* *The research in this report was partially sponsored by the National Science Foundation Grant No. GU-2059 and the Sakko-kai Foundation*
** *and the Institute of Statistical Mathematics, Tokyo.*

ON THE DERIVATION OF ASYMPTOTIC DISTRIBUTION OF THE GENERALIZED HOTELLING'S T_0^2 *

by

Takesi Hayakawa **
Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 692

July, 1970

On the derivation of asymptotic distribution of the generalized Hotelling's T_0^2

Takesi Hayakawa
University of North Carolina at Chapel Hill
Institute of Statistical Mathematics (Tokyo)

Abstract

This paper considers the derivation of the asymptotic expansion of the probability density function of a generalized Hotelling's T_0^2. To treat this problem, we make some important formulas for the generalized Laguerre polynomials and for the univariate Laguerre polynomials.

Classification Number: 40

Key Words and Phrases:

(i) generalized Hotelling's T_0^2

(ii) generalized Laguerre polynomial

On the Derivation of Asymptotic Distribution
of the Generalized Hotelling's T_0^2 *

Takesi Hayakawa
Department of Statistics
University of North Carolina at Chapel Hill
and
Institute of Statistical Mathematics, Tokyo

1. Introduction and Summary.

Let S_1, S_2 be independent $m \times m$ matrices on n_1 and n_2 degrees of freedom, respectively. S_2 having a Wishart distribution and S_1 having a non-central Wishart distribution with the same covariance matrix. Hotelling's generalized T_0^2 statistic is defined as

$$(1) \qquad T = n_2^{-1} T_0^2 = \mathrm{tr}\, S_1 S_2^{-1}.$$

When n_2 becomes large, the distribution of T_0^2 approaches that of χ^2 based on mn_1 degrees of freedom. Ito [6] and Siotani [9] have independently derived asymptotic expansions for the cumulative distribution function (c.d.f.) of T_0^2 in the central case by using the idea of a perturbation as in physics. The non-central distribution was treated by Siotani [10] in the same way and later by Ito [7], who used the integral representation of the characteristic function of T_0^2 due to Hsu [5]. But neigher could obtain the term of order n_2^{-2}. Recently, Siotani [11] has derived the non-central distribution of T_0^2 up to order n_2^{-2} from the expansion of its characteristic function by applying the method of perturbation.

* *The research in this report was partially sponsored by the National Science Foundation Grant No. GU-2059 and the Sakko-kai Foundation.*

The exact distribution of T over the range $0 \leq T < 1$ has been obtained in the general non-central case by Constantine [1], using the method of zonal polynomials and generalized Laguerre polynomials of matrix argument developed by James [8] and Constantine [1]. Constantine's solution has the form

$$(2) \qquad f(T) = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right)}{\Gamma_m\left(\frac{n_2}{2}\right)\Gamma\left(\frac{mn_1}{2}\right)} T^{\frac{1}{2}mn_1-1} \cdot etr(-S) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-T)^k}{k!\left(\frac{mn_1}{2}\right)_k} \left(\frac{n_1+n_2}{2}\right)_\kappa L_\kappa^{\frac{1}{2}n-p}(S),$$

where

$$\Gamma_m(a) = \pi^{\frac{1}{4}m(m-1)} \prod_{\alpha=1}^{m} \Gamma(a - \frac{\alpha-1}{2}), \qquad p = \tfrac{1}{2}(m+1),$$

$$(a)_\kappa = \prod_{\alpha=1}^{m} (a - \frac{\alpha-1}{2})_{k_\alpha}, \qquad (a)_n = a(a+1) \ldots (a+n-1),$$

and $L_\kappa^{\frac{1}{2}n-p}(S)$ is a generalized Laguerre polynomial of matrix argument S, corresponding to a partition κ of k into not more than m parts. Davis [2] suggested that the asymptotic expansion of the c.d.f. of T_0^2 can be obtained from (2) by changing the argument to $T = n_2^{-1}T_0^2$ and integrating term by term. In this paper, we give another method of derivation of the asymptotic expansion (for $0 \leq T < n_2$) of the probability density function (p.d.f.) of T_0^2 up to n_2^{-2} by preparing some formulas for the generalized Laguerre polynomials of matrix argument and for univariate Laguerre polynomials. We think that this method is simpler than those of previous authors.

2. Some useful formulas for generalized Laguerre polynomials and the univariate Laguerre polynomials.

Constantine defined the generalized Laguerre polynomial $L_\kappa^\gamma(S)$, $(\gamma > -1)$, by the Hankel transform of matrix argument in the following way.

Let S and R be positive definite symmetric matrices of order m, then $L_\kappa^\gamma(S)$ is defined by

$$(3) \qquad etr(-S)L_\kappa^\gamma(S) = \int_{R>0} A_\gamma(RS)(\det R)^\gamma\, etr(-R)C_\kappa(R)dR,$$

where $A_\gamma(RS)$ is a Bessel function of matrix argument and $C_\kappa(R)$ is a zonal polynomial corresponding to a partition κ of k into not more than m parts.

The following lemmas are very important for our argument.

LEMMA 1. (Hayakawa [3])

$$(4) \qquad \sum_\kappa L_\kappa^{\frac{1}{2}n-p}(S) = L_k^{\frac{1}{2}mn-1}(trS), \qquad p = \tfrac{1}{2}(m+1),$$

where the left hand side (L.H.S.) is a summation over all partitions κ of k into not more than m parts and the right hand side (R.H.S.) is a univariate Laguerre polynomial.

LEMMA 2. (Sugiura and Fujikoshi [12])

Let $C_\kappa(S)$ be the zonal polynomial of degree k corresponding to a partition $\kappa = \{k_1,\dots,k_m\}$ of k $(k_1 \geq k_2 \geq \dots \geq k_m \geq 0)$ for an $m\times m$ positive definite matrix S. Put

$$(5) \qquad a_1(\kappa) = \sum_{\alpha=1}^m k_\alpha(k_\alpha-\alpha),$$

$$(6) \qquad a_2(\kappa) = \sum_{\alpha=1}^m k_\alpha(4k_\alpha^2-6\alpha k_\alpha+3\alpha^2).$$

Then the following equalities hold

$$(7) \qquad \sum_{k=0}^\infty \frac{x^k}{k!}\sum_\kappa a_1(\kappa)C_\kappa(S) = x^2 trS^2 etr(xS),$$

(8) $$\sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{\kappa} a_2(\kappa)C_\kappa(S) = [4x^3(\mathrm{tr}S^3) + 3x^2\mathrm{tr}S^2 + 3x^2(\mathrm{tr}S)^2 + x\mathrm{tr}S]\mathrm{etr}(xS).$$

(9) $$\sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{\kappa} a_1^2(\kappa)C_\kappa(S) = [x^4(\mathrm{tr}S^2)^2 + 4x^3\mathrm{tr}S^3 + x^2\{\mathrm{tr}S^2+(\mathrm{tr}S)^2\}]\mathrm{etr}(xS).$$

We can give a similar set of results for Laguerre polynomial $L_\kappa^{\frac{1}{2}n-p}(S)$.

LEMMA 3.

Let $L_\kappa^{\frac{1}{2}n-p}(S)$ be a generalized Laguerre polynomial corresponding to a partition $\kappa = \{k_1,\ldots,k_m\}$ of k $(k_1 \geq \ldots \geq k_m \geq 0)$ for an $m\times m$ positive definite matrix S, and $a_1(\kappa)$ and $a_2(\kappa)$ are given by (5) and (6). Then

(10) $$\sum_{\kappa} a_1(\kappa)L_\kappa^{\frac{1}{2}n-p}(S) = k(k-1)\Big[\frac{mn(n+m+1)}{4} L_{k-2}^{\frac{1}{2}mn+1}(\mathrm{tr}S) - (n+m+1)\mathrm{tr}S\, L_{k-2}^{\frac{1}{2}mn+2}(\mathrm{tr}S) + \mathrm{tr}S^2\, L_{k-2}^{\frac{1}{2}mn+3}(\mathrm{tr}S)\Big],$$

(11) $$\begin{aligned}\sum_{\kappa} a_2(\kappa)L_\kappa^{\frac{1}{2}n-p}(S) = {} & k\Big[\frac{mn}{2} L_{k-1}^{\frac{1}{2}mn}(\mathrm{tr}S) - \mathrm{tr}S\, L_{k-1}^{\frac{1}{2}mn+1}(\mathrm{tr}S)\Big] \\ & + 3k(k-1)\Big[\frac{mn\{(m+1)n+m+3\}}{4} L_{k-2}^{\frac{1}{2}mn+1}(\mathrm{tr}S) \\ & - \{(m+1)n + m + 3\}\,\mathrm{tr}S\, L_{k-2}^{\frac{1}{2}mn+2}(\mathrm{tr}S) \\ & + \{(\mathrm{tr}S)^2 + \mathrm{tr}S^2\}\, L_{k-2}^{\frac{1}{2}mn+3}(\mathrm{tr}S)\Big] \\ & + 4k(k-1)(k-2)\Big[\frac{mn\{n^2+3(m+1)n+m^2+3m+4\}}{8} L_{k-3}^{\frac{1}{2}mn+2}(\mathrm{tr}S) \\ & - \frac{3}{4}\Big\{n^2 + 3(m+1)n + m^2 + 3m + 4\Big\}\mathrm{tr}S\, L_{k-3}^{\frac{1}{2}mn+3}(\mathrm{tr}S) \\ & + \frac{3}{2}\Big\{(\mathrm{tr}S)^2 + (n+m+2)\mathrm{tr}S^2\Big\} L_{k-3}^{\frac{1}{2}mn+4}(\mathrm{tr}S) \\ & - \mathrm{tr}S^3\, L_{k-3}^{\frac{1}{2}mn+5}(\mathrm{tr}S)\Big].\end{aligned}$$

$$(12)\qquad \sum_{\kappa} a_1^2(\kappa) L_{\kappa}^{\frac{1}{2}n-p}(S) = k(k-1)\Big[\frac{mn\{(m+1)n+m+3\}}{4} L_{k-2}^{\frac{1}{2}mn+1}(\mathit{tr}S)$$

$$- \{(m+1)n + m + 3\}\, \mathit{tr}S\, L_{k-2}^{\frac{1}{2}mn+2}(\mathit{tr}S)$$

$$+ \{(\mathit{tr}S)^2 + \mathit{tr}S^2\}\, L_{k-2}^{\frac{1}{2}mn+3}(\mathit{tr}S)\Big]$$

$$+ 4k(k-1)(k-2)\Big[\frac{mn}{8}\Big\{n^2+3(m+1)n+m^2+3m+4\Big\} L_{k-3}^{\frac{1}{2}mn+2}(\mathit{tr}S)$$

$$- \frac{3}{4}\Big\{n^2 + 3(m+1)n + m^2 + 3m + 4\Big\}\, \mathit{tr}S\, L_{k-3}^{\frac{1}{2}mn+3}(\mathit{tr}S)$$

$$+ \frac{3}{2}\Big\{(\mathit{tr}S)^2 + (n+m+2)\mathit{tr}S^2\Big\} L_{k-3}^{\frac{1}{2}mn+4}(\mathit{tr}S)$$

$$- \mathit{tr}S^3\, L_{k-3}^{\frac{1}{2}mn+5}(\mathit{tr}S)\Big]$$

$$+ k(k-1)(k-2)(k-3)\Big[\frac{mn}{16}\Big\{mn^3 + 2(m^2+m+4)n^2$$

$$+ (m^3+2m^2+21m+20)n + 4(2m^2+5m+5)\Big\} L_{k-4}^{\frac{1}{2}mn+3}(\mathit{tr}S)$$

$$- \frac{1}{2}\Big\{mn^3 + 2(m^2+m+4)n^2 + (m^3+2m^2+21m+20)n$$

$$+ 4(2m^2+5m+5)\Big\}\, \mathit{tr}S\, L_{k-4}^{\frac{1}{2}mn+4}(\mathit{tr}S)$$

$$+ \frac{1}{2}\Big\{2[(m+n+1)^2 + 6](\mathit{tr}S)^2 + [(mn+20)(n+m+1)$$

$$+ 12]\,\mathit{tr}S^2\Big\} L_{k-4}^{\frac{1}{2}mn+5}(\mathit{tr}S)$$

$$- \{2(n+m+1)\mathit{tr}S\,\mathit{tr}S^2 + 8\mathit{tr}S^3\} L_{k-4}^{\frac{1}{2}mn+6}(\mathit{tr}S)$$

$$+ (\mathit{tr}S^2)^2 L_{k-4}^{\frac{1}{2}mn+7}(\mathit{tr}S)\Big].$$

PROOF: From the definition of $L_{\kappa}^{\frac{1}{2}n-p}(S)$ and Lemma 2, (7), the generating function of $\Sigma_{\kappa}\, a_1(\kappa) L_{\kappa}^{\frac{1}{2}n-p}(S)$ is given by

$$etr(-S)\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\kappa}a_1(\kappa)\,L_{\kappa}^{\frac{1}{2}n-p}(S),\quad |x|<1.$$

$$=\int_{R>0}A_{\frac{1}{2}n-p}(RS)(detR)^{\frac{1}{2}n-p}\,etr(-R)\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\kappa}a_1(\kappa)\,C_{\kappa}(R)dR$$

$$=x^2\int_{R>0}A_{\frac{1}{2}n-p}(RS)(detR)^{\frac{1}{2}n-p}\,etr(-(1-x)R)\;trR^2\;dR$$

$$=\frac{x^2}{(1-x)^{\frac{1}{2}mn+2}}\int_{R>0}A_{\frac{1}{2}n-p}\left[\frac{S}{1-x}R\right](detR)^{\frac{1}{2}n-p}\,etr(-R)\{C_{(2)}(R)-\tfrac{1}{2}C_{(1^2)}(R)\}dR.$$

Hence by the definition of $L_{\kappa}^{\frac{1}{2}n-p}(S)$, again, the R.H.S. is

$$\frac{x^2}{(1-x)^{\frac{1}{2}mn+2}}\,etr\left(-\frac{S}{1-x}\right)\left\{L_{(2)}^{\frac{1}{2}n-p}\left(\frac{S}{1-x}\right)-\frac{1}{2}L_{(1^2)}^{\frac{1}{2}n-p}\left(\frac{S}{1-x}\right)\right\}.$$

Therefore, we have

$$\text{(13)}\qquad \sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\kappa}a_1(\kappa)\,L_{\kappa}^{\frac{1}{2}n-p}(S)$$

$$=x^2(1-x)^{-\frac{1}{2}mn-2}\,exp\left(-\frac{x}{1-x}trS\right)\left[L_{(2)}^{\frac{1}{2}n-p}\left(\frac{S}{1-x}\right)-\frac{1}{2}L_{(1^2)}^{\frac{1}{2}n-p}\left(\frac{S}{1-x}\right)\right].$$

Since from the tables of Constantine [1],

$$L_{(2)}^{\frac{1}{2}n-p}(S)-\frac{1}{2}L_{(1^2)}^{\frac{1}{2}n-p}(S)=\frac{mn(n+m+1)}{4}-(n+m+1)trS+trS^2,$$

the R.H.S. of (13) becomes

$$x^2exp\left(-\frac{x}{1-x}trS\right)\left[\frac{mn(n+m+1)}{4}(1-x)^{-\frac{1}{2}mn}-(n+m+1)trS(1-x)^{-\frac{1}{2}mn-1}+trS^2(1-x)^{-\frac{1}{2}mn-2}\right].$$

Hence by the use of the generating function of a univariate Laguerre polynomial, i.e.

$$(1-x)^{-\alpha-1}\,exp\left(-\frac{x}{1-x}z\right)=\sum_{k=0}^{\infty}\frac{x^k}{k!}L_k^{\alpha}(z),\quad (|x|<1),$$

we can expand the R.H.S. of (13) as a power series in x, and comparing the coefficient of x^k on both sides of (13), we have (10).

The proofs of (11) and (12) are done completely the same way as the one of (10). In these cases, as we need the explicit forms of Laguerre polynomials corresponding to $\mathrm{tr}R$, $\mathrm{tr}R^2+(\mathrm{tr}R)^2$, $\mathrm{tr}R^3$ and $(\mathrm{tr}R^2)^2$, we write them here and will omit the details of the proof.

$$(14) \qquad L^{\frac{1}{2}n-p}_{(1)}(S) = \frac{mn}{2} - \mathrm{tr}S$$

$$(15) \qquad 2L^{\frac{1}{2}n-p}_{(2)}(S) + \frac{1}{2}L^{\frac{1}{2}n-p}_{(1^2)}(S) = \frac{mn\{(m+1)n+m+3\}}{4} - \{(m+1)n+m+3\}\mathrm{tr}S + \mathrm{tr}S^2 + (\mathrm{tr}S)^2.$$

$$(16) \qquad L^{\frac{1}{2}n-p}_{(3)}(S) - \frac{1}{4}L^{\frac{1}{2}n-p}_{(21)}(S) + \frac{1}{4}L^{\frac{1}{2}n-p}_{(1^3)}(S) = \frac{mn}{8}\{n^2+3(m+1)n+m^2+3m+4\} - \frac{3}{4}\{n^2+3(m+1)n+m^2+3m+4\}\,\mathrm{tr}S + \frac{3}{2}\{(\mathrm{tr}S)^2+(n+m+2)\mathrm{tr}S^2\} - \mathrm{tr}S^3.$$

$$(17) \qquad L^{\frac{1}{2}n-p}_{(4)}(S) - \frac{1}{6}L^{\frac{1}{2}n-p}_{(31)}(S) + \frac{7}{12}L^{\frac{1}{2}n-p}_{(2^2)}(S) - \frac{1}{6}L^{\frac{1}{2}n-p}_{(21^2)}(S) + \frac{1}{4}L^{\frac{1}{2}n-p}_{(1^4)}(S)$$

$$= \frac{mn}{16}[mn^3 + 2(m^2+m+1)n^2 + (m^3+2m^2+21m+20)n + 4(2m^2+5m+5)]$$

$$- \frac{1}{2}[mn^3 + 2(m^2+m+1)n^2 + (m^3+2m^2+21m+20)n + 4(2m^2+5m+5)]\,\mathrm{tr}S$$

$$+ \frac{1}{2}\{2[(n+m+1)^2+6](\mathrm{tr}S)^2 + [(mn+20)(n+m+1)+12]\,\mathrm{tr}S^2\}$$

$$- 2\{(n+m+1)\mathrm{tr}S\,\mathrm{tr}S^2 + 8\mathrm{tr}S^3\} + (\mathrm{tr}S^2)^2.$$

LEMMA 4.

Put

$$h_{2\alpha+2j}(x,z) = h_{2\alpha+2j} = \exp(-\frac{x}{2})\frac{x^{\alpha+j-1}}{2^{\alpha+j}\Gamma(\alpha+j)}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+j)_k}.$$

and put

$$g_{2\alpha+2j}(x,z) = g_{2\alpha+2j} = exp(-z)h_{2\alpha+2j}(x,z),$$

i.e., $g_{2\alpha+2j}(x,z)$ is the probability density function of non-central χ^2 distribution with $2\alpha+2j$ degrees of freedom and non-centrality parameter z, and, for convenience

$$(\alpha)(x) = \frac{x^{\alpha-1}}{2^{\alpha}\Gamma(\alpha)} .$$

Then the following equalities hold.

$$(20) \qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha-1}(z)}{k!(\alpha)_k} = h_{2\alpha}$$

$$(21) \qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha}(z)}{k!(\alpha)_k} = -h_{2\alpha+2} + h_{2\alpha}$$

$$(22) \qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha+1}(z)}{k!(\alpha)_k} = h_{2\alpha+4} - 2h_{2\alpha+2} + h_{2\alpha}$$

$$(23) \qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha+2}(z)}{k!(\alpha)_k} = -h_{2\alpha+6} + 3h_{2\alpha+4} - 3h_{2\alpha+2} + h_{2\alpha}$$

$$(24) \qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha+3}(z)}{k!(\alpha)_k} = h_{2\alpha+8} - 4h_{2\alpha+6} + 6h_{2\alpha+4} - 4h_{2\alpha+2} + h_{2\alpha}$$

$$(25) \qquad (\alpha)(x)\sum_{k=1}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha-1}(z)}{(k-1)!(\alpha)_k} = -zh_{2\alpha+4} + (z-\alpha)h_{2\alpha+2}$$

$$(26) \qquad (\alpha)(x)\sum_{k=2}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha-1}(z)}{(k-2)!(\alpha)_k} = z^2h_{2\alpha+8} + 2\{(\alpha+1)z-z^2\}h_{2\alpha+6}$$

$$+ \{z^2-2(\alpha+1)z+\alpha(\alpha+1)\}h_{2\alpha+4}$$

$$(27) \qquad (\alpha)(x)\sum_{k=1}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha}(z)}{(k-1)!(\alpha)_k} = zh_{2\alpha+6} - (2z-\alpha-1)h_{2\alpha+4} + (z-\alpha-1)h_{2\alpha+2}$$

$$(28) \qquad (\alpha)(x) \sum_{k=1}^{\infty} \frac{(-\frac{x}{2})^k L_k^{\alpha+1}(z)}{(k-1)!\,(\alpha)_k} = -zh_{2\alpha+8} + (3z-\alpha-2)h_{2\alpha+6} - (3z-2\alpha-4)h_{2\alpha+4} + (z-\alpha-2)h_{2\alpha+2}.$$

PROOF: (20) is another type of generating function for $L_k^{\alpha-1}(z)$. We can show (21), (22), (23) and (24) in the same way. We prove here as an example, (22).

$$\sum_{k=0}^{\infty} \frac{(-\frac{x}{2})^k L_k^{\alpha-1}(z)}{k!\,(\alpha)_k} = \frac{1}{(\alpha)_2} \sum_{k=0}^{\infty} \frac{(-\frac{x}{2})^k L_k^{\alpha+1}(z)}{k!\,(\alpha+2)_k} \{k(k-1) + 2(\alpha+1)k + \alpha(\alpha+1)\}$$

$$= \frac{1}{(\alpha)_2} \sum_{k=2}^{\infty} \frac{(-\frac{x}{2})^k L_k^{\alpha+1}(z)}{(k-2)!\,(\alpha+2)_k} + \frac{2(\alpha+1)}{(\alpha)_2} \sum_{k=1}^{\infty} \frac{(-\frac{x}{2})^k L_k^{\alpha+1}(z)}{(k-1)!\,(\alpha+2)_k} + \sum_{k=0}^{\infty} \frac{(-\frac{x}{2})^k L_k^{\alpha+1}(z)}{k!\,(\alpha+2)_k}.$$

Hence the third term is

$$(29) \qquad \sum_{k=0}^{\infty} \frac{(-\frac{x}{2})^k L_k^{\alpha+1}(z)}{k!\,(\alpha+2)_k} = \exp(-\tfrac{x}{2}) \sum_{k=0}^{\infty} \frac{(\frac{xz}{2})^k}{k!\,(\alpha+2)_k}.$$

By differentiating both sides of (29) with respect to x and multiplying both sides by x, then

$$\sum_{k=1}^{\infty} \frac{(-\frac{x}{2})^k L_k^{\alpha+1}(z)}{(k-1)!\,(\alpha+2)_k} = \exp(-\tfrac{x}{2}) \left[-\frac{x}{2} \sum_{k=0}^{\infty} \frac{(\frac{xz}{2})^k}{k!\,(\alpha+2)_k} + \sum_{k=1}^{\infty} \frac{(\frac{xz}{2})^k}{(k-1)!\,(\alpha+2)_k} \right].$$

Thus, the second term is obtained.

By differentiating both sides of (29) twice with respect to x and multiplying both sides by x^2, then

$$\sum_{k=2}^{\infty} \frac{(-\frac{x}{2})^k L_k^{\alpha+1}(z)}{(k-2)!\,(\alpha+2)_k} = \exp(-\tfrac{x}{2}) \left[\frac{x^2}{2^2} \sum_{k=0}^{\infty} \frac{(\frac{xz}{2})^k}{k!\,(\alpha+2)_k} - 2 \cdot \frac{x}{2} \sum_{k=1}^{\infty} \frac{(\frac{xz}{2})^k}{(k-1)!\,(\alpha+2)_k} \right.$$

$$+ \sum_{k=2}^{\infty} \frac{(\frac{xz}{2})^k}{(k-2)!(\alpha+2)_k}\Bigg] .$$

Adding these results, we have

$$exp(-\frac{x}{2})\left[\frac{x^2}{2^2(\alpha)_2}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+2)_k} - 2\,\frac{x}{2\alpha}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{(k-1)!(\alpha+2)_k}\left\{\frac{1}{\alpha+1}+\frac{1}{k}\right\}\right.$$

$$\left. + \sum_{k=2}^{\infty}\frac{(\frac{xz}{2})^k}{(k-2)!(\alpha+2)_k}\left\{\frac{1}{(\alpha)_2} + 2\,\frac{1}{k\alpha} + \frac{1}{k(k-1)}\right\}\right]$$

$$= exp(-\frac{x}{2})\left[\frac{x^2}{2^2(\alpha)_2}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+2)_k} - 2\,\frac{x}{2\alpha}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+1)_k} + \sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha)_k}\right].$$

Hence we have (22) by multiplying both sides by $\alpha(x)$.

Formulas (25), (26), (27) and (28) can be obtained in the same way. As an example, we prove (27). From (21),

$$\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha}(z)}{k!(\alpha)_k} = \left[-\frac{x}{2\ \alpha}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+1)_k} + \sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha)_k}\right] exp(-\frac{x}{2}).$$

By **differentiating** both sides of the above equation with respect to x and multiplying both sides by x, we have

$$\sum_{k=1}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha}(z)}{(k-1)!(\alpha)_k} = exp(-\frac{x}{2})\left[\frac{x^2}{2^2\ \alpha}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+1)_k} - \frac{x}{2}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha)_k}\right.$$

$$- \frac{x}{2\alpha}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+1)_k} - \frac{x}{2\alpha}\sum_{k=1}^{\infty}\frac{(\frac{xz}{2})^k}{(k-1)!(\alpha+1)_k}$$

$$\left. + \sum_{k=1}^{\infty}\frac{(\frac{xz}{2})^k}{(k-1)!(\alpha)_k}\right].$$

Since

$$\frac{x^2}{2^2\,\alpha}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+1)_k} = (\alpha+1)\frac{x^2}{2^2(\alpha)_2}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+2)_k} + z\frac{x^3}{2^3(\alpha)_3}\sum_{k=1}^{\infty}\frac{(\frac{xz}{2})^{k-1}}{(k-1)!(\alpha+3)_{k-1}},$$

$$\frac{x}{2}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha)_k} = \alpha\frac{x}{2\alpha}\sum_{k=0}^{\infty}\frac{(\frac{xz}{2})^k}{k!(\alpha+1)_k} + z^2\frac{x^2}{2^2(\alpha)_2}\sum_{k=1}\frac{(\frac{xz}{2})^{k-1}}{(k-1)!(\alpha+2)_{k-1}},$$

(27) follows by simple calculation.

NOTE: In Section 3, we sometimes need the following type of summations. For example,

$$\frac{x^{\alpha-1}}{2^{\alpha}\Gamma(\alpha)}\sum_{k=4}^{\infty}\frac{(-\frac{x}{2})^k L_{k-4}^{\alpha+5}(z)}{(k-4)!(\alpha)_k}, \quad \frac{x^{\alpha-1}}{2^{\alpha}\Gamma(\alpha)}\sum_{k=3}^{\infty}\frac{(-\frac{x}{2})^k L_{k-2}^{\alpha+3}(z)}{(k-3)!(\alpha)_k}, \quad \text{etc.}$$

The first sum is obtained in the following way.

$$\frac{x^{\alpha+3}}{2^{\alpha+4}\Gamma(\alpha+4)}\sum_{k=4}^{\infty}\frac{(-\frac{x}{2})^{k-4} L_{k-4}^{\alpha+5}(z)}{(k-4)!(\alpha+4)_{k-4}} = (\alpha+4)(x)\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha+5}(z)}{k!(\alpha+4)_k}$$

$$= h_{2\alpha+12} - 2h_{2\alpha+10} + h_{2\alpha+8}.$$

The second sum is obtained in the following way.

$$\frac{x^{\alpha+1}}{2^{\alpha+2}\Gamma(\alpha+2)}\sum_{k=3}^{\infty}\frac{(-\frac{x}{2})^k L_{k-2}^{\alpha+3}(z)}{(k-3)!(\alpha+2)_{k-2}} = (\alpha+2)(x)\sum_{k=1}^{\infty}\frac{(-\frac{x}{2})^k L_k^{\alpha+3}(z)}{(k-1)!(\alpha+2)_k}$$

$$= -zh_{2\alpha+12} + (3z-\alpha-4)h_{2\alpha+10}$$

$$-(3z-2\alpha-8)h_{2\alpha+8} + (z-\alpha-4)h_{2\alpha+6}.$$

3. Derivation of the asymptotic probability density function of T_0^2.

In this section, we denote n_1 by n.

Let us write

$$(30) \qquad x = n_2 T = T_0^2.$$

Then the p.d.f. $f(x)$ of $x = T_0^2$ is represented by (31),

$$(31) \qquad \frac{n_2^{-\frac{1}{2}mn}\Gamma_m(\frac{n+n_2}{2})}{\Gamma_m(\frac{n_2}{2})\Gamma(\frac{mn}{2})} \, etr(-S) x^{\frac{1}{2}mn-1} \sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (-\frac{x}{n_2})^k \sum_{\kappa} (\frac{n_1+n_2}{2})_k L_\kappa^{\frac{1}{2}n-p}(S).$$

This series is convergent for $|x| < n_2$.

Using a Stirling-type asymptotic expansion, we have

$$(32) \qquad \frac{n_2^{-\frac{1}{2}mn}\Gamma_m(\frac{n+n_2}{2})}{\Gamma_m(\frac{n_2}{2})\Gamma(\frac{mn}{2})} = \frac{1}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})} \Big[1 + \frac{mn}{4n_2}(n-m-1)$$

$$+ \frac{mn}{96n_2^2}\Big\{3m^3n - 2m^2(3n^2-3n+4)$$

$$+ 3m(n^3-2n^2+5n-4) - 8n^2 + 12n + 4\Big\} + 0(1/n_2^3)\Big],$$

$$(33) \qquad (\frac{n+n_2}{2})_\kappa = (\frac{n_2}{2})^k \Big[1 + \frac{1}{n_2}\Big\{a_1(\kappa) + nk\Big\}$$

$$+ \frac{1}{6n_2^2}\Big\{3a_1^2(\kappa) - a_2(\kappa) + 6n(k-1)a_1(\kappa) + 3n^2k(k-1) + k\Big\}$$

$$+ 0(1/n_2^3)\Big].$$

Therefore, by inserting (32) and (33) into (31), we obtain the asymptotic expansion of the p.d.f. of x up to order 2.

$$(34)\qquad etr(-S)\frac{x^{\frac{1}{2}mn-1}}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})}\sum_{k=0}^{\infty}\frac{1}{k!(\frac{mn}{2})_k}(-\frac{x}{2})^k\Big[\sum_{\kappa}L_{\kappa}^{\frac{1}{2}n-p}(S)$$

$$+\frac{1}{4n_2}\Big\{\sum_{\kappa}\{mn(n-m-1)+4(a_1(\kappa)+nk)\}L_{\kappa}^{\frac{1}{2}n-p}(S)\Big\}$$

$$+\frac{1}{96n_2^2}\Big\{\sum_{\kappa}\Big\{mn\{3m^3n-2m^2(3n^2-3n+4)+3m(n^3-2n^2+5n-4)$$

$$-8n^2+12n+4\}+24mn(n-m-1)\{a_1(\kappa)+nk\}$$

$$+16\{3a_1^2(\kappa)-a_2(\kappa)+6n(k-1)a_1(\kappa)+3n^2k(k-1)+k\}\Big\}L_{\kappa}^{\frac{1}{2}n-p}(S)\Big\}$$

$$+0(1/n_2^3)\Big].$$

The next problem is to calculate each term by using the previous lemmas in Section 2.

(i) *The first term* is obvious from (20) and Lemma 1.

$$(35)\qquad etr(-S)\frac{x^{\frac{1}{2}mn-1}}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})}\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k}{k!(\frac{mn}{2})_k}\sum_{\kappa}L_{\kappa}^{\frac{1}{2}n-p}(S)$$

$$=etr(-S)\frac{x^{\frac{1}{2}mn-1}}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})}exp(-\frac{x}{2})\sum_{k=0}^{\infty}\frac{(\frac{x}{2}trS)^k}{k!(\frac{mn}{2})_k}$$

$$=g_{mn}(x;\,trS).$$

(ii) *The term of order $1/n_2$:*

From (10), (20), (21) and (22),

$$(36)\qquad etr(-S)\frac{x^{\frac{1}{2}mn-1}}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})}\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k}{k!(\frac{mn}{2})_{\kappa}}\sum_{\kappa}a_1(\kappa)\,L_{\kappa}^{\frac{1}{2}n-p}(S)$$

$$= \frac{1}{4}\{mn(n+m+1) - 4(n+m+1)\mathrm{tr}S + 4\mathrm{tr}S^2\}\, g_{mn+4}$$

$$+ \{(n+m+1)\mathrm{tr}S - 2\mathrm{tr}S^2\}\, g_{mn+6} + \mathrm{tr}S^2\, g_{mn+8}\ ,$$

and from Lemma 4, (25),

$$(37) \qquad \mathrm{etr}(-S)\, \frac{x^{\frac{1}{2}mn-1}}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})} \sum_{k=0}^{\infty} \frac{(-\frac{x}{2})^k}{k!(\frac{mn}{2})_\kappa} \sum_{\kappa} k\, L_\kappa^{\frac{1}{2}n-p}(S)$$

$$= -\mathrm{tr}S g_{mn+4} + (\mathrm{tr}S - \frac{mn}{2}) g_{mn+2}\ .$$

Hence, by combining these results, we have the term of order 1.

$$(38) \qquad A_1 = mn(n-m-1)g_{mn} - 2n(mn-2\mathrm{tr}S)g_{mn+2} + \{mn(n+m+1)$$

$$- 4(2n+m+1)\mathrm{tr}S + 4\mathrm{tr}S^2\}g_{mn+4} + 4\{(n+m+1)\mathrm{tr}S$$

$$- 2\mathrm{tr}S^2\}g_{mn+6} + 4\mathrm{tr}S^2 g_{mn+8}\ .$$

(iii) *The term of order 2:*

From Lemma 3, Lemma 4 and Note, we have the following formulas.

$$(39) \qquad \mathrm{etr}(-S)\, \frac{x^{\frac{1}{2}mn-1}}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})} \sum_{k=0}^{\infty} \frac{(-\frac{x}{2})^k}{k!(\frac{mn}{2})_k} \sum_{\kappa} a_1^2(\kappa)\, L_\kappa^{\frac{1}{2}n-p}(S)$$

$$= \frac{1}{4}[mn\{(m+1)n + m + 3\} - 4\{(m+1)n + m + 3\}\mathrm{tr}S$$

$$+ 4\mathrm{tr}S^2 + 4(\mathrm{tr}S)^2]\, g_{mn+4} - \frac{1}{2}[mn\{n^2 + 3(m+1)n + m^2 + 3m + 4\}$$

$$- 2\{3n^2 + 10(m+1)n + 3m^2 + 10m + 15\}\mathrm{tr}S + 4\{4(\mathrm{tr}S)^2$$

$$+ (3m+3n+7)\mathrm{tr}S^2\} - 8\mathrm{tr}S^3]\, g_{mn+6} + \frac{1}{16}[mn\{mn^3 + 2(m^2+m+4)n^2$$

$$+ (m^3+2m^2+20m+21)n + 4(2m^2+5m+5)\} - 8\{mn^3 + 2(m^2+m+7)n^2$$

$$+ (m^3+2m^2+39m+38)n + 2(7m^2+19m+52)\} \, tr S + 16\{n^2 + 2(m+1)n$$

$$+ m^2 + 2m + 20\}(tr S)^2 + 8\{mn^3 + (m^2+m+44)n + 44m + 82\} \, tr S^2$$

$$- 32\{(n+m+1) tr S tr S^2 + 10 tr S^3\} + 16(tr S^2)^2] \, g_{mn+8}$$

$$+ \frac{1}{2} [\{mn^3 + 2(m^2+m+4)n^2 + (m^3+2m^2+21m+20)n + 4(2m^2+5m+5)\} tr S$$

$$- 4\{n^2 + 2(m+1)n + m^2 + 2m + 10\}(tr S)^2 - 2\{mn^2 + (m^2+m+23)n$$

$$+ 23m + 38\} \, tr S^2 + 12\{(m+n+1) tr S tr S^2 + 6 tr S^3\} - 8(tr S^2)^2] g_{mn+10}$$

$$+ \frac{1}{2} [2\{n^2 + 2(m+1)n + m^2 + 2m + 7\}(tr S)^2 + \{mn^2 + (m^2+m+20)n$$

$$+ 20m + 32\} tr S^2 - 4\{3(n+m+1) tr S tr S^2 + 14 tr S^3\}$$

$$+ 12(tr S^2)^2] g_{mn+12} + 2[(n+m+1) tr S tr S^2 + 4 tr S^3 - 2(tr S^2)^2] g_{mn+14}$$

$$+ (tr S^2)^2 g_{mn+16} .$$

$$(40) \qquad etr(-S) \frac{x^{\frac{1}{2}mn-1}}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})} \sum_{k=0}^{\infty} \frac{(-\frac{x}{2})^k}{k!\,(\frac{mn}{2})_k} \sum_{\kappa} a_2(\kappa) \, L_{\kappa}^{\frac{1}{2}n-p} \, (S)$$

$$= \; - \frac{1}{2} \{mn - 2 tr S\} \, g_{mn+2} + \frac{1}{4} [3mn\{(m+1)n + m + 3\}$$

$$- \{3(m+1)n + 3m + 10\} \, tr S + 3\{(tr S)^2 + tr S^2\}] g_{mn+4}$$

$$- \frac{1}{2} [mn\{n^2 + 3(m+1)n + m^2 + 3m + 4\} - 6\{n^2 + 6(m+1)n$$

$$+ m^2 + 4m + 7\} tr S + 12\{2(tr S)^2 + (n+m+3) tr S^2\} - 8 tr S^3] g_{mn+6}$$

$$- 3[\{n^2 + 3(m+1)n + m^2 + 3m + 4\} \, tr S - \{5(tr S)^2$$

$$+ (4m+4n+9) tr S^2\} + 4 tr S^3] g_{mn+8} - 6[(tr S)^2 + (n+m+2) tr S^2$$

$$- 2 tr S^3] g_{mn+10} - 4 tr S^3 \, g_{mn+12} .$$

$$(41) \qquad etr(-S)\,\frac{x^{\frac{1}{2}mn-1}}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})}\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k}{k!\,(\frac{mn}{2})_k}\,k(k-1)\sum_{\kappa}L_{\kappa}^{\frac{1}{2}n-p}(S)$$

$$= (trS)^2 g_{mn+4} + \{(mn+2)trS - 2(trS)^2\}g_{mn+6} + \frac{1}{4}\{mn(mn+2)$$

$$- 4(mn+2)trS + 4(trS)^2\}g_{mn+8}$$

$$(42) \qquad etr(-S)\,\frac{x^{\frac{1}{2}mn-1}}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})}\sum_{k=0}^{\infty}\frac{(-\frac{x}{2})^k}{k!\,(\frac{mn}{2})_k}\,(k-1)\sum_{\kappa}a_1(\kappa)\,L_{\kappa}^{\frac{1}{2}n-p}(S)$$

$$= \frac{1}{4}[mn(n+m+1) - 4(n+m+1)trS + 4trS^2]g_{mn+4} - \frac{1}{8}[mn(n+m+1)(mn+4)$$

$$- 2(n+m+1)(3mn+16)trS + 4\{2(n+m+1)(trS)^2 + (mn+12)trS^2\}$$

$$- 8trStrS^3]g_{mn+6} - \frac{1}{4}[3(n+m+1)(mn+4)trS - 8(n+m+1)(trS)^2$$

$$- 4(mn+9)trS^2 + 12trStrS^2]g_{mn+8} + \frac{1}{2}[2(n+m+1)(trS)^2$$

$$+ (mn+8)trS^2 - 6trStrS^2]g_{mn+10} - trStrS^2\, g_{mn+12}.$$

Combining (35), (36), (37), (39), (40), (41) and (42) with (34), and arranging them appropriately, we obtain the term of order 2 in the following form.

$$(43) \qquad A_2 = mn\{3mn^3 - 2(3m^2+3m+4)n^2 + 3(m^3+2m^2+21m+20)n - 8m^2 - 12m + 4\}g_{mn}$$

$$- 12mn^2(n-m-1)(mn-2trS)g_{mn+2} + 6n[mn\{3mn^2 + 8n - (m+1)(m^2+m-4)\}$$

$$- 4\{4mn^2 - (m^2+m-8)n - (m^3+2m^2+5m+4)\}trS + 8n(trS)^2 + 4\{mn$$

$$- (m^2+m-4)\}trS^2]g_{mn+4} - 4[mn\{3mn^3 + (3m^2+3m+16)n^2 + 24(m+1)n$$

$$+ 4(m^2+3m+5)\} - 6\{6mn^3 + 3(m^2+m+8)n^2 - (m^3+2m^2-27m-28)n$$

$$+ 4(m^2+3m+4)\}trS + 24\{2n^2 + (m+1)n + 2\}(trS)^2$$

$$+ 12\{2mn^2 - (m^2+m-16)n + 4(m+2)\}trS^2 - 24ntrStrS^2$$

$$- 32trS^3]g_{mn+6} + 3[mn\{mn^3 + 2(m^2+m+4)n^2 + (m^3+2m^2+21m+20)n$$

$$+ 4(2m^2+5m+4)\} - 8\{4mn^3 + (5m^2+5m+24)n^2 + (m^3+2m^2+45m+44)n$$

$$+ 4(3m^2+8m+9)\}trS + 16\{6n^2 + 6(m+1)n + m^2 + 2m + 5\}(trS)^2$$

$$+ 16\{3mn^2 + 36n + 18m + 32\}trS^2 - 32\{4n + m + 1\}trStrS^2$$

$$- 256trS^3 + 16(trS^2)^2]g_{mn+8} + 24[\{mn^3 + 2(m^2+m+4)n^2$$

$$+ (m^3+2m^2+21m+20)n + 4(2m^2+5m+4)\}trS - 4\{2n^2 + 3(m+1)n + m^2$$

$$+ 2m + 9\}(trS)^2 - 2\{2mn^2 + (m^2+m+32)n + 8(3m+5)\}trS^2$$

$$+ 12(2n+m+1)trStrS^2 + 64trS^3 - 8(trS^2)^2]g_{mn+10} + 8[6\{n^2$$

$$+ 2(m+1)n + m^2 + 2m + 7\}(trS)^2 + 3\{mn^2 + (m^2+m+20)n + 20m$$

$$+ 32\}trS^2 - 12(4n+3m+3)trStrS^2 - 160trS^3 + 36(trS^2)^2]g_{mn+12}$$

$$+ 96[(n+m+1)trStrS^2 + 4trS^3 - 2(trS^2)^2]g_{mn+14} + 48(trS^2)^2g_{mn+16}.$$

Hence we have obtained an asymptotic expansion of the probability density function of T_0^2 as far as the term in n_2^{-2}.

We summarize our results in

THEOREM.

Let S_1 and S_2 be independent Wishart matrix with n and n_2 degrees of freedom, and let S_1 have a non-centrality parametor matrix S, then the asymptotic expansion of the probability density function of a generalized Hotelling's $T_0^2 = n_2 trS_1S_2^{-1}$ is given by

$$f(x) = g_{mn}(x, trS) + \frac{A_1}{4n_2} + \frac{A_2}{96n_2^2} + O(1/n_2^3),$$

where $g_{mn}(x, trS)$ *is a probability density function of a non-central* χ^2 *variable with* mn *degrees of freedom and non-centrality parametor* trS. A_1 *and* A_2 *are given by (38) and (43), respectively.*

NOTE: It should be noted that the coefficients of each chi-square probability density function are in a slightly different form from the ones of Siotani [11, 1.2.44]. However, by setting $S = \frac{1}{2}\Omega = \frac{1}{2}\Sigma^{-1}MM$ and $\omega^2 = tr\Omega$ in (38) and (43), and by rearranging the variables of each coefficients, we have the same result as Siotani's.

ACKNOWLEDGMENT:

The author wishes to express his sincere thanks to Professor N.L. Johnson who read the original version and gave some advice.

REFERENCES

[1] CONSTANTINE, A.G. (1966). "The distribution of Hotelling's generalized T_0^2." *Ann. Math. Statist.* 37, 215-225.

[2] DAVIS, A.W. (1968). "A system of linear differential equations for the distribution of Hotelling's generalized T_0^2." *Ann. Math. Statist.* 39, 815-832.

[3] HAYAKAWA, T. (1970). "On the distribution of the latent roots of a complex Wishart matrix (non-central case). *Institute of Statistics Mimeo Series* 667, University of North Carolina at Chapel Hill.

[4] HOTELLING, H. (1951). "A generalized T-test and measure of multivariate dispersion." *Proc. Second Berkeley Symp. Math. Statist. Prob.* 23-42, Univ. of California Press.

[5] HSU, P.L. (1945). "On the power functions for the E^2-test and the T^2-test." *Ann. Math. Statist.,* 16, 278-286.

[6] ITO, K. (1956). "Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistics." *Ann. Math. Statist.,* 27, 1091-1105.

[7] ITO, K. (1960). "Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistics, II." *Ann. Math. Statist.,* 31, 1148-1153.

[8] JAMES, A.T. (1964). "Distribution of matrix variates and latent roots derived from normal samples." *Ann. Math. Statist.,* 35, 475-501.

[9] SIOTANI, M. (1956). "On the distribution of Hotelling's T^2." *Proc. Inst. Statist. Math.,* 4, 33-42. (In Japanese, with English abstract.)

[10] SIOTANI, M. (1956). "On the distributions of the Hotelling's T^2-statistics." *Ann. Inst. Statist. Math., Tokyo,* 8, 1-14.

[11] SIOTANI, M. (1968). "Some methods for asymptotic distributions in the multivariate analysis." *Inst. of Statist. Mimeo Series* 595, University of North Carolina at Chapel Hill.

[12] SUGIURA, N. and FUJIKOSHI, Y. (1969). "Asymptotic expansions of the non-central distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence." *Ann. Math. Statist.,* 40, 942-952.