# ON THE MARGINAL DISTRIBUTIONS OF THE LATENT ROOTS OF THE MULTIVARIATE BETA MATRIX

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ABSTRACT

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The marginal distributions of the latent roots of the multivariate beta matrix are shown to constitute a complete system of solutions of an ordinary differential equation (d.e.), which is related to the author's d.e.'s for Hotelling's generalized $T_o^2$ and Pillai's $V^{(m)}$ statistics. Similar results are given for the latent roots of the multivariate F and Wishart matrices ($\Sigma$=I). Pillai's approximations to the distributions of the largest and smallest roots are interpreted as exact solutions, the contributions of higher order solutions being neglected.

# On the marginal distributions of the latent roots of the multivariate beta matrix*

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1. Introduction. Let $S(m\times m)$ and $T(m\times m)$ have independent Wishart distributions $W(q,\Sigma)$ and $W(n,\Sigma)$, respectively, where $\Sigma$ is the population covariance matrix and $q,n \geq m$. The latent roots $\ell_1 > \ldots > \ell_m > 0$ of the multivariate beta matrix $B = S(S+T)^{-1}$ are well known to have the joint density function

$$(1) \qquad \phi_{m;q,n}(\underset{\sim}{\ell}) = K(m;q,n)\Pi_{i=1}^{m}\ell_i^{\frac{1}{2}(q-m-1)}\Pi_{i=1}^{m}(1-\ell_i)^{\frac{1}{2}(n-m-1)}\Pi_{i<j}(\ell_i-\ell_j),$$

where $\underset{\sim}{\ell} = (\ell_1,\ldots,\ell_m)'$ and

$$(2) \qquad K(m;q,n) = \pi^{\frac{1}{2}m}\Pi_{i=0}^{m-1}[\Gamma(\tfrac{1}{2}(q+n-i))/\Gamma(\tfrac{1}{2}(m-i))\Gamma(\tfrac{1}{2}(q-i))\Gamma(\tfrac{1}{2}(n-i))].$$

The marginal distributions of the individual $\ell_i$ have been investigated by Roy [14], [15], who showed that the largest root $\ell_1$ is of basic importance in testing hypotheses and constructing confidence regions in multivariate analysis of variance; also by Pillai [10], Khatri [8], Sugiyama and Fukutomi [17], Sugiyama [16], and Al-Ani [1]. Pillai [11] gave very accurate approximations to the upper and lower tails of the distributions of $\ell_1$ and $\ell_m$, respectively, and $\ell_1$ has been extensively tabulated by Heck [7] for $m\leq 5$, using Pillai's approximation, and Pillai ([11], [12], etc.) for $m\leq 20$. Studies of the non-central distributions have been made by Khatri [9] and Pillai and Dotson [13].

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As $n\to\infty$, $nB\to W_I$, say, having the distribution $W(q,I)$ where $I(m\times m)$ is the unit matrix. Hanumara and Thompson [6] have tabulated the largest and smallest roots of $W_I$ using limiting forms of Pillai's approximations, and discussed their application.

The present author [3], [5] has shown that the null distributions of tr B (Pillai's $V^{(m)}$) and tr F (Hotelling's generalized $T_o^2$), where $F = ST^{-1}$, satisfy certain ordinary linear differential equations (d.e.'s) of order m which are related by a simple transformation. In Section 2 it is shown that the marginal distributions of the $\ell_i$ form a complete system of solutions of a similar d.e. Thus, the power-series of Sugiyama and Fukutomi are solutions at the regular singularities 0 and 1. Pillai's approximations are also shown in Section 4 to be exact solutions of the d.e., but approximations to the distributions insofar as contributions from higher-order solutions are neglected. Corresponding results for the latent roots of F and $W_I$ are readily deduced (Sections 6 and 7).

2. <u>The differential equation</u>. Let $D^r(s,\ell) = \{0<x_r<\ldots<x_s<\ell<x_{s-1}<\ldots<x_1<1\} \subset R^r$, where R is the real line. The marginal density function $f_s(\ell)$ of $\ell_s$ is given by

$$(3) \qquad f_s(\ell) = \int_{D^{m-1}(s,\ell)} \phi_{m;q,n}(x_1,\ldots,x_{s-1},\ell,x_s,\ldots,x_{m-1})d\underset{\sim}{x}$$

where $d\underset{\sim}{x} = \Pi_{i=1}^{m-1} dx_i$; it is proportional to

$$(4) \qquad \ell^{\frac{1}{2}(q-m-1)}(1-\ell)^{\frac{1}{2}(n-m-1)}\int_{D^{m-1}(s,\ell)} \Phi(\underset{\sim}{x})\Pi_{i=1}^{m-1}(\ell-x_i)d\underset{\sim}{x},$$

in which $\Phi$ denotes $\phi_{m-1;q-1,n-1}$. Define

$$\Psi_r(\ell;x) = \Phi(\underset{\sim}{x})\Sigma_\alpha(\ell-x_{\alpha(1)})\dots(\ell-x_{\alpha(m-1-r)}),\quad (r=0,1,\dots,m-1), \tag{5}$$

the summation being extended over the $\binom{m-1}{r}$ selections of integers $\alpha(1) < \dots < \alpha(m-1-r)$ from the set $1,2,\dots,m-1$. When $r=m-1$, the sum is taken to be unity. We now introduce the m functions

$$L_{s,r}(\ell) = \int_{D^{m-1}(s,\ell)} \Psi_r(\ell;\underset{\sim}{x})d\underset{\sim}{x},\quad (r=0,1,\dots,m-1), \tag{6}$$

noting that $f_s(\ell)$ is proportional to $\ell^{\frac{1}{2}(q-m-1)}(1-\ell)^{\frac{1}{2}(n-m-1)}L_{s,0}$. Our object is to show that, for each s, the $L_{s,r}$ are related by a system of first-order differential equations which are independent of s. Differentiating (6),

$$L'_{s,r}(\ell) = -Z^{(1)}_{s,r} + Z^{(2)}_{s,r} + (r+1)L_{s,r+1}, \tag{7}$$

where

$$\begin{aligned} Z^{(1)}_{s,r} &= \int_{D^{m-2}(s-1,\ell)} \Psi_r(\ell;x_1,\dots,x_{s-2},\ell,x_{s-1},\dots,x_{m-2})d\underset{\sim}{x}, \\ Z^{(2)}_{s,r} &= \int_{D^{m-2}(s,\ell)} \Psi_r(\ell;x_1,\dots,x_{s-1},\ell,x_s,\dots,x_{m-2})d\underset{\sim}{x}. \end{aligned} \tag{8}$$

Now let $\beta(1),\dots,\beta(r)$ denote the set of subscripts complementary to $\alpha(1),\dots,\alpha(m-1-r)$. We have

$$\begin{aligned} r\ell\, L_{s,r} &= \int_{D^{m-1}(s,\ell)} \Phi(\underset{\sim}{x})\Sigma_\alpha(\ell-x_{\alpha(1)})\dots(\ell-x_{\alpha(m-1-r)})[(\ell-x_{\beta(1)})+\dots+ \\ &\qquad (\ell-x_{\beta(r)}) + (x_{\beta(1)}+\dots+x_{\beta(r)})]d\underset{\sim}{x} \\ &= (m-r)L_{s,r+1} + \Theta_{s,r}, \end{aligned} \tag{9}$$

say. Integration by parts with respect to the $x_{\beta(i)}$ yields

(10) $$\tfrac{1}{2}(q+n-2m+2)\Theta_{s,r} = \ell(1-\ell)[Z_{s,r}^{(1)}-Z_{s,r}^{(2)}] + \tfrac{1}{2}r(q-m+r)L_{s,r}+\Psi_{s,r},$$

where

(11) $$\Psi_{s,r} = \int_{D^{m-1}(s,\ell)} \Phi(\underset{\sim}{x})\Sigma_\alpha(\ell-x_{\alpha(1)})\cdots(\ell-x_{\alpha(m-1-r)})\sum_{j=1}^{r} x_{\beta(j)}(1-x_{\beta(j)}) \Sigma_{k\neq\beta(j)}(x_{\beta(j)}-x_k)^{-1}d\underset{\sim}{x}.$$

A term similar to (11) occurred in the derivation of a d.e. for Hotelling's generalized $T_o^2$ ([3], Equation (2.13)), and the same approach yields

(12) $$\Psi_{s,r} = \tfrac{1}{2}(m-r)(m+r-3)L_{s,r-1}+\tfrac{1}{2}r(r-1)(1-2\ell)L_{s,r}+\tfrac{1}{2}r(r+1)\ell(1-\ell)L_{s,r+1}.$$

Finally, eliminating the Z's, Θ's and Ψ's from equations (7), (9), (10) and (12), we find that

(13) $$\ell(1-\ell)L'_{s,r} = \tfrac{1}{2}(m-r)(q+n-m+r-1)L_{s,r-1}+\tfrac{1}{2}r[(1-\ell)(q-m+r)-\ell(n-m+r)]L_{s,r} + \tfrac{1}{2}(r+1)(r+2)\ell(1-\ell)L_{s,r+1},\quad (r=0,1,\ldots,m-1),$$

where $L_{s,-1} \equiv L_{s,m} \equiv 0$.

We observe that the system (13) is independent of s, and in principle one could successively eliminate $L_{s,1},\ldots,L_{s,m-1}$, arriving at a homogeneous linear d.e. of order m having each $L_{s,0}$ as a solution. Clearly the $f_s$ will be solutions of a similar d.e.; furthermore, they will constitute a linearly independent and hence complete system of solutions, since as $\ell\to 0+$

(14) $$f_s(\ell)/\ell^{\frac{1}{2}(q-m-1)} \sim k_s(m;q,n)\ell^{\frac{1}{2}(m-s)(q-s+2)},\quad (s=1,\ldots,m),$$

where

(15) $$k_s(m;q,n) = K(m;q,n)/[K(s-1;q+m-s+1,n-m+s-1)K(m-s;q-s,m-s+3)].$$

This is easily proved by writting $x_j = \ell w_j$ $(j=s,\ldots,m-1)$ in (4) and letting $\ell \to 0$. We note in addition that (13) is invariant under

(16) $$q \to n,\ n \to q,\ \ell \to 1-\ell,$$

provided that we also replace $L_{s,r}$ by $(-1)^r L_{s,r}$; this reflects the obvious result that (16) transforms $f_s(\ell)$ into $f_{m+1-s}(\ell)$.

3. Solutions of the d.e. It is convenient to introduce $H_r = (1-\ell)^{-r} L_{s,r}$ $(r=0,1,\ldots,m-1)$ and to express (13) as a matrix d.e. for $H = (H_0,\ldots,H_{m-1})'$:

(17) $$dH/d\ell = [\ell^{-1}A + (1-\ell)^{-1}C]H,$$

where

$$A = \begin{bmatrix} a_0 & & & 0 \\ b_1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & b_{m-1} & a_{m-1} \end{bmatrix}, \quad C = \begin{bmatrix} c_0 & d_0 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & d_{m-2} \\ 0 & & \cdots & c_{m-1} \end{bmatrix}$$

(18)

$$a_r = \tfrac{1}{2}r(q-m+r), \qquad b_r = \tfrac{1}{2}(m-r)(q+n-m+r-1)$$
$$c_r = -\tfrac{1}{2}r(n-m+r+2), \qquad d_r = \tfrac{1}{2}(r+1)(r+2).$$

The d.e. (17) is of Fuchsian type, with regular singularities at $\ell=0,1$ and infinity, and we refer to [2], Chapter 4, for the general theory of such d.e.'s. Assuming a series solution $H = \sum_{r=0}^{\infty} h_r \ell^{\rho+r}$ in $|\ell|<1$, we obtain $Ah_0 = \rho h_0$, so that $\rho$ must be one of the latent roots $a_0,\ldots,a_{m-1}$ of A, and $h_0$ the corresponding latent vector. To relate this fundamental set of solutions to the $f_s(\ell)$, we first obtain a non-singular transformation H=PM, where P(m×m) is independent of $\ell$, such that $P^{-1}AP = \mathrm{diag}(a_r)$. A suitable choice is

$$(19)\qquad P = \{p_{ij}\},\ p_{ij} = (-1)^{i-j}\binom{m-j-1}{i-j}\Pi_{r=j}^{i-1}(q+n-m+r)/(q-m+j+r+1),$$

with inverse

$$(20)\qquad P^{-1} = \{p^*_{ij}\},\ p^*_{ij} = \binom{m-j-1}{i-j}\Pi_{r=j}^{i-1}(q+n-m+r)/(q-m+i+r).$$

Both P and $P^{-1}$ are lower triangular, and $M_0=H_0$. It may be shown that

$$(21)\qquad P^{-1}CP = G = \begin{bmatrix} \mu_0, & \nu_0 & & & 0 \\ \lambda_1, & \mu_1 & \nu_1 & & \\ & \ddots & \ddots & \ddots & \nu_{m-2} \\ 0 & & & \lambda_{m-1} & \mu_{m-1} \end{bmatrix}$$

where

$$(22)\qquad \begin{aligned} \lambda_i &= (m-i)(n-i)(q+i-1)(q-m+i-1)(q-m+i-2)(q+n-m+i-1)/[2(q-m+2i-2) \\ &\qquad (q-m+2i-1)^2(q-m+2i)], \\ \mu_i &= i^2-\tfrac{1}{2}i(m+n-3)-m+1 + \tfrac{1}{2}i(i+1)(m-i)(n-i)/(q-m+2i-1) \\ &\qquad -\tfrac{1}{2}(i+1)(i+2)(m-i-1)(n-i-1)/(q-m+2i+1), \\ \nu_i &= \tfrac{1}{2}(i+1)(i+2). \end{aligned}$$

The d.e. (17) now takes the form

$$(23)\qquad dM/d\ell = [\ell^{-1}\,\mathrm{diag}(a_r) + (1-\ell)^{-1}G]M,$$

and assuming a solution $M = \sum_{r=0}^{\infty} \eta_r \ell^{a_p+r}$ corresponding to the latent root $a_p$ of A, we obtain the following recurrence relations for the components $(\eta_{0,r},\ldots,\eta_{m-1,r})$ of the $\eta_r$:

$$\eta_{p,0} = 1, \quad \eta_{i,0} = 0 \ (i \neq p),$$

(24) $$(r-a_i+a_p)\eta_{i,r} = \lambda_i\eta_{i-1,r-1}+[\mu_i+(r-1)-a_i+a_p]\eta_{i,r-1}+\nu_i\eta_{i+1,r-1},$$

$$(i=0,\ldots,m-1;\ r=1,2,\ldots).$$

This form of solution unfortunately breaks down if $a_i-a_p$ is a positive integer for some i. In fact, $a_{p+1}-a_p = \frac{1}{2}(q-m+1)+p$ $(p \leq m-2)$, which is an integer if q-m is odd, while $a_{p+2}-a_p = q-m+2(p+1)$ $(p \leq m-3)$ is always an integer. Generally in such situations the solution must be obtained by limiting procedures which may produce logarithmic terms. However, it may be seen from (6) that the $L_{s,r}$ are in fact representable by power series, and it appears that if $a_i-a_p$ is a positive integer for i>p, then the right-hand side of the ith equation in (24) vanishes identically when $r = a_i-a_p$. Thus $\eta_{i,r}$ is an undetermined constant introducing the $a_i$-solution at this stage, and the power series form is preserved.

We also see from (24) that the $\eta_{0,r}$ are zero for r<p in the $a_p$-solution, while

(25) $$\eta_{0,p} = (p+1)!/\Pi_{i=1}^{p}(q-m+p+i+1) \doteq \xi_p, \text{ say.}$$

Hence $M_0(\ell) = O(\ell^{a_p+p})$ as $\ell \to 0+$, and since $a_{m-s}+(m-s) = \frac{1}{2}(m-s)(q-s+2)$, it follows from (14) that $L_{0,s}$ must be some linear combination of the $a_{m-s}, a_{m-s+1}, \ldots,$ and $a_{m-1}$-solutions. The coefficient of the $a_{m-s}$-solution is clearly $k_s(m;q,n)/\xi_{m-s}$, but the remaining coefficients have not been determined for general s. However, the density function $f_1(\ell)$ of the largest root corresponds to the largest root $a_{m-1}$ of A, and is thus completely specified by (24). The resulting power series coincides with the result of Sugiyama and Fukutomi [17].

4. Pillai's approximations. A particular solution corresponding to the smallest root $a_o=0$ of A may be given explicitly as an (m-1)th degree polynomial. Writing $(z)_i = z(z+1)\ldots(z+i-1), (z)_{-i} = z(z-1)\ldots(z-i+1)$, it may be shown that

$$\eta_{i,r} = (-1)^r \binom{m-1}{r}\binom{r}{i}(q)_i (n-1)_{-i}(q+n-m)_r / [(q-m+1)_{i+r}(q-m+i)_i]. \tag{26}$$

Thus we obtain the following approximation to the lower tail of $f_m(\ell)$ for large q by neglecting the $a_1,\ldots,a_{m-1}$ solutions (i.e., terms of order $\ell^{q-m+1}$ at least):

$$f_m(\ell) \approx k(m;q,n)\ell^{\frac{1}{2}(q-m-1)}(1-\ell)^{\frac{1}{2}(n-m-1)} \sum_{r=0}^{m-1}\binom{m-1}{r}(q+n-m)_r(q-1)_{-(m-1-r)}(-\ell)^r, \tag{27}$$

where

$$k(m;q,n) = \pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}(q+n-m+1))/[2^{m-1}\Gamma(\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}q)\Gamma(\tfrac{1}{2}n)]. \tag{28}$$

Using (16), a corresponding approximation to the upper tail of $f_1(\ell)$ for large n (near the regular singularity $\ell=1$) is obtained. The result is found to be simply the right-hand side of (27) multiplied by $(-1)^{m-1}$.

The integrated form of the approximation was arrived at by Pillai [11] using a different approach, and used in a series of tabulations of the upper 5% and 1% points of $\ell_1$. Its accuracy to essentially five places of decimals when $n_2 \geq m+11$ was demonstrated at least for $m \leq 10$ by substituting in explicit expressions for the distribution function [10]. In order to investigate the usefulness of the d.e. (23), some percentage points were calculated by following the $a_{m-1}$-solution out from the origin, using the same computation procedure as in [4]. The method appeared to be effective at least up to m=7, since on comparing the 1% points, i.e., the less accurate results of the d.e. and the more accurate results of the approximation, these were generally found to differ by no more than a unit in the fifth decimal place. On the other hand, the 5% points obtained from the d.e. tended to exceed Pillai's by about three units in the fifth decimal place. The d.e. approach should be more accurate at lower significance levels, and a tabulation of upper 10% points has been made.

5. The median root for m=3. The success of the Pillai approximation suggests a similar approach to the tails of the distributions of the other roots, approximating the lower tail of $f_s$ by the $a_{m-s}$-solution for large q, and deducing a corresponding result for the upper tail of $f_{m-s+1}$ when n is large using (16). However, so far it has proved difficult to obtain the solutions in closed forms analogous to (27), except in the case of the median root $\ell_2$ when m=3. The $a_1$-solution is

(29)
$$M_0 = \ell^{\frac{1}{2}q}(1-\ell)^{\frac{1}{2}n}; \quad M_1 = (\tfrac{1}{2}q+\ell[2(n-1)/(q-2)-\tfrac{1}{2}(q+n-4)])\ell^{\frac{1}{2}q-1}(1-\ell)^{\frac{1}{2}n},$$
$$M_2 = -[(n-2)(q-3)(q+n-2)/2q(q-1)]\ell^{\frac{1}{2}q}(1-\ell)^{\frac{1}{2}n}.$$

Hence, for large q and $\ell$ near zero,

(30)
$$f_2(\ell) \approx [B(q-1,n-1)]^{-1}\,\ell^{q-2}(1-\ell)^{n-2},$$

the beta density with parameters q-1 and n-1. The same approximation holds for the upper tail and large n, and has been used to compute upper 5% and 1% points of $\ell_2$. The results are identical to five decimal places with those published by Pillai and Dotson [13], except where the latter have employed extrapolation, in which case differences of up to 0.006 occur for the 5% points and up to .003 for the 1% points.

6. <u>The latent roots of the multivariate F matrix</u>. Substituting $\ell = v/(v+1)$ in (23), we obtain the d.e.

(31) $$dM^*/dv = (v^{-1}\,\mathrm{diag}(a_r) + (1+v)^{-1}\,[G-\mathrm{diag}(a_r)])M^*,$$

which has regular singularities at 0, -1 and infinity. The density function $f_s^*(v)$ of the <u>sth</u> largest root $v_s$ of F is a certain linear combination of the $a_{m-s},\ldots,a_{m-1}$-solutions of (31), multiplied by the factor $v^{\frac{1}{2}(q-m-1)}/(v+1)^{\frac{1}{2}(q+n)-m+1}$. As $v \to 0+$

(32) $$f_s^*(v)/v^{\frac{1}{2}(q-m-1)} \sim k_s(m;q,n)\; v^{a_{m-s}+(m-s)},$$

where $k_s$ is defined by (14).

7. <u>The latent roots of the Wishart matrix $W_I$</u>. As $n \to \infty$, the densities of $n\ell_s$ and $nv_s$ converge to $\bar{f}_s(u)$, say, the density function of the <u>sth</u> largest root $u_s$ of $W_I$. To obtain a d.e. for the $\bar{f}_s$, we substitute $\ell = u/n$ and $M_i = n^i\,\bar{M}_i$ $(i=0,\ldots,m-1)$ in (23); leting $n \to \infty$ it is found that

(33) $$d\bar{M}/du = [u^{-1}\,\mathrm{diag}(a_r) + \bar{G}]\,\bar{M},$$

where $\bar{M} = (\bar{M}_0,\ldots,\bar{M}_{m-1})'$, and

(34) $$\bar{G} = \begin{bmatrix} \bar{\mu}_0 & \nu_0 & & & 0 \\ \bar{\lambda} & \bar{\mu} & \nu_1 & & \\ & \ddots & \ddots & \ddots & \nu_{m-2} \\ & & \ddots & \ddots & \\ 0 & & & \bar{\lambda}_{m-1} & \bar{\mu}_{m-1} \end{bmatrix}$$

where

$$(35)\quad \begin{aligned} \bar{\lambda}_i &= (m-i)(q+i-1)(q-m+i-1)(q-m+i-2)/[2(q-m+2i-2)(q-m+2i-1)^2(q-m+2i)] \\ \bar{\mu}_i &= -[i(q-m+i)(q+m-5)+2(m-1)(q-m-1)]/[2(q-m+2i-1)(q-m+2i+1)]. \end{aligned}$$

The d.e. (33) has a regular singularity at u=0, but an irregular singularity at infinity. As before, $\bar{f}_s$ is a linear combination of the $a_{m-s},\ldots,a_{m-1}$-solutions at the origin, multiplied by $e^{-\frac{1}{2}u}\, u^{\frac{1}{2}(q-m-1)}$. In particular, as $u\to 0+$,

$$(36)\qquad \bar{f}_s(u)/u^{\frac{1}{2}(q-m-1)} \sim \bar{k}_s(m;q)\, u^{a_{m-s}+(m-s)},$$

where

$$(37)\quad \begin{aligned} \bar{k}_s(m;q) &= \bar{K}(m;q)/[\bar{K}(s-1;q+m-s+1)K(m-s;q-s,m-s+3)] \\ \bar{K}(m;q) &= \pi^{\frac{1}{2}m}/[2^{\frac{1}{2}mq}\, \Pi_{i=0}^{m-1}\, \Gamma(\tfrac{1}{2}(m-i))\Gamma(\tfrac{1}{2}(q-i))]. \end{aligned}$$

Taking limits in (27), we find that $\bar{f}_1(u) \approx (-1)^{m-1}\, g_{m,q}(u)$ for large u, and $\bar{f}_m(u) \approx g_{m,q}(u)$ for u near zero, where

$$(38)\quad \begin{aligned} g_{m,q}(u) &= \kappa(m;q)e^{-\frac{1}{2}u}u^{\frac{1}{2}(q-m-1)}\, \textstyle\sum_{r=0}^{m-1}\binom{m-1}{r}(q-1)_{-(m-1-r)}(-u)^r, \\ \kappa(m;q) &= \pi^{\frac{1}{2}}/[2^{\frac{1}{2}(q+m-1)}\Gamma(\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}q)]. \end{aligned}$$

Again, $g_{m,q}$ corresponds to a particular $a_0$-solution of (33) at the origin. Hanumara and Thompson [6] used integrated forms of (38) in their tabulation of $u_1$ and $u_m$. Some percentage points of $u_1$ were calculated using the $a_{m-1}$-solution of the d.e., and these differed from the results of the approximation by no more than a unit in the second decimal place.

The d.e. (33) is in fact closely connected with the author's d.e. [5] for the moment generating function of Pillai's $V^{(m)}$ = tr B. If we write

$\lambda_{m;q,n}(s) = \mathcal{L} \exp(-s\, V^{(m)})$, then it is easily seen that $\bar{f}_1(u)$ is proportional to $e^{-\frac{1}{2}u}\, u^{\frac{1}{2}mq-1}\, \lambda_{m-1;q-1,m+2}(\frac{1}{2}u)$.

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REFERENCES

[1] Al-Ani, S. (1968). On the distribution of the ith latent root under null hypotheses concerning complex multivariate normal populations. Mimeo Series 145, Department of Statistics, Purdue University.

[2] Coddington, E. A. and Levinson, N. (1955). Theory of Ordinary Differential Equations. McGraw-Hill, New York.

[3] Davis, A. W. (1968). A system of linear differential equations for the distribution of Hotelling's generalized $T_o^2$. Ann. Math. Statist. 39, 815-832.

[4] Davis, A. W. (1970). Exact distributions of Hotelling's generalized $T_o^2$. Biometrika 57, 187-191.

[5] Davis, A. W. (1970). On the null distribution of the sum of the roots of a multivariate beta distribution. To appear in Ann. Math. Statist.

[6] Hanumara, R. C. and Thompson, W. A. (1968). Percentage points of the extreme roots of a Wishart matrix. Biometrika 55, 505-512.

[7] Heck, D. L. (1960). Charts of some upper percentage points of the distribution of the largest characteristic root. Ann. Math Statist., 31, 625-642.

[8] Khatri, C. G. (1964). Distribution of the largest or the smallest characteristic root under null hypothesis concerning complex multivariate normal populations. Ann. Math. Statist. 35, 1807-1810.

[9] Khatri, C. G. (1969). Non-central distributions of ith largest characteristic roots of three matrices concerning complex multivariate normal populations. Ann. Inst. Statist. Math. 21, 23-32.

[10] Pillai, K. C. S. (1954). On some distribution problems in multivariate analysis. Mimeo Series 88, Institute of Statistics, University of North Carolina.

[11] Pillai, K. C. S. (1965). On the distribution of the largest characteristic root of a matrix in multivariate analysis. Biometrika 52, 405-414.

[12] Pillai, K. C. S. (1967). Upper percentage points of the largest root of a matrix in multivariate analysis. Biometrika 54, 189-194.

[13] Pillai, K. C. S. and Dotson, C. O. (1969). Power comparisons of tests of two multivariate hypotheses based on individual characteristic roots. Ann Inst. Statist. Math. 21, 49-66.

[14] Roy, S. N. (1945). The individual sampling distribution of the maximum, the minimum and any intermediate of the p-statistics on the null hypothesis. Sankhyā 7, 133-158.

[15] Roy, S. N. (1957). Some Aspects of Multivariate Analysis. Wiley, New York.

[16] Sugiyama, T. (1967). Distribution of the largest latent root and the smallest latent root of the generalized B statistic and F statistic in multivariate analysis. Ann. Math. Statist. 38, 1152-1159.

[17] Sugiyama, T. and Fukutomi, K. (1966). On the distribution of the extreme characteristic roots of the matrices in multivariate analysis. Reports of statistical application on research, union of Japanese scientists and engineers, 13.