mn)(det AB)^m(det\Sigma)^n}\ exp(-\frac{1}{2p}T)\ T^{mn-1}$$

$$\sum_{k=0}^{\infty}\frac{1}{k!(mn)_k}\ T^k\ P_k(\mu(\Sigma^{-1}\otimes B^{-1}A^{-\frac{1}{2}})D^{-\frac{1}{2}},\ D),$$

where $D = \Sigma^{-1}\otimes C^{-1} - I_m\otimes I_n/p$ and $||AB|| < p$.

Theorem 5 and 6 can be proved in the same way as the real variate case. However, since the procedure is exactly the same, we will omit it.

ACKNOWLEDGMENT. The author wishes to express his sincere thanks to Professor Norman L. Johnson who read carefully the original version and gave some advice.

REFERENCES.

[1]. Goodman, N.R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction). *Ann. Math. Statist.*, 34, 152-177.

[2]. Hayakawa, T. (1966). On the distribution of a quadratic form in a multivariate normal sample. *Ann. Inst. Statist. Math., Tokyo*, 18, 191-201.

[3]. Hayakawa, T. (1969). On the distribution of the latent roots of a positive definite random symmetric matrix I. *Ann. Math. Statist. Math., Tokyo*, 21, 1-21.

[4]. Hayakawa, T. (1970). *On the distribution of the latent roots of a complex Wishart matrix (Non-central case).* Institute of Statist. Mimeo Series No. 667. University of North Carolina at Chapel Hill.

[5]. James, A.T. (1964). Distribution of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.*, 35, 475-501.

[6]. Khatri, C.G. (1966). On certain distribution problems based on positive definite quadratic functions in normal vectors. *Ann. Math. Statist.*, 37, 468-479.

[7]. Kotz, S., Johnson, N.L. and Boyd, D.W. (1967). Series representations of distributions of quadratic forms in normal variable II. Non-central case.
Ann. Math. Statist., 38, 838-848.

[8]. Ruben, H. (1962). Probability content of regions under spherical normal distribution IV: the distribution of homogeneous and non-homogeneous quadratic functions of normal variables.
Ann. Math. Statist., 33, 542-570.