SEQUENTIAL RANK TESTS FOR LOCATION

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1. Introduction. Recent years have seen the development of sequential rank tests. Some strong theoretical motivations for the use of such tests may be found in Hall, Wijsman and Ghosh [9]. Though, still at an early stage, quite a few research papers have come up in the area of two sample sequential rank tests ([2,3,4,13,21]). Most of these tests, however, are based on Lehmann alternatives, and as a result, are not applicable for the location problem. Besides, with the exception of termination with probability 1 (wp 1) and finite moment generating function (mgf) (for Wilcoxon scores), any other characteristics like the OC and ASN functions of such tests are very little known. Recently, Professor W.J.Hall [8] has suggested a sequential Wilcoxon test for the two sample location problem. His procedure achieves asymptotically the prescribed strength (α,β) and possesses the Wald [19] optimality for logistic shift. On the contrary, in the specification of the alternative hypothesis, it involves the use of a functional of the unknown distribution and thereby demands its knowledge ; this, however, does not appear to be very realistic.

Compared to the two sample location problem, relatively little attention has been paid to the so called one sample problem. Efforts in this direction made by Weed [20] (see also [5]) are subject to the same criticisms as in the two sample problem dealing with Lehmann alternatives.

In the present paper, we develop a general class of sequential rank tests for the one and two sample location problems. In the classical two sample location problem, once observations are taken in pairs at each stage of experimentation and we work with their differences (which are distributed symmetrically about the difference of the two locations) , the problem reduces to the corresponding one sample case. Hence, we shall only deal specifically with the one sample problem while the above remark permits us to handle equally the two sample case.

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In section 2, we start with an asymptotically equivalent form of the Wald [19] SPRT based on the maximum likelihood estimator (MLE) of location. Since this procedure demands the knowledge of the underlying distribution, in section 3, we consider an alternative procedure based on the sample means which along with some mild regularity conditions requires a strongly consistent estimator of the population variance. Both these procedures are vulnerable to gross errors or outliers, and, in addition, the second one may be quite inefficient for distributions with 'heavy tails'. In section 4, we work with robust estimators of location based on rank statistics, and after using the asymptotic linearity of rank statistics, we ultimately express our 'stopping rule' in terms of some well known rank statistics. It is shown that the proposed procedure terminates wp 1 when the underlying score functions are square integrable. Section 5 deals with the OC and ASN of the proposed tests, while section 6 is devoted to the study of the allied ARE (asymptotic relative efficiency) results. The last section includes, by way of remarks, a comparison of the proposed procedure with some alternative ones suggested by Albert [1] and others; it appears that the others may achieve the asymptotic optimality of the fixed sample size procedure but not of the SPRT.

2. An asymptotically equivalent form of the Wald SPRT. Let $\{X_1, X_2, \ldots \text{ ad inf}\}$ be a sequence of iidrv (independent and identically distributed random variables) with a df (distribution function) $F_\theta(x) = F(x - \theta)$, θ (ε Θ) is an unknown parameter. We want to test

(2.1) $H_0: \theta = 0$ against $H_1: \theta = \Delta$, where Δ is small.

We assume that (i) $F(x) + F(-x) = 1$ for all real x i.e., F is symmetric about 0, (ii) F is absolutely continuous wrt Lebesgue measure admitting a density f, (iii) f is strongly unimodal i.e., $-\log f(x)$ is a convex function of x, and (iv) $h(x) = -f'(x)/f(x)$ is absolutely continuous wrt Lebesgue measure. This implies that (a) $h(x)$ is $\uparrow$ in x , and (b)

(2.2) $I(f) = \int_{-\infty}^{\infty}(f'/f)^2 dF = \int_{-\infty}^{\infty} h^2(x)dF(x) = \int_{-\infty}^{\infty} h'(x)dF(x) < \infty$.

Finally, we assume that $h'(x)$ is uniformly continuous in x, so that

(2.3) $\lim_{\delta \to 0} \{ \sup_x | h'(x+\delta) - h'(x) | \} = 0.$

These assumptions are all met by a broad class of df's including the normal, logistic, double exponential, Cauchy and many other df's.

Consider now the Wald [19] SPRT of strength (α, β), and define

(2.4) $\lambda_m = \Sigma_{i=1}^{m} \log \{ f(X_i - \Delta)/f(X_i) \}, \quad m \geq 1,$

and two numbers A and B such that $0 < B < 1 < A < \infty$. Then, the stopping variable $N(\Delta)$ is defined to be the first positive integer for which the inequality

(2.5) $\log B < \lambda_m < \log A$

is vitiated. In the event $\lambda_{N(\Delta)} \leq \log B$, H_0 is accepted, while if $\lambda_{N(\Delta)} \geq \log A$, H_1 is accepted. From the results of Wald [19] it follows that $B \geq \beta/(1-\alpha)$ and $A \leq (1-\beta)/\alpha$. For small Δ, the excess over the boundaries can be neglected, and we have $B \simeq \beta/(1-\alpha)$, $A \simeq (1-\beta)/\alpha$, and

(2.6) $P\{\text{Type I error}\} \simeq \alpha$ and $P\{\text{Type II error}\} \simeq \beta$.

Next we propose an asymptotically (as $\Delta \to 0$) equivalent form of SPRT. Suppose θ is the true value of the location parameter, and write $\theta = \phi\Delta$, where we assume that

(2.7) $\phi \in I = \{ x : |x| \leq K$, where $K (1 < K < \infty)$ is fixed.$\}$

In the following lemma we give an asymptotic order of $N(\Delta)$ as $\Delta \to 0$, where P_ϕ stands for the probability computed under $\theta = \phi\Delta$.

<u>Lemma 2.1</u>. <u>Under</u> (2.2) <u>and</u> (2.3), <u>for every</u> $\varepsilon > 0$, <u>there exist finite positive constants</u> $c_1(\varepsilon)$, $c_2(\varepsilon)$, <u>such that</u>

(2.8) $\lim_{\Delta \to 0} [\sup_{\phi \in I} P_\phi\{ \Delta^{-2} c_1(\varepsilon) < N(\Delta) < \Delta^{-2} c_2(\varepsilon) \}] \geq 1 - \varepsilon.$

<u>Proof</u>. Let $Z_i(\Delta) = \log[f(X_i - \Delta)/f(X_i)]$, $i \geq 1$, and $Z_n^*(\Delta) = \Sigma_{i=1}^{n} Z_i(\Delta)$. Then,

(2.9) $$P_\phi[N(\Delta) > n] \leq P_\phi[\log B < Z_n^*(\Delta) < \log A]$$
$$= P_\phi \left[\frac{(\log B - E_\phi Z_n^*(\Delta))}{(n V_\phi(Z_1(\Delta)))^{\frac{1}{2}}} < \frac{(Z_n^*(\Delta) - E_\phi Z_n^*(\Delta))}{(n V_\phi(Z_1(\Delta)))^{\frac{1}{2}}} < \frac{(\log A - E_\phi Z_n^*(\Delta))}{(n V_\phi(Z_1(\Delta)))^{\frac{1}{2}}} \right].$$

Since, $Z_n^*(\Delta)$ involves a sum of iidrv's with a non-zero and finite variance, by the classical central limit theorem, one gets that for large n, $(Z_n^*(\Delta) - E_\phi Z_n^*(\Delta))/(nV_\phi(Z_1(\Delta)))^{\frac{1}{2}}$

is $\sim$ N(0,1). Hence, it suffices to show that for $n \simeq c_2(\varepsilon)/\Delta^2$ ($\to \infty$ as $\Delta \to 0$),

(2.10) $$\log(AB^{-1})/(\, n\, V_\phi(Z_1(\Delta))\,)^{\frac{1}{2}} < \sqrt{2\pi}\,\varepsilon \quad \text{as} \quad \Delta \to 0.$$

Now, when Δ is small and (2.3) holds, $\log\{f(x-\Delta)/f(x)\} = \Delta h(x-\Delta/2) + O(\Delta^2)$, uniformly in real x. So, for $\phi \in I$, Δ small, under (2.2) and (2.3),

(2.11) $$V_\phi(Z_1(\Delta)) \simeq \Delta^2 I(f) \;\Rightarrow\; nV_\phi(Z_1(\Delta)) \simeq c_2(\varepsilon)I(f) .$$

Thus, the lhs of (2.10) is approximately equal to (for small Δ) $\log(AB^{-1})/(c_2(\varepsilon)I(f))^{\frac{1}{2}}$, and this can be made smaller than $(2\pi)^{\frac{1}{2}}\varepsilon$ by proper choice of $c_2(\varepsilon)$. Again, on writing $2d = \min(-\log B, \log A)$, we have for small Δ ,

(2.12) $$P_\phi\{N(\Delta) \le n\} = P_\phi\{Z^*_m(\Delta) \notin (\log B, \log A), \text{ for some } m: 1 \le m \le n\}$$
$$\le P_\phi\{\max_{1\le m\le n} |Z^*_m(\Delta)| > 2d\} \le P_\phi\{\max_{1\le m\le n} |Z^*_m(\Delta) - E_\phi Z^*_m(\Delta)| > d\},$$

as for all $m \le n \simeq c_1(\varepsilon)/\Delta^2$, $|E_\phi Z^*_m(\Delta)| \le m\,\Delta^2(|\phi-\frac{1}{2}|\, I(f) + o(1)) \le c_1(\varepsilon)|\phi-\frac{1}{2}|I(f) + o(1)$ can be made smaller than d by proper choice of $c_1(\varepsilon)$. Again, since $\{Z^*_m(\Delta) - E_\phi Z^*_m(\Delta), \mathcal{B}_m\}$ forms a martingale sequence, where $\mathcal{B}_m$ is the σ-field generated by $Z_1(\Delta),\ldots, Z_m(\Delta)$ (and hence is $\uparrow$ in m), using the Kolmogorov inequality for martingales (see [11, p.386]), one gets by taking $n \simeq \Delta^{-2}c_1(\varepsilon)$ that

(2.13) $$P_\phi\{\max_{1\le m\le n} |Z^*_m(\Delta) - E_\phi Z^*_m(\Delta)| > d\} \le d^{-2}V_\phi(Z^*_n(\Delta))$$
$$= d^{-2}\{n\,\Delta^2 I(f)\,[1+o(1)]\} = d^{-2}c_1(\varepsilon)I(f)\{1+o(1)\} \le \varepsilon/2,$$

by proper choice of $c_1(\varepsilon)$. This leads to (for all $\phi \in \mathbf{I}$)

(2.14) $$P_\phi\{c_1(\varepsilon)\,\Delta^{-2} \le N(\Delta) \le c_2(\varepsilon)\Delta^{-2}\} \ge 1-\varepsilon , \text{ as } \Delta \to 0. \text{ Q.E.D.}$$

Define now $Y_i(\Delta) = Y_i = X_i - \Delta/2$, $i \ge 1$, $U_m(\Delta) = \Sigma_{i=1}^m h(Y_i(\Delta))$. Then, for small Δ ,

(2.15) $$\lambda_m = \Sigma_{i=1}^m \log\{f(Y_i - \Delta/2)f(Y_i + \Delta/2)\}$$
$$= \Delta U_m(\Delta) + m(\Delta^2/8).[m^{-1}\Sigma_{i=1}^m\{h'(Y_i + a_1\Delta/2) - h'(Y_i - a_2\Delta/2)\}] ,$$

$0 < a_1, a_2 < 1$. Again, when θ is the true location, let $\hat{\theta}_m$ be the MLE of it ; by the assumed strong unimodality of f, the MLE exists, is unique and is a strongly consistent estimator of θ. Now,

(2.16) $$0 = m^{-1}\Sigma_{i=1}^m h(X_i - \hat{\theta}_m) = m^{-1}\Sigma_{i=1}^m h(X_i - \Delta/2) + (\Delta/2 - \hat{\theta}_m)(m^{-1}\Sigma_{i=1}^m h'(X_i - \Delta/2))$$
$$+ (\Delta/2 - \hat{\theta}_m)\{m^{-1}\Sigma_{i=1}^m[h'(X_i - \Delta/2 + \eta(\Delta/2 - \hat{\theta}_m)) - h'(X_i - \Delta/2)]\}, \quad 0 < \eta < 1.$$

For $\theta = \phi\Delta$, $\phi \in I$, Δ small, on using (2.2), (2.3), the strong consistency of $\hat{\theta}_m$ and the Kintchine law of large numbers, one gets that $m^{-1}\Sigma_{i=1}^m h'(X_i - \Delta/2) \to I(f)$ a.s. as $m \to \infty$, and on defining $W_m(\Delta) = (\hat{\theta}_m - \Delta/2)\{m^{-1}\Sigma_{i=1}^m h'(X_i - \Delta/2)\}$,

(2.17) $\quad m^{-1}U_m(\Delta) - W_m(\Delta) = o(\Delta)$ a.s.; $\quad \lambda_m - m\Delta W_m(\Delta) = o(m\,\Delta^2)$ a.s., as $\Delta \to 0$.

We now have the following lemma.

Lemma 2.2. For every $\varepsilon > 0$, there exists a $\Delta_0 = \Delta_0(\varepsilon)$, such that

(2.18) $\quad P_\phi\{ \mid \lambda_{N(\Delta)} - \Delta N(\Delta).W_{N(\Delta)}(\Delta) \mid \geq \varepsilon \} \leq \varepsilon$ for all $\Delta \leq \Delta_0$, $\phi \in I$.

Proof. The lhs of (2.18) is equal to

(2.19) $\quad P_\phi\{ \mid \lambda_{N(\Delta)} - \Delta N(\Delta) W_{N(\Delta)}(\Delta) \mid \geq \varepsilon,\ N(\Delta) \notin (c_\varepsilon^{(1)}\Delta^{-2}, c_\varepsilon^{(2)}\Delta^{-2}) \}$ +

$\quad P_\phi\{ \mid \lambda_{N(\Delta)} - \Delta N(\Delta) W_{N(\Delta)}(\Delta) \mid \geq \varepsilon \quad N(\Delta) \in (c_\varepsilon^{(1)}\Delta^{-2}, c_\varepsilon^{(2)}\Delta^{-2}) \}$;

by lemma 2.1, the first term is bounded by ε , while the second term is bounded by

(2.20) $\quad 1 - P_\phi\{|\lambda_m - \Delta m W_m(\Delta)| \leq \varepsilon$ for all $m \in (c_\varepsilon^{(1)}\Delta^{-2}, c_\varepsilon^{(2)}\Delta^{-2}) \}$

$\quad \leq 1 - P_\phi\{|\lambda_m - \Delta m W_m(\Delta)| \leq m\delta\Delta^2$ for all $m \in (c_\varepsilon^{(1)}\Delta^{-2}, c_\varepsilon^{(2)}\Delta^{-2})\}$,

provided $m\delta\Delta^2 < \varepsilon$. Again, (2.17) implies that for any $\delta > 0$, $\Delta_0(\varepsilon)$ can be so chosen that for all $\Delta \leq \Delta_0(\varepsilon)$,

(2.21) $\quad P_\phi\{|\lambda_m - m\Delta W_m(\Delta)| < \delta m^2$, for all $m \in (c_\varepsilon^{(1)}\Delta^{-2}, c_\varepsilon^{(2)}\Delta^{-2})\} \geq 1 - \varepsilon$.

From (2.19), (2.20) and (2.21) the lemma follows directly. Q.E.D.

Consider now another stopping variable $N^*(\Delta)$ defined to be the first integer for which the following inequalities are vitiated :

(2.22) $\quad (\log B)/(m\,\Delta) < W_m(\Delta) < (\log A)/(m\Delta)$;

if $N^*(\Delta)W_{N^*(\Delta)}(\Delta) \leq \Delta^{-1}\log B$ we accept H_0, while H_1 is accepted if $N^*(\Delta)W_{N^*(\Delta)}(\Delta) \geq \Delta^{-1}\log A$.

When Δ is small, $W_m(\Delta)$ is also infinitesimally small for each $m \geq 1$, so that the excess over the boundaries can be neglected. If $L_W^{(F)}(\phi\Delta)$ and $L_*^{(F)}(\phi\Delta)$ denote the OC functions of the two stopping rules $N(\Delta)$ and $N^*(\Delta)$ respectively, by lemmas 2.1 and 2.2,

(2.23) $\quad \lim_{\Delta \to 0} |L_W^{(F)}(\phi\Delta) - L_*^{(F)}(\phi\Delta)| = 0$ for all $\phi \in I$.

Since $L_W^{(F)}(0) \simeq 1 - \alpha$ and $L_W^{(F)}(\Delta) \simeq \beta$, as $\Delta \to 0$, we have immediately for $N^*(\Delta)$,

(2.24) $\quad \lim_{\Delta \to 0} P\{\text{Type I error}\} = \alpha$, $\quad \lim_{\Delta \to 0} P\{\text{Type II error}\} = \beta$.

Finally, we shall see in section 6 that this procedure retains asymptotically (as $\Delta \to 0$) the Wald-optimality of the SPRT.

3. The sample mean procedure. The procedures considered above assume the knowledge of the df F. In the absence of this prior information on F, we may consider the following procedure based on the sample mean and variance. It remains valid for a broad class of df with finite mgf in the neighbourhood of 0.

Now, $\bar{X}_m = m^{-1}\Sigma_{i=1}^{m} X_i$ is an unbiased estimator (in fact, the BLUE) of θ. In the definition of $W_m(\Delta)$, $\{m^{-1}\Sigma_{i=1}^{m} h'(X_i - \Delta/2)\}^{-1}$ can be interpreted as an estimate of $\{I(f)\}^{-1}$, which is the variance of the asymptotic distribution of $m^{\frac{1}{2}}(\hat{\theta} - \theta)$. Hence, noting that the variance of $m^{\frac{1}{2}}(\bar{X}_m - \theta)$ is σ^2which can be unbiasedly estimated by the sample variance $s_m^2 = (m-1)^{-1} \Sigma_{i=1}^{m}(X_i - \bar{X}_m)^2$, and keeping in mind the stopping variable in (2.22), we may define the following procedure based on the sample means:

Continue sampling as long as

(3.1) $$(s_m^2 \log B)/(m\,\Delta) < (\bar{X}_m - \Delta/2) < (s_m^2 \log A)/(m\,\Delta);$$

if for the first time at the mth stage, $\bar{X}_m - \Delta/2 \le (s_m^2 \log B)/(m\,\Delta)$ **accept** H_0, and accept H_1 when $\bar{X}_m - \Delta/2 \ge (s_m^2 \log A)/(m\,\Delta)$; the corresponding stopping variable is denoted by $N_M(\Delta)$.Had σ^2 been known, then the SPRT of Wald[19] reduces to (3.1) when F is normal, provided we replace s_m^2 by σ^2. In that case, Wald's technique for the SPRT remains equally applicable for the mean procedure as here also we deal with iidrv's; we denote the OC and ASN functions by $L_\sigma^{(F)}(\phi\Delta)$ and $\zeta_\sigma(\phi\Delta)$. The OC and ASN for the actual stopping variable $N_M(\Delta)$ are denoted by $L_M^{(F)}(\phi\Delta)$ and $\zeta_M(\phi\Delta)$, respectively. Then, since, for $\sigma^2 < \infty$, $s_m^2 \to \sigma^2$ a.s. as $m \to \infty$, along the same line as in lemmas 2.1 and 2.2, it follows that whenever σ^2 exists,

(3.2) $$\lim_{\Delta \to 0} | L_M^{(F)}(\phi\Delta) - L_\sigma^{(F)}(\phi\Delta)| = 0 , \forall \phi \in I,$$

which ensures that $N_M(\Delta)$ has asymptotically (as $\Delta \to 0$) the strength (α,β). It will be seen in section 6 that if F has a finite mgf, then

(3.3) $$\lim_{\Delta \to 0} \zeta_M(\phi\Delta) / \zeta_\sigma(\phi\Delta) = 1, \forall \phi \in I,$$

and we have the following theorem on the termination probability for $N_M(\Delta)$.

Theorem 3.1. If $\sigma^2 < \infty$, then for every (fixed) θ (= $\phi\Delta$), Δ not necessarily small,

(3.4) $$\lim_{n \to \infty} P_\theta \{ N_M(\Delta) > n \} = 0.$$

Proof. $P_\theta \{N_M(\Delta) > n\} \le P_\theta \{(s_n^2 \log B)/(n^{\frac{1}{2}}\Delta) < n^{\frac{1}{2}}(\bar{X}_n - \Delta/2) < (s_n^2 \log A)/(n^{\frac{1}{2}}\Delta)\}$. If $\theta = \Delta/2$, $n^{\frac{1}{2}}(\bar{X}_n - \Delta/2) \sim N(0,\sigma^2)$ (as $n \to \infty$), while the two limits both converge to 0 (a.s.) as $n \to \infty$; hence the probability can be made arbitrarily small for large n. If $\theta \ne \Delta/2$, $n^{\frac{1}{2}}(\bar{X}_n - \Delta/2) = n^{\frac{1}{2}}(\theta - \Delta/2) + n^{\frac{1}{2}}(\bar{X}_n - \theta) = n^{\frac{1}{2}}(\theta - \Delta/2) + O_p(1)$, and hence the proof follows trivially by noting that the two limits in (3.1) converge a.s. to 0 while for $\theta >$ (or $<$) $\Delta/2$, $n^{\frac{1}{2}}(\theta - \Delta/2) \to \infty$ (or $-\infty$)as $n \to \infty$.

The two procedures based on $\bar{X}_m$ and $\hat{\theta}_m$ will be compared in section 6.

4. The proposed rank order procedure. Because of the vulnerability of the sample mean and variance to gross errors, outliers, and their inefficiency for df's with heavy tails, the procedure in section 3 is not so robust as compared with the alternative rank procedure to be posed below.

For each $n \geq 1$ and real b ($-\infty < b < \infty$), define

(4.1) $$R_{ni}(b) = \tfrac{1}{2} + \Sigma_{j=1}^{n} c(|X_i - b| - |X_j - b|),\ i=1,\ldots,n;\ n \geq 1,$$

where $c(u) = 1, \tfrac{1}{2}$ or 0 according as $u >, =$ or <0. Consider now n scores $J_n(i/(n+1))$ $\{= EJ(U_{ni})\}$, $i=1,\ldots,n$, where $U_{n1} <\ldots< U_{nn}$ are the ordered random variables in a sample of size n from the rectangular (0,1) distribution, and $J(u) : 0 \leq u <1$ is the score function characterized by a known symmetric df G with a finite Fisher information I(g),(defined as in (2.2) with f replaced by g,) in the following manner :

(4.2) $$J(u) = -g'(G^{-1}((1+u)/2))/g(G^{-1}((1+u)/2)),\ 0 \leq u <1.$$

For example, when G is normal or logistic, J(u) is the inverse of the chi distribution with 1 degree of freedom or u, and the corresponding $J_n(i/(n+1))$, $i=1,\ldots,n$ are known as the normal or the Wilcoxon scores. Assume that

(4.3) $J(u)$ is $\uparrow$ in $u : 0 \leq u <1$ and is not a constant ;

this holds when g is strongly unimodal, as is the case with normal or logistic or many other G. Let then

(4.4) $$\mu = \int_0^1 J(u)du \quad \text{and} \quad \nu^2 = \int_0^1 J^2(u)du = I(g) < \infty .$$

For each real b, consider the usual one sample rank order statistic

(4.5) $$T_n(b) = \Sigma_{i=1}^{n} s(X_i - b)\ J_n(R_{ni}(b)/(n+1))\ ;\ s(u) = 2c(u) - 1.$$

By (4.3), $T_n(b)$ is $\downarrow$ in $b : -\infty < b < \infty$. When θ is the true parameter, $T_n(\theta)$ is distributed symmetrically about 0. Hence, as in [10], we consider the following robust, median-unbiased and translation-invariant estimator θ_n^* of θ :

(4.6) $$\theta_{n1}^* = \sup\{ b: T_n(b)> 0 \},\ \theta_{n2}^* = \inf \{b: T_n(b)< 0 \};\quad \theta_n^* = (\theta_{n1}^* + \theta_{n2}^*)/2.$$

Now, asymptotically (as $n \to \infty$), $n^{\frac{1}{2}}(\theta_n^*-\theta) \sim N(0,\tau^2)$, where $\tau^2 = \nu^2/C^2(F)$,

(4.7) $$C(F) = 2 \int_0^\infty (d/dx)J(2F(x) - 1)dF(x) \quad (> 0),$$

and it is assumed (as in [12]) that the following holds :

(4.8) $$\lim_{x \to \infty} (d/dx)\ J(2F(x)-1) \text{ is bounded.}$$

Also, we assume that both f(x) and f'(x) are bounded almost everywhere. Finally, we assume that for some finite positive K,

(4.9) $J'(u) \leq K(1-u)^{-1}$, $0 \leq u < 1$, $\Rightarrow \int_0^1 \exp\{t\,J(u)\}du < \infty$, for all $t \leq t_o (> 0)$.

Note that (4.9) is comparable to Wald's requirement of finite mgf for $Z_i(\Delta)$, defined after (2.8), and it holds when G is normal, logistic, double exponential or many other df.

Since, when $\theta = 0$, the distribution of $T_n(0)$ is known and is symmetric about 0, we can always select an α_n ($\simeq \alpha$, specified) and a $T_{n,\alpha}$, such that

(4.10) $P_\theta\{ |T_n(\theta)| \leq T_{n,\alpha} \} = P_0 \{ |T_n(0)| \leq T_{n,\alpha} \} = 1 - \alpha_n \simeq 1 - \alpha$.

Let then

(4.11) $\theta^*_{U,n} = \inf\{ b: T_n(b) < -T_{n,\alpha} \}$, $\theta^*_{L,n} = \sup\{ b: T_n(b) > T_{n,\alpha} \}$;

(4.12) $\hat{C}_n = 2T_{n,\alpha} / \{ n(\theta^*_{U,n} - \theta^*_{L,n})\}$.

It is shown in [15] that $\hat{C}_n$ is a strongly consistent estimator of C(F). Thus, a strongly consistent estimator of τ^2 is $\hat{\tau}^2_n = \nu^2/\hat{C}^2_n$. Then, analogous to the procedures described in the preceeding two sections, a stopping rule may be formulated as follows:

Continue sampling as long as

(4.13) $(\nu^2 \log B)/(m\,\Delta\hat{C}^2_m) < \theta^*_m - \Delta/2 < (\nu^2 \log A)/(m\Delta\,\hat{C}^2_m)$;

if $\theta^*_m \leq \Delta/2 + (\nu^2 \log B)/(m\Delta\,\hat{C}^2_m)$, accept H_0, while if $\theta^*_m \geq \Delta/2 + (\nu^2 \log A)/(m\Delta\,\hat{C}^2_m)$, accept H_1. Now, with a view to simplifying further the above procedure, consider the asymptotic (as $\Delta \to 0$) linear relationship proved in Sen and Ghosh [15] :

(4.14) $m^{-\frac{1}{2}} \{T_m(\theta^*_m) - T_m(\Delta/2)\} - m^{\frac{1}{2}}(\Delta/2 - \theta^*_m)C(F) = 0$ a.s., as $m \to \infty$,

which along with (4.6) and some simplifications lead to the following proposed rule:

Continue sampling as long as

(4.15) $(\nu^2 \log B)(\Delta\hat{C}_m)^{-1} < T_m(\Delta/2) < (\nu^2 \log A)(\Delta\,\hat{C}_m)^{-1}$;

if $T_m(\Delta/2) \leq (\nu^2 \log B)(\Delta\hat{C}_m)^{-1}$, accept H_0, while if $T_m(\Delta/2) \geq (\nu^2 \log A)(\Delta\,\hat{C}_m)^{-1}$, accept H_1; the corresponding stopping variable is denoted by $N_J(\Delta)$. In this as well as in the next section, where there is no confusion, we shall write $N_J(\Delta) = N(\Delta)$.

We are particularly interested in the above procedure when Δ is small, $\theta = \phi\Delta$, the OC function of $N_J(\Delta)$ by $L_J^{(F)}(\phi\Delta)$. Then, we have the following theorem.

Theorem 5.1. Under (4.8) and (4.9), the proposed sequential rank tests have asymptotically (as $\Delta \to 0$) the same OC function as of the Wald SPRT i.e.,

(5.1) $$\lim_{\Delta \to 0} | L_J^{(F)}(\phi\Delta) - L_W^{(F)}(\phi\Delta)| = 0 \text{ for all } \phi \in I.$$

Remark. Note that from the results of Wald [19] , we have

(5.2) $$\lim_{\Delta \to 0} L_W^{(F)}(0) = 1 - \alpha \quad \text{and} \quad \lim_{\Delta \to 0} L_W^{(F)}(\Delta) = \beta \quad ,$$

and hence, from (5.1) and (5.2), we have

(5.3) $$\lim_{\Delta \to 0} L_J^{(F)}(0) = 1 - \alpha \quad \text{and} \quad \lim_{\Delta \to 0} L_J^{(F)}(\Delta) = \beta \ ,$$

a property which may be termed as the asymptotic (as $\Delta \to 0$) consistency of the proposed sequential rank order tests (SROT).

Proof of the theorem. By definition, $|T_n(b)| \leq n\mu$, for all $n \geq 1$ and b. Also, it follows from the results of Sen and Ghosh [15]that as $n \to \infty$, for every (fixed) positive s,

(5.4) $$P \{ |C(F)/ \hat{C}_n - 1| > O(n^{-\frac{1}{4}}(\log n)^3 \} \leq O(n^{-s}), \text{ where we take } s> 2.$$

Hence, on defining $d = \nu^2\{\mu\, C(F)\}^{-1}(\min [-\log B, \log A])$, one gets **from** (4.15) and (5.4) that

(5.5) $$N(\Delta) > \Delta^{-1}d \ (\to \infty), \text{ in probability, as } \Delta \to 0.$$

Again, from theorem 4.3 of [15], we have for $n \to \infty$, with probability $\geq 1 - O(n^{-s})$,

(5.6) $$\sup_{b \in I_n^*} |n^{-\frac{1}{2}} \{T_n(\theta) - T_n(\theta + n^{-\frac{1}{2}}b)\} - bC(F)| = O(n^{-\frac{1}{4}}(\log n)^2),$$

where $I_n^* = \{b : |b| <(\log n)^k, k \geq 1\}$. Further, on defining

(5.7) $$n^*(\Delta) \simeq K \Delta^{-2}(-\log\Delta), \quad K < \infty , \text{ as } \Delta \to 0,$$

we have for every $\phi \in I$,

(5.8) $$P_\phi\{ N(\Delta) > n^*(\Delta)\} \leq P_\phi\{\nu^2(\log B)/(\Delta\hat{C}_{n^*(\Delta)}) < T_{n^*(\Delta)}(\Delta/2) < \nu^2(\log A)/(\Delta\hat{C}_{n^*(\Delta)})\},$$

where on using (5.4)-(5.7) and the asymptotic normality of $\{n^*(\Delta)\}^{-\frac{1}{2}}T_{n^*(\Delta)}(\phi\Delta)$ (under P_ϕ , this has the same distribution as of $\{n^*(\Delta)\}^{-\frac{1}{2}}T_{n^*(\Delta)}(0)$ under H_0), it readily follows that the rhs of (5.8) converges to 0 as $\Delta \to 0$. Thus, as $\Delta \to 0$, $d/\Delta \leq N(\Delta) \leq n^*(\Delta)$, in probability, and since in this interval (5.6) holds,

(5.9) $$\{N(\Delta)\}^{-\frac{1}{2}} \{T_{N(\Delta)}(\Delta/2) - T_{N(\Delta)}(\phi\Delta)\} - \{N(\Delta)\}^{\frac{1}{2}}\Delta(\phi - \tfrac{1}{2})C(F) = o_p(1), \text{ as } \Delta \to 0.$$

Since, when θ holds, $\{T_n(\theta), n \geq 1 \}$ forms a martingale sequence (cf. [15]), it follows from theorem 4.4 of Strassen [17] (after verifying the needed regularity conditions as in [16]) that on writing

(5.10) $\tilde{T}_t = (n+1-t)T_n(\theta) + (t-n)T_{n+1}(\theta)$, for $n \leq t < n+1$, $n \geq 0$, $T_0(\theta) = 0$,

and denoting by $\{ \xi(t),\ t \geq 0 \}$ a standard Brownian motion on $(0, \infty)$, we have

(5.11) $t^{-\frac{1}{2}} \tilde{T}_t = \nu t^{-\frac{1}{2}} \xi(t) + o(1)$ a.s. , as $t \to \infty$.

Finally, noting that for all $n \geq d/\Delta$, the rhs of (5.4) is $O(\Delta^{\frac{1}{4}}(-\log\Delta)^3) = o(1)$ as $\Delta \to 0$, we have

(5.12) $\lim_{\Delta \to 0} | L_J^{(F)}(\phi\Delta) - P_{\Delta'}(\xi, \phi)| = 0$, for all $\phi \in I$,

where $\Delta' = \Delta/\tau$, $\tau = \nu/C(F)$, and

(5.13) $P_{\Delta'}(\xi, \phi) = P$ {A standard Brownian motion $\xi(t) : 0 < t < \infty$, first crosses the line $(\log B)/\Delta' + t(\frac{1}{2} - \phi)\Delta'$ before crossing the line $(\log A)/\Delta' + t(\frac{1}{2} - \phi)\Delta'$ } .

Since, $\{ Z_n^*(\Delta),\ n \geq 1 \}$, defined after (2.8), also forms a martingale sequence, proceeding in an analogous way, we have

(5.14) $\lim_{\Delta \to 0} | L_W^{(F)}(\phi\Delta) - P_{\Delta}(\xi, \phi)| = 0$, for all $\phi \in I$.

Note that if we let $\Delta \to 0$, for the SPRT, we obtain that

(5.15) $\lim_{\Delta \to 0} L_W^{(F)}(\phi\Delta) = \lim_{\Delta \to 0} P_{\Delta}(\xi, \phi) = P(\phi)$ exists for all $\phi \in I$,

where $P(0) = 1-\alpha$ and $P(1) = \beta$. Hence, the proof is completed from (5.12)-(5.15).

Next, we investigate the asymptotic ASN function of the proposed SROT. We consider the case of $\theta = \phi\Delta$, $\phi \in I$, Δ small, and first, we let $\phi \neq \frac{1}{2}$. Then, we have the following theorem.

Theorem 5.2. For every $\phi\ (\neq \frac{1}{2}) \in I$, under (4.8) and (4.9),

(5.16) $\lim_{\Delta \to 0}\{\Delta^2 E_\phi(N_J(\Delta))\} = \tau^2\{P(\phi)\log B + [1-P(\phi)]\log A\}\{(\phi-\frac{1}{2})^{-1}\} = \Psi(\phi, \tau)$.

Proof. For some arbitrarily small positive ε , define

(5.17) $n_1(\Delta) = [\varepsilon\Delta^{-2}]$ and $n_2(\Delta) = [\{C^2 \nu^2(\log AB^{-1})(-\log\Delta)(\Delta^2 C(F))^{-1}\}]$, $C < \infty$,

where C will be chosen later on. Then, we have

(5.18) $\Delta^2 E_\phi\{N_J(\Delta)\} = \Delta^2\{ \Sigma_{n \leq n_1(\Delta)} + \Sigma_{n_1(\Delta) < n \leq n_2(\Delta)} + \Sigma_{n > n_2(\Delta)}\ nP_\phi[N_J(\Delta) = n] \}$,

where the first term on the rhs of (5.18) is bounded by ε . Also, by using (5.4), we obtain for small Δ,

$$P_\phi\{ N(\Delta) > n_2(\Delta)\} \le P\{ |C(F)/\hat{C}_{n_2(\Delta)} - 1| > \varepsilon \} +$$

(5.19) $$P_\phi\{(\nu^2 \log B)/(\Delta\hat{C}_{n_2(\Delta)}) < T_{n_2(\Delta)}(\Delta/2) < (\nu^2\log A)/(\Delta\hat{C}_{n_2(\Delta)}), |C(F)/\hat{C}_{n_2(\Delta)}-1| < \varepsilon\}$$
$$\le P_\phi\{(\nu^2\log B)(1+\varepsilon)/(\Delta C(F)) < T_{n_2(\Delta)}(\Delta/2) < (\nu^2\log A)(1+\varepsilon)/(\Delta C(F))\} + O(\{n_2(\Delta)\}^{-s}).$$

Again, for any $\phi \in I$, $|\phi\Delta - \Delta/2| = |\phi - \tfrac{1}{2}|\Delta$, and $\{n_2(\Delta)\}^{\frac{1}{2}} |\phi - \tfrac{1}{2}|\Delta = C\nu\{(\log AB^{-1})(-\log \Delta)\}^{\frac{1}{2}}$. $|\phi - \tfrac{1}{2}| = \{O(-\log\Delta)\}^{\frac{1}{2}} = O(\log n_2(\Delta))^{\frac{1}{2}}$. Hence, on using (5.6) and noting that $(\{n_2(\Delta)\}^{\frac{1}{2}}.\ \Delta C(F))^{-1} = O((-\log \Delta)^{-\frac{1}{2}}) = o(1)$, as $\Delta \to 0$, we have

(5.20) $$P_\phi\{ N(\Delta) > n_2(\Delta)\} \le P_\phi\{[n_2(\Delta)]^{\frac{1}{2}}(\tfrac{1}{2} - \phi)\Delta C(F)(1+ o(1)) < [n_2(\Delta)]^{-\frac{1}{2}} T_{n_2(\Delta)}(\phi\Delta) < [n_2(\Delta)]^{\frac{1}{2}}\Delta C(F)(1+o(1))(\tfrac{1}{2} - \phi)\} + O([n_2(\Delta)]^{-s})$$

$$= P_0\{C\nu(\tfrac{1}{2} - \phi)\{(\log AB^{-1})(\tfrac{1}{2}\log n_2(\Delta) + O(1))\}^{\frac{1}{2}}(1 + o(1)) < [n_2(\Delta)]^{-\frac{1}{2}} T_{n_2(\Delta)}(0) < C\nu(\tfrac{1}{2} - \phi)\{(\log AB^{-1})(\tfrac{1}{2}\log n_2(\Delta) + O(1))\}^{\frac{1}{2}}(1+ o(1))\} + O([n_2(\Delta)]^{-s}).$$

Now, it follows from Sen and Ghosh [16, section 3] that for $c_n = c(\log n)^{\frac{1}{2}}$, $c > 0$,

(5.21) $$P_0\{ n^{-\frac{1}{2}} T_n(0) > c_n\} = P_0\{n^{-\frac{1}{2}} T_n(0) < -c_n\} \le \exp(-c_n^2/2\nu^2) = n^{-c^2/2\nu^2} \le n^{-s},$$

by choosing $c^2 \ge 2s\nu^2$. Thus, it follows that by proper choice of C in (5.17), the rhs of (5.20) can be made smaller than $O(\{n_2(\Delta)\}^{-s})$, for Δ sufficiently small. Now,

$$\Sigma_{n > n_2(\Delta)}\, nP_\phi\{N(\Delta) = n\} = (n_2(\Delta) + 1)P_\phi\{ N(\Delta) > n_2(\Delta)\} + \Sigma_{n_2(\Delta)+1}^{\infty} P_\phi\{ N(\Delta) > n \}$$

(5.22) $$= \Sigma_{n_2(\Delta)+1}^{\infty} P_\phi\{ N(\Delta) > n \} + O(\{n_2(\Delta)\}^{-s+1}), \text{ as } \Delta \to 0.$$

Consider now n: $kn_2(\Delta) \le n < (k+1)n_2(\Delta)$, k being a positive integer. The same proof as in above leads to

(5.23) $$P_\phi\{ N(\Delta) > n \} \le P_\phi\{ N(\Delta) > kn_2(\Delta) \} = O(\{kn_2(\Delta)\}^{-s}) = O(n^{-s}), \text{ for } k=1,2,\ldots,$$

which leads to

(5.24) $$\Sigma_{n > n_2(\Delta)} P_\phi\{ N(\Delta) > n\} \le O(\{n_2(\Delta)\}^{-s+1}),\ s > 1, \text{ as } \Delta \to 0.$$

Thus, using (5.17), (5.18), (5.22), (5.24) and the fact that

(5.25) $$\Sigma_{n_1(\Delta)+1}^{n_2(\Delta)} nP_\phi\{ N(\Delta) = n\} = \Sigma_{n_1(\Delta)+1}^{n_2(\Delta)} P_\phi\{ N(\Delta) > n \} + \{n_1(\Delta)+1\}P_\phi\{N(\Delta) > n_1(\Delta)\} - \{n_2(\Delta) + 1\} P_\phi\{ N(\Delta) > n_2(\Delta) \},$$

it suffices to show that for every $\varepsilon > 0$,

(5.26) $$\lim_{\Delta \to 0} \left| \Delta^2 \Sigma_{n_1(\Delta)+1}^{n_2(\Delta)} P_\phi\{ N(\Delta) > n\} - \Psi(\phi,\tau) \right| < \varepsilon \quad \forall \phi (\neq \tfrac{1}{2}) \in I.$$

Now, from (5.6), one gets that as $\Delta \to 0$,

$$(5.27)\quad \sup_{n_0(\Delta)\le n\le n_2(\Delta)} n^{-\frac{1}{2}}|T_n(\Delta/2) - T_n(\phi\Delta) - n(\phi - \tfrac{1}{2})\Delta C(F)| = O(\Delta^{\frac{1}{4}}(-\log\Delta)^3) = o(1),$$

with a probability $\ge 1 - O(\{n_0(\Delta)\}^{-s+1}) = 1 - O(\Delta^{s-1}) = 1 - o(1)$, where

$$(5.28)\quad n_0(\Delta) = \nu^2\{\min(-\log B, \log A)\}(1+\varepsilon)/\{C(F)\Delta\mu\} \quad (\to\infty \text{ as } \Delta\to 0).$$

Note that for Δ sufficiently small, $n_0(\Delta)\le n_1(\Delta)$. Since, $|T_m(\Delta/2)|\le m\mu$, for all $m\ge 1$, by (5.19), (5.28), and the definition of $N(\Delta)$, we have for all $n_1(\Delta)\le n\le n_2(\Delta)$, as $\Delta\to 0$,

$$P_\phi\{N(\Delta)>n\} \le P_\phi\{(\nu^2\log B)/(\Delta\hat{C}_m) < T_m(\Delta/2) < (\nu^2\log A)/(\Delta\hat{C}_m),\ n_0(\Delta)\le m\le n\}$$

$$\le P_\phi\{(\nu^2(1+\varepsilon)\log B)/(\Delta C(F)) < T_m(\Delta/2) < (\nu^2(1+\varepsilon)\log A)/(\Delta C(F)),\ n_0(\Delta)\le m\le n_2(\Delta)\} + O(\Delta^s)$$

$$(5.29)\quad = P_\phi\{\frac{\nu^2(1+\varepsilon)\log B}{\Delta\, C(F)} < T_m(\Delta/2) < \frac{\nu^2(1+\varepsilon)\log A}{\Delta\, C(F)},\ 1\le m\le n\} + O(\Delta^s),$$

which, by the use of (5.27), reduces to

$$(5.30)\quad P_\phi\{\frac{(\nu^2(1+\varepsilon)\log B)}{\Delta\, m^{\frac{1}{2}}\, C(F)} < \tilde{Z}_m(\Delta) < \frac{(\nu^2(1+\varepsilon)\log A)}{\Delta\, m^{\frac{1}{2}} C(F)},\ 1\le m\le n\} + O(\Delta^{s-1}),$$

where

$$(5.31)\quad \tilde{Z}_m(\Delta) = m^{\frac{1}{2}}\{T_m(\phi\Delta) + m(\phi - \tfrac{1}{2})\Delta\, C(F)\},\ m\ge 1.$$

Essentially retracing steps backwards, one gets that for $n_1(\Delta)\le n\le n_2(\Delta)$, as $\Delta\to 0$,

$$(5.32)\quad P_\phi\{N(\Delta)>n\} \ge P_\phi\{\frac{\nu^2(1-\varepsilon)\log B}{\Delta\, m^{\frac{1}{2}}\, C(F)} < \tilde{Z}_m(\Delta) < \frac{\nu^2(1-\varepsilon)\log A}{\Delta\, m^{\frac{1}{2}} C(F)},\ 1\le m\le n\} + O(\Delta^{s-1}).$$

Thus, if we define two stopping variables $\tilde{N}_1(\Delta)$ (and $\tilde{N}_2(\Delta)$) by the least positive integers for which $\tilde{Z}_m(\Delta)$ first crosses either of the two curves $(\nu^2(1+\varepsilon)\log B)/(m^{\frac{1}{2}}\Delta C(F))$ or $(\nu^2(1+\varepsilon)\log A)/(m^{\frac{1}{2}}\Delta C(F))$ (and $(\nu^2(1-\varepsilon)\log B)/(m^{\frac{1}{2}}\Delta C(F))$ or $(\nu^2(1-\varepsilon)\log A)/(m^{\frac{1}{2}}\Delta C(F))$), we have from (5.29) - (5.32) that as $\Delta\to 0$,

$$(5.33)\quad \Delta^2(\Sigma_{n_1(\Delta)+1}^{n_2(\Delta)} P_\phi\{\tilde{N}_1(\Delta)>n\} + o(1) \ge \Delta^2(\Sigma_{n_1(\Delta)+1}^{n_2(\Delta)} P_\phi\{N(\Delta)>n\}$$

$$\ge \Delta^2(\Sigma_{n_1(\Delta)+1}^{n_2(\Delta)} P_\phi\{\tilde{N}_2(\Delta)>n\}) + o(1).$$

Note that $\Delta^2(\Sigma_{n=1}^{n_1(\Delta)} P_\phi\{\tilde{N}_i(\Delta)>n\} < \varepsilon$, and for $n\ge n_2(\Delta)$, as $\Delta\to 0$,

$$(5.34)\quad P_\phi\{\tilde{N}_i(\Delta)>n\} \le P_\phi\{\frac{\nu^2(1-(-1)^i\varepsilon)\log B}{\Delta\, n^{\frac{1}{2}} C(F)} < \tilde{Z}_n(\Delta) < \frac{\nu^2(1-(-1)^i\varepsilon)\log A}{\Delta\, n^{\frac{1}{2}}\, C(F)}\}$$

$$= P_0\{n^{\frac{1}{2}}(\tfrac{1}{2}-\phi)\Delta C(F) + (\nu^2(1-(-1)^i\varepsilon)\log B)/(n^{\frac{1}{2}}\Delta C(F)) < n^{-\frac{1}{2}}T_n(0) <$$
$$n^{\frac{1}{2}}(\tfrac{1}{2}-\phi)\Delta C(F) + (\nu^2(1-(-1)^i\varepsilon)\log A)/(n^{\frac{1}{2}}\Delta C(F))\}$$

$$= O(n^{-s}),\ \text{by (5.21), for } i=1,2.$$

Hence, from (5.26), (5.33) and (5.34), it suffices to show that for every $\eta > 0$, there exists a positive ε, such that for all $\phi\ (\ne \tfrac{1}{2}) \in I$, (5.35) $\lim_{\Delta \to 0} \left| \Delta^2 \Sigma_{n=1}^{\infty} P_{\phi}\{ \tilde{N}_i(\Delta) = n\} - \Psi(\phi,\tau) \right| < \eta$, for i=1,2.

We shall prove (5.35) only for $\tilde{N}_1(\Delta)$, as the proof for $\tilde{N}_2(\Delta)$ is identical. Note that $|T_n(\phi\Delta)| \leq n\mu$, and $|\tilde{Z}_n(\Delta)| \leq n^{\frac{1}{2}}(\mu + |\phi - \frac{1}{2}|)$. Hence, using (5.34), we have

(5.36) $\liminf \int_{[N>n]} |\tilde{Z}_n(\Delta)| \, dP_{\phi} = 0$, $\liminf \int_{[N>n]} n^{\frac{1}{2}} |T_n(\phi\Delta)| \, dP_{\phi} = 0$.

Let now $\mathcal{B}_n^*(\theta)$ be the σ-field generated by $\text{sgn}(X_i-\theta)$ and $R_{ni}(\theta)$, i=1,...,n; $n \geq 1$. Then, from theorem 4.6 of [15] , we note that for $\theta = \phi\Delta$, $\{ T_n(\phi\Delta), \mathcal{B}_n^*(\theta); n \geq 1 \}$ forms a martingale sequence with $E_{\phi}\{T_n(\phi\Delta)\} = 0$, for all n. Thus, using the elegant result of Chow, Robbins and Teicher [6] , we obtain that

(5.37) $E_{\phi}\{ T_{\tilde{N}_1(\Delta)}(\phi\Delta)\} = E_0 \{T_{\tilde{N}_1(\Delta)}(0) \} = 0$,

which along with (5.31) implies that

(5.38) $E_{\phi}\{ \tilde{N}_1(\Delta)\} = E_{\phi}[\{ \tilde{N}_1(\Delta)\}^{\frac{1}{2}} \tilde{Z}_{\tilde{N}_1(\Delta)}(\Delta)][(\phi - \frac{1}{2})\Delta\, C(F)]^{-1}$.

Now, neglecting excess over the boundaries (permissible for $\Delta \to 0$), $\{\tilde{N}_1(\Delta)\}^{\frac{1}{2}}\tilde{Z}_{\tilde{N}_1(\Delta)}(\Delta)$ can only assume the two values $(\nu^2(1+\varepsilon)\log B)(\Delta C(F))^{-1}$ and $(\nu^2(1+\varepsilon)\log A)(\Delta C(F))^{-1}$ with respective probabilities , say, $P_1^*(\phi\Delta)$ and $P_2^*(\phi\Delta)$. By the method of proof of theorem 5.1, it follows that

(5.39) $\lim_{\Delta \to 0} P_1^*(\phi\Delta) = P(\phi) = \lim_{\Delta \to 0}\{1- P_2^*(\phi\Delta)\}$,

where $P(\phi)$ is defined in (5.15). Hence, from (5.37) - (5.39), we obtain that

(5.40) $$\lim_{\Delta\to 0} \{\Delta^2 E_{\phi}[\tilde{N}_1(\Delta)]\} = \frac{\tau^2 \, [(1+\varepsilon)P(\phi)\log B + (1+\varepsilon)[1-P(\phi)]\log A]}{(\phi - \frac{1}{2})C(F)} = (1+\varepsilon)\Psi(\phi,\tau).$$

Since ε is arbitrarily small, (5.35) follows from (5.40). Q.E.D.

The above proof fails when $\phi = \frac{1}{2}$, nor the Wald technique (see [19, p.176]) seems readily applicable. However, one may note that if

(5.41) $(dP(\phi)/d\phi)|_{\frac{1}{2}-0} = (dP(\phi)/d\phi)|_{\frac{1}{2}+0} = P'(\frac{1}{2})$ exists

then considering a sequence of ϕ values, say $\frac{1}{2} \pm \varepsilon_r$, $\varepsilon_r(> 0) \to 0$ as $r \to \infty$, using theorem 5.2 and the L'Hôpital rule, one gets that

(5.42) $\lim_{\Delta \to 0} [\Delta^2 E_{\Delta/2}\{N(\Delta)\}] = \tau^2 P'(\frac{1}{2})\log AB^{-1}$.

6. ARE results. We want to compare the performance of the proposed SROT with that of the MLE and the mean procedures proposed in sections 2 and 3. For two sequential procedures Q and R for testing H_0 against H_1 (as in (2.1)), denote by $N_Q(\Delta)$ and

$N_R(\Delta)$ the corresponding stopping variables. Then, if both the procedures are asymptotically (as $\Delta \to 0$) of strength (α,β), the ARE of the procedure Q wrt the procedure R when $\theta = \phi\Delta$, is defined by

(6.1) $$e(Q, R) = \lim_{\Delta \to 0} \{E_\phi[N_R(\Delta)]\} / E_\phi[N_Q(\Delta)] .$$

Note that under the assumption of finite mgf of X's, all moments of X^2 exist, and then $E\,|s_n^2 - \sigma^2|^{2k} = O(n^{-k})$. Taking $k = 2 + \delta$, one finds that on applying the Markov inequality, $P_\phi\{\,|s_n^2 - \sigma^2| > \varepsilon\,\} \leq \varepsilon^{-4-2\delta} O(n^{-2-\delta})$. For the mean procedure based on the true σ^2, again neglecting excess over the boundaries for small Δ , Wald's equations ([19], (3.57) on p.53 and (A.99) on p.176) hold. Hence, proceeding as in section 5, we obtain that

(6.2) $$\lim_{\Delta \to 0} E_\phi\{\Delta^2 N_M(\Delta)\} = \begin{cases} \sigma^2\{P(\phi)\log B + (1-P(\phi))\log A\}/(\phi - \tfrac{1}{2}), & \phi \neq \tfrac{1}{2}, \\ -\sigma^2 \log A \log B, & \phi = \tfrac{1}{2}. \end{cases}$$

Also, when $\phi = \tfrac{1}{2}$, in the same way as in (5.41) one gets that $\lim_{\Delta \to 0} E\{\Delta^2 N_M(\Delta)\} = \sigma^2 P'(\tfrac{1}{2})\log(AB^{-1})$. Thus, $P'(\tfrac{1}{2}) = -\log A \log B / (\log AB^{-1})$. Then, we get from (5.42) that

(6.3) $$\lim_{\Delta \to 0} (\Delta^2 E_{\frac{1}{2}}\{N_J(\Delta)\}) = -\tau^2 \log A \,\mathrm{Log} B.$$

Hence, from (5.16), (6.2) and (6.3), the ARE of the **proposed** SROT wrt the sample mean procedure is given by

(6.4) $$e(J, M) = \sigma^2 C^2(F)/\nu^2 .$$

The above is the Pitman efficiency of a general rank order test wrt the Student t-test. In the particular case of $J(u) = u$, i.e., of the Wilcoxon signed rank statistic, (6.4) equals to $12\,\sigma^2(\int_{\infty}^{\infty} f^2(x)dx)^2$, and this is (i) $\geq$ 0.864 uniformly in the class of df with finite second moment, (ii) equal to $3/\pi = 0.955$ when F is normal and (iii) greater than 1 for many non-nornal F, including the class of heavy tail df's. Again, when $J(u) = \Phi^{-1}((1+u)/2)$, Φ being the standard normal df, i.e., for the normal scores statistic, (6.4) is bounded below by 1, where the lower bound is attained only when F is also normal. This, clearly indicates the asymptotic supremacy of the normal scores procedure over the standard normal theory procedure.

Now, since $E_\phi[\log\{f(X_1 - \Delta)/f(X_1)\}] = \Delta^2(\phi - \tfrac{1}{2})I(f) + o(\Delta^2)$ and $V_\phi[\log\{f(X_1 - \Delta)/f(X_1)\}] = \Delta^2 I(f) + o(\Delta^2)$, neglecting excess over the boundaries and denoting by W the Wald SPRT procedure, it follows from Wald [19] that

$$(6.5)\qquad \lim_{\Delta\to 0}\,[\Delta^2\, E_\phi\{\, N_W(\Delta)\}] = \begin{cases}\{P(\phi)\log B + (1-P(\phi))\log A\}/\{(\phi-\tfrac12)I(f)\}, & \phi \neq \tfrac12,\\ -(\log A\,\log B)/I(f), & \phi = \tfrac12.\end{cases}$$

From (5.16), (6.3) and (6.5), we obtain that

$$(6.6)\qquad e(\,J,\,W\,) = \{\,I(f)\,\tau^2\}^{-1} = C^2(F)/\{I(f)\,\nu^2\} = \rho^2\ ,$$

where $\rho = \{\int_0^1 \psi(u)J(u)du\}/\{(\int_0^1 \psi^2(u)du)(\int_0^1 J^2(u)du)\}^{\frac12}$, and $\psi(u) = -f'(F^{-1}((1+u)/2))/f(F^{-1}((1+u)/2))$, $0 \le u < 1$. For the equivalent representation of the central term in (6.6) in terms of ρ^2, we may refer to Hájek and Šidák [7,p.236].In this situation when $F \equiv G$ (up to a scale variation), i.e., the true and the assumed df's differ only in scale parameters, $\rho = 1$, and hence (6.6) equals to 1. Thus, the <u>normal scores</u> and <u>Wilcoxon signed rank</u> statistics lead to SROT which are <u>asymptotically</u> (as $\Delta \to 0$) <u>optimal</u> when the underlying df is <u>normal and logistic</u>. In fact, when F is normal, the SPRT , the mean procedure and the normal scores SROT are all asymptotically optimal.

We conclude this section by the following theorem which reveals the asymptotic optimality of the MLE procedure ; in fact, it shares asymptotically the same properties as of the Wald SPRT.

<u>Theorem 6.1</u>. <u>Under the conditions in section</u> 2, <u>the MLE procedure satisfies the ASN equation in</u> (6.5) <u>with</u> $N_W(\Delta)$ <u>replaced by</u> $N^*(\Delta)$.

<u>Proof</u>. In the definition of the stopping rule (2.22), upon replacing $m\Delta W_m(\Delta)$ by $\Delta U_m(\Delta)$, defined just before (2.15), we get a parallel stopping rule whose stopping variable is denoted by $N_U(\Delta)$. We start with $\phi \neq \frac12$, and let

$$(6.7)\qquad n_1 = n_1(\Delta,\varepsilon) = [\,\varepsilon\Delta^{-2}],\ \text{and}\ \ n_2 = n_2(\Delta,\varepsilon) = [\,K_\varepsilon\Delta^{-2}\,],$$

where ε (>0) is arbitrary and K_ε is so chosen that (6.11) (to follow) holds.Note that

$$(6.8)\qquad P_\phi\{\,N_U(\Delta) > n\,\} = P_\phi\{\,\log B < \Delta U_m(\Delta) < \log A\ ,\ \text{for all } 1 \le m \le n\},$$

where $U_m(\Delta)$ involves sum of iidrv's. We define $r' \sim \Delta^{-2}$ as $\Delta \to 0$, and let $\chi_k = \Delta.(\Sigma_{i=(k-1)r'+1}^{kr'} h(X_i - \Delta/2))$, k=1,2,..., then $E_\phi(\chi_k) \sim (\phi-\frac12)I(f) + o(1)$ and $V_\phi(\chi_k) \sim I(f)$ as $\Delta \to 0$. Since χ_k involves sum of iidrv's with finite variance, by the classical central limit theorem, $\{\chi_k - E_\phi(\chi_k)\}\{V_\phi(\chi_k)\}^{-\frac12} \sim N(0,\,1)$ as $\Delta \to 0$. Hence,

$$(6.9)\qquad P_\phi\{\ |\chi_k| > (\,\log A - \log B\,)\} \ge \eta > 0,\ \text{for all}\ \ 0 < \Delta \le \Delta_0.$$

Hence, by using the same technique as in Stein [17] , it follows that for all n: $kr' \le n < (k+1)r'$, $P_\phi\{ N_U(\Delta) > n\} \le P_\phi\{ N_U(\Delta) > kr'\} \le (1-\eta)^k$, for k=1,2,....; hence

(6.10) $\Delta^2 \Sigma_{n>n_2} P_\phi\{ N_U(\Delta) > n \} \le \Delta^2 r' \Sigma_{k=K_\varepsilon+1}^\infty (1-\eta)^k \le (1-\eta)^{K_\varepsilon}\eta^{-1} \le \varepsilon$,

where K_ε is so chosen that

(6.11) $\eta^{-1}(1-\eta)^{K_\varepsilon} \le \varepsilon$ and $K_\varepsilon > (1+\varepsilon)\{ \max(-\log B, \log A)\} / \{|\phi - \tfrac{1}{2}| I(f) \}$.

Since $\Delta^2\Sigma_{n<n_1} n P_\phi\{ N_U(\Delta) = n\}$ and $\Delta^2\Sigma_{n<n_1} n P_\phi\{ N^*(\Delta) = n\}$ are both $\le \varepsilon$, and by the Wald technique on $\Delta U_m(\Delta)$, $\Delta^2 E_\phi\{ N_U(\Delta)\} \to \Psi(\phi, I^{-\frac{1}{2}}(f))$, as $\Delta \to 0$, to prove the theorem it suffices to show that for every $\varepsilon > 0$,

(6.12) $\lim_{\Delta \to 0} (\Delta^2 \Sigma_{n_2+1}^\infty P_\phi\{ N^*(\Delta) > n \}) < \varepsilon$,

(6.13) $\lim_{\Delta \to 0} (\Delta^2 \Sigma_{n_1+1}^{n_2} n[P_\phi\{N_U(\Delta) = n\} - P_\phi\{ N^*(\Delta) = n\}]) = 0$.

Now, $\Delta^2\Sigma_{n_2+1}^\infty P_\phi\{ N^*(\Delta) > n\}$ can be written as

(6.14) $\Delta^2\Sigma_{n_2+1}^\infty [P_\phi\{ N^*(\Delta) > n , |\hat{\theta}_n - \theta| < C\Delta\} - P_\phi\{ N^*(\Delta) > n, |\hat{\theta}_n - \theta| > C\Delta\}]$,

where C is some positive constant. Then,

(6.15) $P_\phi\{ N^*(\Delta) > n, |\hat{\theta}_n - \theta| > C\Delta \} \le P_\phi\{|\hat{\theta}_n - \theta| > C\Delta \}$

$\le P_0\{ \Sigma_{i=1}^n h(X_i - C\Delta) > 0\} + P_0\{ \Sigma_{i=1}^n h(X_i + C\Delta) < 0 \}$.

Since for $\Delta \to 0$, $E_0\{ h(X_1 - C\Delta)\} \sim -C\Delta I(f) + o(\Delta)$, $V_0(h(X_1 - C\Delta)) \sim I(f) + o(1)$, and by Wald's assumption on SPRT, the mgf M(t) of $h(X_1 - C\Delta)$ exists, we have

(6.16) $P_0\{ \Sigma_{i=1}^n h(X_i - C\Delta) > 0\} \le \inf_{t>0} \{\exp [nE_0 h(X_1 - C\Delta)t + n\log M(t)]\}$

$\le [\exp(-ntC\Delta I(f)(1+o(1)) + n\log(1 + \tfrac{1}{2}t^2 I(f) + o(t^2)))]|_{t=Cn^{-\frac{1}{2}}}$

$= \exp \{- n^{\frac{1}{2}} \Delta C^2 I(f)[1 + o(1) + O([n^{\frac{1}{2}}\Delta]^{-1})] \}$.

Since for $n \ge n_2$, $n^{\frac{1}{2}}\Delta \ge K_\varepsilon^{\frac{1}{2}}$, the rhs of (6.16) (for $\Delta \to 0$) can be made smaller than

(6.17) $6n^{-3/2} \Delta^{-3} C_1^{-6} I^3(f)$, where $0 < C_1 < C$.

A similar bound holds for $P_0\{ \Sigma_{i=1}^n h(X_i + C\Delta) < 0 \}$. Hence, from (6.14), (6.15) and (6.17), it follows that the second term on the rhs of (6.14) can be made smaller than $24(K_\varepsilon^{\frac{1}{2}} C_1^6 I^3(f))^{-1} < \varepsilon'(>0)$, where ε' (depending on ε) is also arbitrarily small. Also, by (2.3) and (2.16),

(6.18) $P_\phi\{ N^*(\Delta) > n, |\hat{\theta}_n - \theta| < C\Delta \} \le P_\phi\{\log B < n\Delta W_n(\Delta) < \log A, |\hat{\theta}_n - \theta| < C\Delta \}$

$\le P_\phi\{ \log B - n\xi_\Delta < \Delta U_n(\Delta) < \log A - n\xi_\Delta \}$, $\phi \in I$, where $0 < \xi_\Delta < (C + \frac{1}{2})\Delta^2\eta_\Delta$ and $\eta_\Delta \to 0$ as $\Delta \to 0$. Since $E_\phi(\Delta U_n(\Delta)) = n\Delta^2(\phi - \frac{1}{2})I(f) + o(\Delta^2)$, and

(6.19) $\log B$ (or $\log A$) $\pm n\xi_\Delta - E_\phi(\Delta U_n(\Delta)) = (\frac{1}{2} - \phi)I(f)n\Delta^2\{1 + O(\eta_\Delta) + O(\{n\Delta^2\}^{-1})\} = (\frac{1}{2} - \phi)I(f)n\Delta^2\{1 + o(1) + O(K_\varepsilon^{-1})\}$, as $n\Delta^2 \geq K_\varepsilon$,

we can proceed again as in (6.16) (as $U_n(\Delta) - E_\phi\{U_n(\Delta)\}$ has finite mgf), and obtain a similar bound as in (6.17). This leads to the asymptotic (as $\Delta \to 0$) negligibility of (6.14).

To prove (6.13), we note that

(6.20) $$\Delta^2(\Sigma_{n_1+1}^{n_2} n P_\phi\{N^*(\Delta) = n\}) = \Delta^2(\Sigma_{n_1+1}^{n_2} P_\phi\{N^*(\Delta) > n\}) + \Delta^2(n_1+1)P_\phi\{N^*(\Delta) > n_1\} - \Delta^2(n_2+1)P_\phi\{N^*(\Delta) > n_2\}.$$

The second term on the rhs of (6.20) is bounded by $\varepsilon^2 + O(\Delta^2)$, while upon noting that $n^{-1}\Sigma_{i=1}^{n} h(X_i - \Delta/2) \overset{a.s.}{\to} I(f)$, $n_2^{-\frac{1}{2}}\log B$, $n_2^{-\frac{1}{2}}\log A$ both converge to 0 as $\Delta \to 0$, and $n_2^{\frac{1}{2}}(\hat{\theta}_{n_2} - \Delta/2) \sim N((\phi - \frac{1}{2})K_\varepsilon^{\frac{1}{2}}, I^{-1}(f))$ as $\Delta \to 0$, we readily conclude that

(6.21) $$\Delta^2(n_2+1)P_\phi\{N^*(\Delta) > n_2\} \leq (K_\varepsilon + O(\Delta^2))\, P_\phi\{\log B < n_2 \Delta W_{n_2}(\Delta) < \log A\}$$
$$= (K_\varepsilon + O(\Delta^2))\, P_\phi\{n_2^{-\frac{1}{2}}\log B < n_2^{\frac{1}{2}}(\hat{\theta}_{n_2} - \Delta/2)(n_2^{-1}\Sigma_{i=1}^{n_2} h(X_i - \Delta/2)) < n_2^{-\frac{1}{2}}\log A\}$$
$$\to 0 \text{ as } n_2 \to \infty \text{ (i.e., } \Delta \to 0 \text{).}$$

It remains only to consider the term $\Delta^2\Sigma_{n_1+1}^{n_2} P_\phi\{N^*(\Delta) > n\}$. One can write

(6.22) $$\Delta^2\Sigma_{n_1+1}^{n_2} P_\phi\{N^*(\Delta) > n\} = \Delta^2\Sigma_{n_1+1}^{n_2} P_\phi\{N^*(\Delta) > n, \sup_{n_1 \leq n' \leq n} |\hat{\theta}_{n'} - \theta| > C\Delta\}$$
$$+ \Delta^2\Sigma_{n_1+1}^{n_2} P_\phi\{N^*(\Delta) > n, \sup_{n \leq n' \leq n} |\hat{\theta}_{n'} - \theta| < C\Delta\};$$

(6.23) $$P_\phi\{\sup_{n_1 \leq n' \leq n} |\hat{\theta}_{n'} - \theta| > C\Delta\} = P_0\{\Sigma_{i=1}^{n'} h(X_i - C\Delta) > 0, \text{ for some } n' : n_1 \leq n' \leq n\}$$
$$+ P_0\{\Sigma_{i=1}^{n'} h(X_i + C\Delta) < 0, \text{ for some } n' : n_1 \leq n' \leq n\}.$$

Write $Y_i = h(X_i - C\Delta)$, $i \geq 1$, and $\xi = E_0 Y_1$. Then,

(6.24) $$P_0\{\Sigma_{i=1}^{n'} Y_i > 0, \text{ for some } n_1 \leq n' \leq n\} = P_0\{\Sigma_{i=1}^{n'}(Y_i - \xi) > -n'\xi, \text{for some } n_1 \leq n' \leq n\}.$$

Since for small Δ, $\xi = -C\Delta I(f) + o(\Delta)$, $V_0(Y_1) = I(f) + O(\Delta)$, and $c_n = (-n\xi)^{-1}$ is (> 0) non-increasing in n, by the well known Hájek-Rényi inequality, we have from (6.24) that the sum of the two terms in (6.23) is bounded above by

(6.25) $$2\xi^{-2}\{V_0(Y_1)\}(n_1^{-1} + (n_1^{-1} - n^{-1})) \leq 4\xi^{-2} n_1^{-1}\{V_0(Y_1)\} \leq \frac{4(1 + o(1))}{C^2 I(f)},$$

which can be made smaller than ε, by proper choice of C. It follows from (6.22) and

(6.25) that

(6.26) $$\Delta^2\Sigma_{n_1+1}^{n_2} P_\phi\{ N^*(\Delta) > n\} = \Delta^2\Sigma_{n_1+1}^{n_2} P_\phi\{ N^*(\Delta) > n, \sup_{n_1\le n'\le n}|\hat{\theta}_{n'} - \theta| < C\Delta \} + o(1).$$

Again, if $\sup_{n_1\le n'\le n}|\hat{\theta}_{n'} - \theta| < C\Delta$, $(1-\varepsilon)\log B < \Delta U_{n'}(\Delta) < (1-\varepsilon)\log A \Rightarrow \log B < n'\Delta W_{n'}(\Delta) < \log A \Rightarrow (1-\varepsilon)\log B < \Delta U_{n'}(\Delta) < (1+\varepsilon)\log A$, for all $n_1 \le n' \le n_2$. Hence, on defining $N_U^{(i)}(\Delta)$, $i=1,2$, as two stopping variables analogous to $N_U(\Delta)$, with $\log B$, $\log A$ replaced by $(1+(-1)^i\varepsilon)\log B$ and $(1+(-1)^i\varepsilon)\log A$ respectively, for $i=1,2$, one gets

(6.27) $$\begin{aligned}&\Delta^2\Sigma_{n_1+1}^{n_2} P_\phi\{ N_U^{(1)}(\Delta) > n, \sup_{n_1\le n'\le n}|\hat{\theta}_{n'} - \theta| < C\Delta \}\\ &\le \Delta^2\Sigma_{n_1+1}^{n_2} P_\phi\{ N^*(\Delta) > n, \sup_{n_1\le n'\le n}|\hat{\theta}_{n'} - \theta| < C\Delta \}\\ &\le \Delta^2\Sigma_{n_1+1}^{n_2} P_\phi\{ N_U^{(2)}(\Delta) > n, \sup_{n_1\le n'\le n}|\hat{\theta}_{n'} - \theta| < C\Delta \}.\end{aligned}$$

Now, for each $N_U^{(i)}(\Delta)$, retracing backwards the steps (6.20) - (6.25) one obtains that the first and the third terms in (6.27) converge (as $\Delta \to 0$) to $(1+(-1)^i\varepsilon)\Psi(\phi, I^{-\frac{1}{2}}(f))$ $i=1,2$ respectively, where we note that for each of them the Wald technique holds for $E_\phi N_U^{(i)}(\Delta)$. Since ε is arbitrarily small, the proof of the theorem is completed.

7. Concluding remarks. An alternative test procedure for the testing problem in (2.1) has been proposed by Albert [1] following a suggestion of H. Robbins based on the dual problem of bounded length confidence interval for the parameter under test. This procedure has the strength ($\alpha(\Delta)$, $\beta(\Delta)$) with $\alpha(\Delta) \le \beta(\Delta)$. An alternative test procedure of asymptotic strength (α, β) can be formulated as follows :

Under H_0: $\theta = 0$, $n^{-\frac{1}{2}}(T_n(0))$ is $\sim N(0,\nu^2)$, while for small values of Δ, under H_1: $\theta = \Delta$, $n^{-\frac{1}{2}}(T_n(0) - \Delta C(F))$ is $\sim N(0,\nu^2)$. Since ν is known, had $C(F)$ been known, by considering the one sided test and equating its first and second kinds of errors to α and β respectively, we obtain that the required sample size n is equal to

(7.1) $$n \simeq n(\alpha, \beta) = [\nu^2\{ \tau_\alpha + \tau_\beta \}^2\{ C(F)\}^2],$$

where τ_ε is defined in (4.16). But, in practice, the df F as well as the functional $C(F)$ is unknown. As in Sen and Ghosh [15], we have under the conditions of section 4,

(7.2) $$(\nu\,\tau_\gamma)/\{ n^{\frac{1}{2}}(\hat{\theta}_{U,n} - \hat{\theta}_{L,n})\} \xrightarrow{a.s.} C(F) \quad \text{as} \quad n \to \infty,$$

where $1-2\gamma$ is the confidence coefficient of the corresponding bounded length confidence interval problem. Hence, one can propose a sequential procedure where the stopping variable N is the first positive integer n ($\geq n_0$) for which

(7.3) $\hat{\theta}_{U,n} - \hat{\theta}_{L,n} \leq \Delta\tau_\gamma / (\tau_\alpha + \tau_\beta)$;

accept H_1 if $N^{-\frac{1}{2}} T_N(0) \geq \nu\, \tau_\alpha$, and accept H_0 otherwise.

Both the procedures considered in this section are based on the dual problem of confidence interval for θ . However, they suffer from the drawback that their asymptotic ASN are the same as the corresponding fixed sample size test had C(F) been known , but not of the sequential tests proposed here or the SPRT. Thus, these procedures are usually less efficient than the ones considered in the previous sections. For example, when F is normal, the ARE of the procedures considered in this section with respect to the Wald SPRT or our normal scores SROT will be only about 50% for the usual levels $\alpha = \beta = 0.05$. For brevity, the details of this aspect are omitted.

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