SOME DISTRIBUTIONS ON THE POSITIVE REAL LINE WHICH HAVE NO MOMENTS

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A recent letter to the editor (Gross, 1969) prompts us to report on a distribution, defined on the positive real line, which has no moments. Some generalizations of this distribution are also considered. This distribution is defined as

(1) $$f(X) = \frac{\alpha}{(X+\alpha)^2}, \quad X \geq 0.$$

Clearly, $\int_0^\infty \frac{X}{(X+\alpha)^2}\, dX$ diverges, so no moments exist. More generally, we may show that the distribution defined as

(2) $$f(X) = \frac{C}{(X+\alpha)^{r+1}} \quad \text{where} \quad C = r\alpha^r$$

has moments up to order r-1.

The distribution (2) has the interesting property that its percentiles are very simple to obtain. For

(3) $$P = \int_0^{t_p} \frac{r\alpha^r}{(X+\alpha)^{r+1}}\, dX = 1 - \frac{\alpha^r}{(t_p+\alpha)^r}$$

and hence

(4) $$t_p = \alpha \frac{1-V}{V} \quad \text{where} \quad V = (1-P)^{1/r}$$

Note that for distribution (1), this gives $t_p = \alpha \frac{P}{1-P}$ for the pth percentile.

The distribution (1) was not created arbitrarily. It arose when we were considering the distribution of the ratio of two independent exponential variables, say y_1 and y_2. Let

(5) $$f(y_1) = \frac{1}{\beta_1} e^{-y_1/\beta_1}$$

$$f(y_2) = \frac{1}{\beta_2} e^{-y_2/\beta_2}$$

and

(6) $$X = y_1/y_2 \qquad y_1 = XZ$$

$$Z = y_2 \qquad y_2 = Z \ .$$

Then

(7) $$f(y_1, y_2) = \frac{1}{\beta_1\beta_2} e^{-y_1/\beta_1 - y_2/\beta_2}$$

(8) $$|J| = \begin{vmatrix} Z & X \\ 0 & 1 \end{vmatrix} = Z.$$

So

(9) $$f(X,Z) = \frac{1}{\beta_1\beta_2} Z\, e^{-XZ/\beta_1 - Z/\beta_2}$$

and

(10) $$f(X) = \int_0^\infty f(X,Z)\,dZ = \frac{1}{\beta_1\beta_2} \int_0^\infty Ze^{-Z(X/\beta_1+1/\beta_2)} dZ$$

$$= \frac{1}{\beta_1\beta_2} \cdot \frac{e^{-Z(X/\beta_1+1/\beta_2)}}{(X/\beta_1+1/\beta_2)^2} \left(-Z(X/\beta_1+1/\beta_2)-1\right)\Big|_0^\infty$$

$$= \frac{1}{\beta_1\beta_2} \frac{1}{(X/\beta_1+\beta_1/\beta_2)^2} = \frac{1}{\beta_1\beta_2 \frac{1}{\beta_1^2} (X + \beta_1/\beta_2)^2}$$

$$= \frac{(\beta_1/\beta_2)}{(X+\beta_1/\beta_2)^2} = \frac{\alpha}{(X+\alpha)^2}$$

where $\alpha = \beta_1/\beta_2$.

This leads us to a "second" generalization of (1). Suppose that y_i, are gamma variables with parameters θ_i, β_i, i=1,2. That is,

(11) $$f(y_i) = \frac{y_i^{\theta_i-1} e^{-y_i/\beta_i}}{\Gamma(\theta_i)\beta_i^{\theta_i}}$$

(Note that when $\theta_i=1$, $f(y_i)$ is the exponential distribution). In this case the distribution of $X=y_1/y_2$ is given by

(12) $$f(X) = \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)\alpha} \left(\frac{X}{X+\alpha}\right)^{\theta_1-1} \left(\frac{\alpha}{X+\alpha}\right)^{\theta_2+1}$$

where $\alpha=\beta_1/\beta_2$. It is easy to show that the distribution of $V = X/X+\alpha$ is a beta distribution with parameters θ_1 and θ_2. This fact, of course, is well known. This relationship allows us to calculate the percentiles of the distribution of X easily. If t_p is the pth percentile of the beta distribution with parameters θ_1 and θ_2, then the pth percentile of the distribution of X is $\alpha t_p/(1-t_p)$. At this point we observe that if $\theta_1=1$, equation (12) becomes

(13) $$f(X) = \theta_2 \alpha^{\theta_2}/(X+\alpha)^{\theta_2+1}$$

which is identical to equation (2).

Next, we briefly consider estimation of parameters for (12). Maximum likelihood leads to the following set of equations:

$$\frac{n\hat{\theta}_2}{\hat{\alpha}} - (\hat{\theta}_1+\hat{\theta}_2) \sum \frac{1}{X_i+\hat{\alpha}} = 0$$

(14) $$n\,\Psi(\hat{\theta}_1+\hat{\theta}_2) - n\,\Psi(\hat{\theta}_1) + \sum \ln X_i - \sum \ln(X_i+\hat{\alpha}) = 0$$

$$n\,\Psi(\hat{\theta}_1+\hat{\theta}_2) - n\,\Psi(\hat{\theta}_2) + n \ln \hat{\alpha} - \sum \ln(X_i+\hat{\alpha}) = 0$$

where $\Psi(z) = \frac{d}{dz} \ln \Gamma(z+C)$, which is the digamma function. Needless to say, these equations cannot be solved without the aid of a computer. When moments exist, moment estimators may be used. For the special case $\theta_1=\theta_2=1$, we have

(15) $$\frac{n}{\hat{\alpha}} - 2 \sum \frac{1}{X_i+\hat{\alpha}} = 0$$

If n=1, then $\hat{\alpha}=X_1$. If n=2, $\hat{\alpha}=\sqrt{X_1 X_2}$. The geometric mean is not the MLE in general. However, it seems to be a good starting value for an iterative procedure. Other methods of estimation use the median of the observations or the median of the midranges. The last method calculates all possible midranges

$$Z_i = \frac{1}{2}(X_{(j)}+X_{(k)}) \quad j=1,\dots,n \quad k=j,\ j+1,\dots,n$$

where $X_{(j)}$ is the jth order statistic from the sample. The median of the n(n+1)/2 midranges is then used as an estimate of α. This method was suggested by Moses (2). Some sampling experiments were done using these four methods. Table 1 gives the combinations of n and α that were used in the study of the estimates.

Table 1

Values of n for given α

α = 1	α = 5
5	5
10	10
20	20
50	50

Twenty estimates of α were computed for each (α,n) pair. Some results are given in Table 2.

Table 2

Results of Sampling Experiment[1]

α	n	Method[2]	Mean	S.D.	Skewness	Kurtosis
1	5	1	1.25	1.17	2.39	8.8
		2	1.66	1.12	.83	2.94
		3	1.62	1.16	1.21	4.18
		4	2.30	2.08	1.84	5.55
5	5	1	3.94	3.67	2.23	8.01
		2	4.51	2.77	.72	2.06
		3	4.83	3.09	.87	2.94
		4	8.29	7.53	2.25	8.34
1	10	1	1.23	.86	.79	2.07
		2	1.16	.60	.91	2.88
		3	1.25	.72	1.05	3.01
		4	2.24	1.58	1.14	3.26
5	10	1	4.82	4.49	2.08	6.87
		2	6.35	4.86	1.06	3.04
		3	5.95	4.73	1.31	3.73
		4	10.66	11.40	2.50	9.44
1	20	1	.79	.28	-.16	1.77
		2	.95	.35	.59	2.31
		3	.94	.36	.95	3.43
		4	1.46	.57	.82	3.53
5	20	1	4.19	1.99	.85	3.59
		2	4.38	2.12	1.42	5.06
		3	4.55	2.18	1.50	5.22
		4	7.27	3.58	1.24	4.35
1	50	1	.96	.23	-.09	2.00
		2	1.04	.29	1.15	4.03
		3	1.05	.25	.36	2.41
		4	1.69	.45	-.12	1.90
5	50	1	5.00	1.40	.41	2.25
		2	5.02	1.19	.30	4.15
		3	5.03	1.22	.01	2.46
		4	8.17	2.17	-.17	2.02

Notes: (1) These results are based on 20 samples at each (α,n) combination
(2) The methods are coded as
1 - Median
2 - Geometric Mean
3 - Maximum Likelihood
4 - Median of Midranges

These results indicate the following:

1) The median of the midranges is not a good estimator since it has larger bias and larger standard deviation than the other three estimators. The positive bias is to be expected since the parent distribution is skewed to the right.

2) For samples of size 5, 10 and 20, the expectation of the geometric mean estimator and the maximum likelihood estimator are approximately equal, whereas the expectation of the median is slightly below these two. For samples of size 50, these three estimates have virtually the same expectation.

3) The standard deviation of all the estimators except the median of the midranges are approximately equal, regardless of sample size.

4) For samples of size 5 and 10, the median is generally more skewed than the geometric mean or the maximum likelihood estimators. For larger sample sizes the median is less skewed or has approximately the same skewness as the other two estimators.

5) For samples of size 5 or 10, the geometric mean and the maximum likelihood estimators seem to have minimum kurtosis. For samples of size 20 and 50, the median seems to have less kurtosis than the other estimators.

One of us (P.A.L.) has found the distribution (1) useful as a teaching example of a distribution with no moments, but whose percentiles are extremely simple to obtain.

References

1. Gross, A. Letter to the Editor, The American Statistician, Dec. 1969, V. 23, No. 5.

2. Moses, L. Query 10 Confidence Limits from Rank Tests, Technometrics, 1965, v. 7, No. 2.