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ON EQUIVALENCE OF PROBABILITY MEASURES

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ABSTRACT

Let H be a real and separable Hilbert space, Γ the Borel σ-field of H sets, and μ_1 and μ_2 two probability measures on (H,Γ). μ_1 and μ_2 are equivalent (mutually absolutely continuous) if, for $A \in \Gamma$, $\mu_1(A) = 0 \Longleftrightarrow \mu_2(A) = 0$. Several sufficient conditions for equivalence are obtained in this paper. Some of these results do not require that μ_1 and μ_2 be Gaussian. The conditions obtained are applied to show equivalence for some specific measures when H is $L_2[T]$.

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INTRODUCTION

Conditions for equivalence of probability measures on the Borel σ-field of a Hilbert space have been the subject of much research during recent years [1] - [9]. For two Gaussian measures μ_1, μ_2, either μ_1 and μ_2 are equivalent (mutually absolutely continuous, denoted by $\mu_1 \sim \mu_2$) or else μ_1 and μ_2 are orthogonal ($\mu_1 \perp \mu_2$), and general necessary and sufficient conditions for equivalence have been obtained (e.g., [5]).

When one or both of the measures is not Gaussian, few conditions for equivalence are known. Moreover, even when both measures *are* Gaussian the general conditions for equivalence are often difficult to verify, requiring one to prove existence of a Hilbert-Schmidt operator with prescribed spectral properties.

In this paper, several conditions for equivalence are given. Most of these conditions are stated in terms of sample function properties. Several results do not require that both measures be Gaussian; when the two measures *are* Gaussian, the sufficient conditions given here will often be easier to verify than those previously obtained.

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Definitions and Problem Statement

Let H be a real and separable Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and Borel σ-field Γ. Let (Ω,β,P) be a probability space, and suppose that S and N are β/Γ measurable mappings (e.g., $S^{-1}(A) \in \beta$ for all $A \in \Gamma$) of Ω into H. Following Mourier [10], a β/Γ measurable mapping of Ω into H will be called a "random element" in H. Let $(H\times H, \Gamma\times\Gamma)$ be the usual product measurable space; $\Gamma\times\Gamma$ is the smallest σ-field containing all measurable rectangles $A\times B$, $A,B\in\Gamma$. $H\times H$ is a separable Hilbert space under the inner product defined by $\langle (\underset{\sim}{x}_1,\underset{\sim}{x}_2), (\underset{\sim}{y}_1,\underset{\sim}{y}_2)\rangle_{H\times H} = \langle \underset{\sim}{x}_1,\underset{\sim}{y}_1\rangle + \langle \underset{\sim}{x}_2,\underset{\sim}{y}_2\rangle$ for $\underset{\sim}{x}_1,\underset{\sim}{x}_2,\underset{\sim}{y}_1,\underset{\sim}{y}_2$ in H [11].

Define the map $(S,N)\colon \Omega \to H\times H$ by $(S,N)(\omega) = (S(\omega),N(\omega))$; this map is $\beta/\Gamma\times\Gamma$ measurable, since for A,B in Γ

$$\{\omega: \ (S(\omega),N(\omega)) \in A\times B\} = \{\omega: \ S(\omega) \in A\} \cap \{\omega: \ N(\omega) \in B\}.$$

Hence (S,N) induces from P a measure $\mu_{S,N}$ on $(H\times H, \Gamma\times\Gamma)$ defined by $\mu_{S,N}[C] = P\{\omega: \ (S(\omega),N(\omega)) \in C\}$, C in $\Gamma\times\Gamma$. Further, S and N induce measures μ_S and μ_N, respectively, on Γ; e.g., for A in Γ, $\mu_S(A) = P[S^{-1}(A)]$. S and N are independent if and only if $\mu_{S,N} = \mu_S \otimes \mu_N$, where $\mu_S \otimes \mu_N[A\times B] = \mu_S(A)\mu_N(B)$, A, B in Γ.

Consider the map $f\colon H\times H \to H$, $f(\underset{\sim}{x},\underset{\sim}{y}) = \underset{\sim}{x}+\underset{\sim}{y}$. f is continuous relative to the norm topologies of $H\times H$ and H, so that f is $\Gamma\times\Gamma/\Gamma$ measurable. Hence S+N is β/Γ measurable, and induces a measure μ_{S+N} from P,

$$\mu_{S+N}[A] = \mu_{S,N}[f^{-1}(A)] = P\{\omega: (S(\omega),N(\omega)) \in f^{-1}(A)\}.$$

Let $\underset{\sim}{v}$ denote any fixed element of H. Define $f_{\underset{\sim}{v}}\colon H\to H$ by $f_{\underset{\sim}{v}}(\underset{\sim}{y}) = \underset{\sim}{v}+\underset{\sim}{y}$; $f_{\underset{\sim}{v}}$ is the section of f at $\underset{\sim}{v}$, therefore is Γ/Γ measurable and for A in Γ we define the measure $\mu_{\underset{\sim}{v}+N}$ by $\mu_{\underset{\sim}{v}+N}(A) = \mu_N[f_{\underset{\sim}{v}}^{-1}(A)]$.

Two measures μ_1 and μ_2 on (H,Γ) are said to be equivalent $(\mu_1 \sim \mu_2)$ if for $A \epsilon \Gamma$ $\mu_1(A) = 0 \iff \mu_2(A) = 0$. They are orthogonal $(\mu_1 \perp \mu_2)$ if there exists A in Γ such that $\mu_1(A) = 0$, $\mu_2(A) = 1$.

The problem considered in this paper is that of obtaining sufficient conditions for equivalence of μ_N and μ_{S+N}. Several of the conditions obtained are stated in terms of the measures $\mu_{\underset{\sim}{v}+N}$, $\underset{\sim}{v} \epsilon H$.

Covariance Operators; Gaussian Measures

Suppose $E||S(\omega)||^2 < \infty$; then there exists [10] an element $\underset{\sim}{m}_S$ of H and an operator R_S in H such that $\langle \underset{\sim}{m}_S, \underset{\sim}{u}\rangle = E\langle S(\omega), \underset{\sim}{u}\rangle$, $\langle R_S \underset{\sim}{u}, \underset{\sim}{v}\rangle = E\langle S(\omega)-\underset{\sim}{m}_S, \underset{\sim}{u}\rangle \cdot \langle S(\omega)-\underset{\sim}{m}_S, \underset{\sim}{v}\rangle$, for all $\underset{\sim}{u}$, $\underset{\sim}{v}$ in H. The operator R_S is a "covariance" operator; i.e., it is linear, bounded, non-negative, self-adjoint, and trace-class.

If $E||N(\omega)||^2 < \infty$, then S and N have a "cross-covariance" operator R_{SN}: $H \to H$, defined by $\langle R_{SN}u, v\rangle = E\langle S(\omega)-\underset{\sim}{m}_S, \underset{\sim}{v}\rangle \langle N(\omega)-\underset{\sim}{m}_N, \underset{\sim}{u}\rangle$ for all $\underset{\sim}{u}$, $\underset{\sim}{v}$ in H; moreover, $R_{SN} = R_S^{\frac{1}{2}} V R_N^{\frac{1}{2}}$ for a bounded linear operator V, with $||V|| \leq 1$ [12]. R_{SN} is thus trace class, and $R_{NS} = R_{SN}^*$, where $*$ denotes adjoint.

S is said to be a Gaussian element, and μ_S a Gaussian measure, if $\langle S, \underset{\sim}{u}\rangle$ is a Gaussian random variable for all $\underset{\sim}{u}$ in H. μ_S then has a covariance operator R_S and a mean element m_S, and $E||S(\omega)||^2 < \infty$ [10].

If μ_N and μ_{S+N} are Gaussian measures, then they are either equivalent or orthogonal [2,3]. A number of conditions for equivalence have been given by various authors; the conditions most useful for our purposes are the following [5]:

Lemma 1. $\mu_N \sim \mu_{S+N}$ if and only if

(a) $\underset{\sim}{m}_S - \underset{\sim}{m}_N$ is in the range of $R_N^{\frac{1}{2}}$;

(b) $R_{S+N} = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}} + R_N$, where W is a Hilbert-Schmidt operator that does not have -1 as an eigenvalue.

In dealing with the equivalence of Gaussian measures, one often needs the following results on the range of square roots of covariance operators [13], [14]:

Lemma 2. Suppose R_1 and R_2 are covariance operators. Then

(1) range $(R_1^{\frac{1}{2}}) \subset$ range $(R_2^{\frac{1}{2}})$ if and only if the following (equivalent) conditions are satisfied:

(a) $R_1^{\frac{1}{2}} = R_2^{\frac{1}{2}}G$ for G linear and bounded;

(b) $R_1 = R_2^{\frac{1}{2}}QR_2^{\frac{1}{2}}$ for Q linear and bounded;

(c) $\langle R_1\underset{\sim}{u},\underset{\sim}{u}\rangle \le k\langle R_2\underset{\sim}{u},\underset{\sim}{u}\rangle$ for all $\underset{\sim}{u}$ in H and some finite scalar k.

(2) range $(R_1^{\frac{1}{2}})$ = range $(R_2^{\frac{1}{2}})$ if and only if the following (equivalent) conditions are satisfied:

(a) $R_1^{\frac{1}{2}} = R_2^{\frac{1}{2}}G$, for G linear and bounded with bounded inverse

(b) $R_1 = R_2^{\frac{1}{2}}QR_2^{\frac{1}{2}}$ for Q linear and bounded with bounded inverse.

APPLICATIONS

In most applications H is $L_2[T]$ (Lebesgue measure) for some compact interval T of the real line. In such cases, S and N are random functions corresponding to measureable stochastic processes (S_t), (N_t), whose sample functions belong almost surely to $L_2[T]$. It is easy to verify that a measurable stochastic process (S_t) with sample functions a.s. in $L_2[T]$ is a β/Γ measurable function; one uses the facts that $\langle S, \underset{\sim}{u} \rangle$ is $\beta/B[R]$ measurable ($B[R] \equiv$ Borel sets of the real line) for all $\underset{\sim}{u}$ in $L_2[T]$, and that Γ is the smallest σ-field such that all bounded linear functionals on $L_2[T]$ are $\Gamma/B[R]$ measurable.

In many signal detection problems (see, e.g., [6], Chapter 20), one observes a sample function $X(\omega)$ belonging to $L_2[T]$ and must determine whether the sample function is "noise" $(X(\omega) = N(\omega))$ or "signal-plus-noise" $(X(\omega) = S(\omega)+N(\omega))$, these two possibilities being mutually exclusive and exhaustive. If $\mu_{S+N} \perp \mu_N$, there exists a statistical test such that the probability of making an incorrect classification is zero. However, if $\mu_{S+N} \sim \mu_N$, then the probability of deciding correctly when signal is present can be unity only if the probability of deciding incorrectly is unity when signal is absent. Equivalence holds in virtually all practical signal detection problems. Thus conditions for equivalence are of interest to aid in constructing valid mathematical models of signal detection problems.

Equivalence Conditions for Independent S, N

In this section it is assumed that $\mu_{S,N} = \mu_S \otimes \mu_N$.

Theorem 1. If $\mu_{\underset{\sim}{v}+N} \sim \mu_N$ a.e. $d\mu_S(\underset{\sim}{v})$, then $\mu_{S+N} \sim \mu_N$.

Proof: For A in Γ,

$$\mu_{S+N}(A) = \mu_{S,N}[f^{-1}(A)] = \mu_S \otimes \mu_N[f^{-1}(A)]$$

$$= \int_H \mu_N\,[f_{\underset{\sim}{v}}^{-1}(A)]\; d\mu_S(\underset{\sim}{v})$$

$$= \int_H \mu_{\underset{\sim}{v}+N}(A)\; d\mu_S(\underset{\sim}{v}) \;. \qquad (*)$$

Suppose $\mu_{\underset{\sim}{v}+N} \sim \mu_N$ a.e. $d\mu_S(\underset{\sim}{v})$. Then, $\mu_N(A) = 0 \Rightarrow \mu_{\underset{\sim}{v}+N}(A) = 0$ a.e. $d\mu_S(\underset{\sim}{v})$ $\Rightarrow \mu_{S+N}(A) = 0$, from (*). Also, $\mu_{S+N}(A) = 0 \Rightarrow \mu_{\underset{\sim}{v}+N}(A) = 0$ a.e. $d\mu_S(\underset{\sim}{v}) \Rightarrow$ $\mu_N(A) = 0$. Hence $\mu_{S+N} \sim \mu_N$. (Note that $\mu_{\underset{\sim}{v}+N} \perp \mu_N$ a.e. $d\mu_S(\underset{\sim}{v})$ does not imply $\mu_{S+N} \perp \mu_N$, since the set $A_{\underset{\sim}{v}}$ satisfying $\mu_{\underset{\sim}{v}+N}(A_{\underset{\sim}{v}}) = 1-\mu_N(A_{\underset{\sim}{v}}) = 0$ can vary with $\underset{\sim}{v}$.)

Although the above result assumes that $\mu_{S,N} = \mu_S \otimes \mu_N$, it can be applied to yield conditions for equivalence when S and N are not independent. For example, suppose that $N = N_1+N_2$, where N_1 and N_2 are β/Γ measurable transformations inducing measures μ_{N_1} and μ_{N_2}. If $\mu_{N_1,N_2} = \mu_{N_1} \otimes \mu_{N_2}$ and $\mu_{S,N_2} = \mu_S \otimes \mu_{N_2}$, one can use the above result to determine if $\mu_N \sim \mu_{N_2}$ and $\mu_{S+N} \sim \mu_{N_2}$. If both these equivalences hold, then by the chain rule for Radon-Nikodym derivatives one must have $\mu_{S+N} \sim \mu_N$. This simple modification should be useful in many applications, especially in practical signal detection problems, where the noise usually contains an additive Gaussian component that is independent of the signal and of the remainder of the noise.

In order to apply Theorem 1, one must first determine sufficient conditions for $\mu_{\underset{\sim}{v}+N} \sim \mu_N$, $\underset{\sim}{v} \epsilon H$. Several such conditions are known when μ_N is Gaussian, and are utilized to obtain the following corollary.

COROLLARY. Suppose that μ_N is Gaussian with $||\underset{\sim}{m}_N|| = 0$, and $\mu_{S,N} = \mu_S \otimes \mu_N$. Then $\mu_{S+N} \sim \mu_N$ if any of the following conditions is satisfied:

(1) $\underset{\sim}{v} \epsilon$ range $(R_N^{\frac{1}{2}})$ a.e. $d\mu_S(\underset{\sim}{v})$;

(2) $E||S(\omega)||^2 < \infty$, $\underset{\sim}{m}_S \epsilon$ range $(R_N^{\frac{1}{2}})$, and $R_S = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$ with W trace-class;

(3) H is $L_2[T]$ for a compact interval T, μ_N is the measure induced by a measurable mean-square-continuous stationary stochastic process with rational spectral density $\hat{R}_N$, $E||S(\omega)||^2 < \infty$, and

(a) there exists a rational spectral density function $\hat{R}_0$ such that $\langle R_S \underset{\sim}{u}, \underset{\sim}{u} \rangle \le k \int_{-\infty}^{\infty} \hat{R}_0(\lambda) |\hat{\underset{\sim}{u}}(\lambda)|^2 \, d\lambda$ for all $\underset{\sim}{u}$ in H ($\hat{\underset{\sim}{u}}$ is the Fourier transform of $\underset{\sim}{u}$, $\underset{\sim}{u}(t) \equiv 0$ for $t \notin T$) and some finite scalar k, with

$$\int_{-\infty}^{\infty} \frac{\hat{R}_0(\lambda)}{\hat{R}_N(\lambda)} \, d\lambda < \infty,$$

(b) $\underset{\sim}{m}_S \epsilon$ range $(R_N^{\frac{1}{2}})$.

(4) H is as in (3), μ_N is induced by a measurable mean-square continuous stationary stochastic process with a spectral density function, $\hat{R}_N$, and

$$\int_{-\infty}^{\infty} \frac{|\hat{\underset{\sim}{v}}(\lambda)|^2}{\hat{R}_N(\lambda)} \, d\lambda < \infty \quad \text{a.e.} \quad d\mu_S(\underset{\sim}{v}) .$$

PROOF: (1) From Lemma 1, $\underset{\sim}{v}$ in range $(R_N^{\frac{1}{2}}) \iff \mu_{\underset{\sim}{v}+N} \sim \mu_N$.

(2) Let $\{\lambda_n\}$, $\{\underset{\sim}{e}_n\}$ be the non-zero eigenvalues and an associated set of orthonormal eigenvectors of R_N. Then

$$E \sum_{1}^{N} \frac{\langle S(\omega), \underset{\sim}{e}_n\rangle^2}{\lambda_n} = \sum_{1}^{N} \left\{ \frac{\langle R_S \underset{\sim}{e}_n, \underset{\sim}{e}_n\rangle}{\lambda_n} + \frac{\langle \underset{\sim}{m}_S, \underset{\sim}{e}_n\rangle^2}{\lambda_n} \right\}$$

and (2) follows, since $S(\omega) \in \overline{\text{range } (R_N^{\frac{1}{2}})}$ almost surely.

(3) According to Hajek [4],

$$\int_{-\infty}^{\infty} \frac{\hat{R}_0(\lambda)}{\hat{R}_N(\lambda)} d\lambda < \infty$$

implies that $R_0 = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$, W trace-class. By Lemma 2, the condition $\langle R_S \underset{\sim}{u}, \underset{\sim}{u}\rangle \le k \int_{-\infty}^{\infty} \hat{R}_0(\lambda) |\hat{\underset{\sim}{u}}(\lambda)|^2 \, d\lambda$ implies that range $(R_S^{\frac{1}{2}}) \subset$ range $(R_0^{\frac{1}{2}})$, and this last condition implies (Lemma 2) that $R_S^{\frac{1}{2}} = R_0^{\frac{1}{2}} G$ for G bounded; while the representation $R_0 = R_N^{\frac{1}{2}} W R_N^{\frac{1}{2}}$ with W trace-class implies $R_0^{\frac{1}{2}} = R_N^{\frac{1}{2}} Q$, Q Hilbert-Schmidt. Hence $R_S = R_N^{\frac{1}{2}} QGG^*Q^* R_N^{\frac{1}{2}}$, and QGG^*Q^* is trace-class, so that (3) follows from (2).

(4) This result is due to Kelly, Reed, and Root [15], and can be proved quickly as follows. By Lemma 2, $\underset{\sim}{v}$ is in range $(R_N^{\frac{1}{2}})$ if and only if there exists a finite scalar k such that $\langle \underset{\sim}{v}, \underset{\sim}{u}\rangle^2 \le k \langle R_N \underset{\sim}{u}, \underset{\sim}{u}\rangle$, all $\underset{\sim}{u}$ in H. The condition given in (4) implies that

$$\langle \underset{\sim}{v}, \underset{\sim}{u}\rangle^2 \le \int_{-\infty}^{\infty} \frac{|\hat{\underset{\sim}{v}}(\lambda)|^2}{\hat{R}_N(\lambda)} d\lambda \, \langle R_N \underset{\sim}{u}, \underset{\sim}{u}\rangle \quad \text{a.e.} \quad d\mu_S(\underset{\sim}{v}),$$

so that $\underset{\sim}{v} \in$ range $(R_N^{\frac{1}{2}})$ a.e. $d\mu_S(\underset{\sim}{v})$.

The conditions cited above can be used to determine equivalence when $N = N_1 + N_2$, where $\mu_{N_1,N_2} = \mu_{N_1} \otimes \mu_{N_2}$, $\mu_{N_2,S} = \mu_{N_2} \otimes \mu_S$, and μ_{N_2} is Gaussian, as discussed previously.

Dr. T.T. Kadota has kindly provided the author with a preprint of a forthcoming paper [16]. In that paper, part (1) of the above corollary is proved for $H = L_2[0,b]$, $b < \infty$, under the additional assumptions that μ_N has continuous covariance function and R_N is strictly positive definite.

DEPENDENT S, N

When the assumption that $\mu_{S,N} = \mu_S \otimes \mu_N$ is not valid, the equivalence of μ_{S+N} and μ_N is not implied by $\mu_{\underset{\sim}{v}+N} \sim \mu_N$ a.e. $d\mu_S(\underset{\sim}{v})$. As a counterexample, suppose that μ_N is Gaussian and that $S(\omega) = kR_N^{1/p}N(\omega)$, for p a positive integer and some scalar k. S is then Gaussian, and for $p = 1$ or 2, $S(\omega) \in$ range $(R_N^{1/p})$ a.e. $dP(\omega)$, implying $\mu_{\underset{\sim}{v}+N} \sim \mu_N$ a.e. $d\mu_S(\underset{\sim}{v})$. One has $R_{S+N} = R_N^{\frac{1}{2}}[k^2R_N^{2/p}+2kR_N^{1/p}+I]R_N^{\frac{1}{2}}$. From Lemma 1 and Lemma 2, $\mu_{S+N} \perp \mu_N$ if and only if $k^2R_N^{2/p} + 2kR_N^{1/p} + I$ has zero as an eigenvalue; this will occur if and only if $k = -\lambda_i^{-1/p}$ for some non-zero eigenvalue λ_i of R_N. As a specific example, $\mu_{S+N} \perp \mu_N$ if H is $L_2[0,1]$, μ_N is Wiener measure, and

$$S_t(\omega) = -((2n+1)^2/4)\pi^2 \int_0^t \int_u^1 N_v(\omega)dvdu = -((2n+1)^2/4)\pi^2[R_NN]_t(\omega) ,$$

for any integer n.

Dependent S, N: Gaussian Measures

In this section, it is assumed that μ_N, μ_S, and μ_{S+N} are Gaussian. As previously noted, this implies the existence of covariance operators R_N, R_S, and R_{S+N} and mean elements $\underset{\sim}{m}_N$, $\underset{\sim}{m}_S$, and $\underset{\sim}{m}_{S+N}$; we assume $||\underset{\sim}{m}_N|| = 0$. Let $\mu_{(S+N)}$ denote the Gaussian measure defined by

$$\mu_{(S+N)}(A) = \int_{f^{-1}(A)} d\mu_S \otimes \mu_N \quad \text{for} \quad A \in \Gamma$$

$(f(\underset{\sim}{u},\underset{\sim}{v}) = \underset{\sim}{u}+\underset{\sim}{v})$. $\mu_{(S+N)}$ has covariance operator $R_S + R_N$ and mean element $\underset{\sim}{m}_S$. Finally, we assume that the range of R_N is dense in H; in cases where this is not satisfied, one can obtain the results given in this section by defining H to be the closure of range (R_N). We proceed to obtain some new sufficient conditions for equivalence of μ_{S+N} and μ_N.

Lemma 3. Suppose $\mu_S\{\text{range } (R_N^{1/2})\} = 1$. Then $R_S = R_N^{1/2} W R_N^{1/2}$ for a covariance operator W, and $\underset{\sim}{m}_S \in \text{range } (R_N^{1/2})$.

Proof: By the corollary to Theorem 1, $\mu_S\{\text{range } (R_N^{1/2})\} = 1 \Rightarrow \mu_{(S+N)} \sim \mu_N$; from Lemma 1 this implies $R_S = R_N^{1/2} W R_N^{1/2}$, W Hilbert-Schmidt, and $\underset{\sim}{m}_S \in$ range $(R_N^{1/2})$. Let $\underset{\sim}{g} \equiv R_N^{-1/2}\underset{\sim}{m}_S$.

Since μ_S is Gaussian, $\langle S, \underset{\sim}{u}\rangle$ is a Gaussian random variable for all $\underset{\sim}{u} \in H$. Define $Y: \Omega \to H$ by

$$Y(\omega) = \begin{cases} R_N^{-1/2} S(\omega) & \text{for } S(\omega) \in \text{range } (R_N^{1/2}) \\ \underset{\sim}{0} & \text{for } S(\omega) \notin \text{range } (R_N^{1/2}) . \end{cases}$$

For any $A \in \Gamma$, $R_N^{1/2}[A] \equiv \{\underset{\sim}{x}: \underset{\sim}{x} = R_N^{1/2}\underset{\sim}{u}$ for $\underset{\sim}{u} \in A\}$ is an element of Γ, since $R_N^{1/2}$ is a continuous map. Moreover, $Y^{-1}(A) = \{\omega: S(\omega) \in R_N^{1/2}[A]\}$ if $\underset{\sim}{0} \notin A$; if $\underset{\sim}{0} \in A$, then $Y^{-1}(A) = \{\omega: S(\omega) \in R_N^{1/2}[A]\} \cup \{\omega: S(\omega) \notin \text{range } (R_N^{1/2})\}$. In either case, $Y^{-1}(A) \in \beta$, so that Y is β/Γ measurable and thus induces from P a measure μ_Y on (H,Γ). For any $\underset{\sim}{u}$ in H we show that $\langle Y,\underset{\sim}{u}\rangle$ is a Gaussian random variable; first, note that there exists $\{\underset{\sim}{u}_n\}$ such that $R_N^{\frac{1}{2}}\underset{\sim}{u}_n \to \underset{\sim}{u}$. Using $R_S = R_N^{\frac{1}{2}}WR_N^{\frac{1}{2}}$, W bounded, and $\underset{\sim}{m}_S = R_N^{\frac{1}{2}}\underset{\sim}{g}$, one has that $\langle S,\underset{\sim}{u}_n\rangle \to \langle Y,\underset{\sim}{u}\rangle$ almost surely and in $L_2(\Omega,\beta,P)$. Hence $\langle Y,\underset{\sim}{u}\rangle$ is a Gaussian r.v. for all $\underset{\sim}{u}$ in H, μ_Y is Gaussian with covariance operator R_Y and mean element $\underset{\sim}{m}_Y$, $R_S = R_N^{\frac{1}{2}}R_YR_N^{\frac{1}{2}}$, and $\underset{\sim}{m}_S = R_N^{\frac{1}{2}}\underset{\sim}{m}_Y$.

THEOREM 2. If $\mu_S\{\text{range } (R_N^{\frac{1}{2}})\} = 1$, then $\mu_{S+N} \sim \mu_N$ if and only if range $(R_{S+N}^{\frac{1}{2}}) \supset$ range $(R_N^{\frac{1}{2}})$.

PROOF: $\mu_S[\text{range } (R_N^{\frac{1}{2}})] = 1$ implies, by Lemma 3, that $m_S \in$ range $(R_N^{\frac{1}{2}})$ and $R_S = R_N^{\frac{1}{2}}WR_N^{\frac{1}{2}}$, W trace class. From Lemma 1, $\mu_{S+N} \sim \mu_N$ if and only if $R_{S+N} = R_N^{\frac{1}{2}}(I+Q)R_N^{\frac{1}{2}}$, where Q is Hilbert-Schmidt and does not have -1 as an eigenvalue. $R_S = R_N^{\frac{1}{2}}WR_N^{\frac{1}{2}}$ implies $R_S^{\frac{1}{2}} = R_N^{\frac{1}{2}}G$, G Hilbert-Schmidt; hence

$$R_{S+N} = R_S + R_{SN} + R_{NS} + R_N = R_N^{\frac{1}{2}}[W+GV+V^*G^*+I]R_N^{\frac{1}{2}} ,$$

where V is an operator of norm ≤ 1 satisfying $R_{SN} = R_S^{\frac{1}{2}}VR_N^{\frac{1}{2}}$ [12]. Now $Z \equiv W+GV+V^*G^*$ is Hilbert-Schmidt, so that $\mu_{S+N} \sim \mu_N$ if and only if Z does not have -1 as an eigenvalue. From Lemma 2, this is precisely the condition for range $(R_{S+N}^{\frac{1}{2}}) \supset$ range $(R_N^{\frac{1}{2}})$.

The results summarized in Lemma 2 can be used to determine whether range $(R_N^{\frac{1}{2}}) \subset$ range $(R_{S+N}^{\frac{1}{2}})$, and thus $\mu_{S+N} \sim \mu_N$ whenever $\mu_{\underset{\sim}{v}+N} \sim \mu_N$ a.e. $d\mu_S(\underset{\sim}{v})$. The following corollary gives two additional conditions for $\mu_{S+N} \sim \mu_N$.

COROLLARY. $\mu_{S+N} \sim \mu_N$ if $\mu_S\{\text{range } (R_N^{\frac{1}{2}})\} = 1$ and either of the two following conditions is satisfied:

(a) There exists no non-null $\underset{\sim}{u}$ in H satisfying $G^*\underset{\sim}{u} = -V\underset{\sim}{u}$ and

$V^*G^*u = -u$, where G,V are operators satisfying $R_S^{1/2} = R_N^{1/2}G$, $||V|| \le 1$, $R_{SN} = R_S^{1/2}VR_N^{1/2}$. (This condition is also necessary for $\mu_{S+N} \sim \mu_N$.)

(b) H is $L_2[T]$, T a compact interval, μ_N is induced by a measurable mean-square continuous stochastic process, (S_t) is measurable, and for all $t \in T$, S_t is independent of N_v for all $v \ge t$.

PROOF: (a) From the theorem, it is sufficient to show that the stated condition is necessary and sufficient for range $(R_{S+N}^{1/2}) \supset$ range $(R_N^{1/2})$, or, by Lemma 2, for $(I+GG^*+GV+V^*G^*)\underset{\sim}{u} = \underset{\sim}{0}$ to imply $||\underset{\sim}{u}|| = 0$. If $(I+GG^*+GV+V^*G^*)\underset{\sim}{u} = \underset{\sim}{0}$, then $||\underset{\sim}{u}||^2+||G^*\underset{\sim}{u}||^2+2\langle G^*\underset{\sim}{u}, V\underset{\sim}{u}\rangle = 0$. The LHS of this last equality is $\ge ||\underset{\sim}{u}||^2 + ||G^*\underset{\sim}{u}||^2 - 2||G^*\underset{\sim}{u}||\ ||V\underset{\sim}{u}|| \ge ||\underset{\sim}{u}||^2 + ||G^*\underset{\sim}{u}||^2 - 2||G^*\underset{\sim}{u}||\ ||\underset{\sim}{u}|| = (||\underset{\sim}{u}||-||G^*\underset{\sim}{u}||)^2 \ge 0$, with equality throughout if and only if $G^*\underset{\sim}{u} = -V\underset{\sim}{u}$ and $||G^*\underset{\sim}{u}|| = ||\underset{\sim}{u}||$. Further, $||\underset{\sim}{u}||^2 + ||G^*\underset{\sim}{u}||^2 + 2\langle G^*\underset{\sim}{u}, V\underset{\sim}{u}\rangle \ge ||\underset{\sim}{u}||^2 + ||G^*\underset{\sim}{u}||^2 - 2||V^*G^*\underset{\sim}{u}||\ ||\underset{\sim}{u}|| \ge ||\underset{\sim}{u}||^2 + ||G^*\underset{\sim}{u}||^2 - 2||G^*\underset{\sim}{u}||\ ||\underset{\sim}{u}|| = (||\underset{\sim}{u}||-||G^*\underset{\sim}{u}||)^2 \ge 0$, with equality throughout if and only if $||G^*\underset{\sim}{u}|| = ||\underset{\sim}{u}||$ and $V^*G^*\underset{\sim}{u} = -\underset{\sim}{u}$. Hence, if $G^*\underset{\sim}{u} = -V\underset{\sim}{u}$ and $V^*G^*\underset{\sim}{u} = -\underset{\sim}{u}$ only for $\underset{\sim}{u} = \underset{\sim}{0}$, range $(R_{S+N}^{1/2}) \supset$ range $(R_N^{1/2})$. It is clear that $V^*G^*\underset{\sim}{u} = -\underset{\sim}{u}$ and $G^*\underset{\sim}{u} = -V\underset{\sim}{u}$ together imply $(I+GG^*+GV+V^*G^*)\underset{\sim}{u} = \underset{\sim}{0}$, so that the condition is also necessary for range $(R_{S+N}^{1/2}) \supset$ range $(R_N^{1/2})$.

(b) We show that the assumptions imply range $(R_{S+N}^{1/2}) \supset$ range $(R_N^{1/2})$. Suppose not. Then by Lemma 2 there must exist $\{\underset{\sim}{u}_n\}$ in H such that $R_N^{1/2}\underset{\sim}{u}_n \to \underset{\sim}{u}$, $||\underset{\sim}{u}|| = 1$, and $R_{S+N}^{1/2}\underset{\sim}{u}_n \to \underset{\sim}{0}$. This implies the existence of a random variable X in $L_2(\Omega,\beta,P)$ such that $X = \ell.i.m.\langle\underset{\sim}{u}_n, N\rangle = \ell.i.m.\langle -\underset{\sim}{u}_n, S-\underset{\sim}{m}_S\rangle$ ($\ell.i.m.$ for $L_2(\Omega,\beta,P)$). Since (N_t) is mean-square continuous, the subspace of $L_2(\Omega,\beta,P)$ spanned by $\{\langle N,\underset{\sim}{u}\rangle, \underset{\sim}{u} \in H\}$ is identical to the subspace spanned by $\{N_t, t \in T\}$. Moreover, this subspace is separable since it is isometric to the closure of the range of $R_N^{1/2}$ (under the map taking $R_N^{1/2}\underset{\sim}{u}$ into $\langle\underset{\sim}{u},N\rangle$). Hence, there are scalars $\{\alpha_k^M\}$, $k = 1,2,\dots,M$; $M = 1,2,\dots$ such that $X = \ell.i.m. \sum_{k=1}^{M}\alpha_k^M N_{t_k}$;

let $X_M \equiv \sum_{k=1}^{M} \alpha_k^M N_{t_k}$. Note that the set $\{t_k\}$ can be assumed independent of M, since $\{N_t\}$, $t \in T$, spans a separable subspace of $L_2(\Omega,\beta,P)$.

We now show that $EX(\omega)[S_t(\omega) - m_{S_t} + N_t(\omega)] = 0$ for all $t \in T$. To see this, one notes that $EX(\omega)[S_t(\omega) - m_{S_t} + N_t(\omega)] = -\lim_n E\langle \underset{\sim}{u}_n, S(\omega) - m_S \rangle [S_t(\omega) - m_{S_t} + N_t(\omega)]$ $= -\lim_n [R_S \underset{\sim}{u}_n + R_{NS} \underset{\sim}{u}_n](t)$. In $L_2[T]$, one has $R_S \underset{\sim}{u}_n + R_{NS} \underset{\sim}{u}_n = R_N^{\frac{1}{2}}(GG^* + V^*G^*) R_N^{\frac{1}{2}} \underset{\sim}{u}_n$ $\to R_N^{\frac{1}{2}}(GG^* + V^*G^*) \underset{\sim}{u} = \underset{\sim}{0}$, from the assumptions on $\underset{\sim}{u}$ and part (a) of this corollary. Since R_N has continuous kernel $R(t,u)$, this implies that $[R_S \underset{\sim}{u}_n + R_{NS} \underset{\sim}{u}_n](t) \to 0$ uniformly for all t in T. To prove this, let $R_0(t,u)$ be the kernel of $R_N^{\frac{1}{2}}$. Then

$$([R_S \underset{\sim}{u}_n + R_{NS} \underset{\sim}{u}_n](t))^2 = \left\{\int_0^T R_0(t,s)[(GG^* + V^*G^*) R_N^{\frac{1}{2}} \underset{\sim}{u}_n](s)\,ds\right\}^2$$

$$\leq \int_0^T R_0^2(t,s)\,ds \; ||(GG^* + V^*G^*) R_N^{\frac{1}{2}} \underset{\sim}{u}_n||^2$$

$$= R(t,t) \; ||(GG^* + V^*G^*) R_N^{\frac{1}{2}} \underset{\sim}{u}_n||^2$$

$$\leq [\sup_{t \in T} R(t,t)] \; ||(GG^* + V^*G^*) R_N^{\frac{1}{2}} \underset{\sim}{u}_n||^2 \to 0.$$

Hence $EX(\omega)[N_t(\omega) + S_t(\omega) - m_{S_t}] = 0$ for all $t \in T$.

For X_M defined as above,

$$E[S_{t_1}(\omega) - m_{S_{t_1}}]X_M(\omega) = \sum_{k=1}^{M} \alpha_k^M R_{SN}(t_1, t_k) = 0,$$

since $R_{SN}(t_1, t_k) = 0$ for $k \geq 1$ (we are assumming that $t_k \geq t_1$ for $k \geq 1$). Hence

$$E[S_{t_1}(\omega) - m_{S_{t_1}}]X(\omega) = 0 \implies EN_{t_1}(\omega)X(\omega) = 0$$

$\implies X$ is contained in the subspace of $L_2(\Omega,\beta,P)$ spanned by $\{N_{t_k}\}$, $k = 2,3,\ldots$ Suppose next that $X_M \to X$ in $L_2(\Omega,\beta,P)$ with $X_M = \sum_{k=m}^{M} \alpha_k^M N_{t_k}$, where $1 < m \leq M$, m fixed and independent of M. Then $EX_M(\omega)[S_{t_m}(\omega)-m_{S_{t_m}}] = 0$ for all M $\Rightarrow EX_M N_{t_m} = 0$, all $M \Rightarrow X$ is contained in the linear subspace spanned by $\{N_{t_k}\}$, $k \geq m+1$. Hence $EX(\omega)\sum_1^M \alpha_k^M N_{t_k}(\omega) = 0$, all M

$$\Rightarrow EX^2(\omega) = 0 \Rightarrow \lim E\left\langle N(\omega), u_n\right\rangle^2 = 0$$

$$\Rightarrow \lim_n ||R_N^{1/2} u_n||^2 = 0 \Rightarrow ||u|| = 0$$

$$\Rightarrow \text{range } (R_{S+N}^{1/2}) \supset \text{range } (R_N^{1/2}) .$$

Examples

Suppose that H is $L_2[0,b]$, $b < \infty$, and that μ_N is Gaussian with null mean function and covariance function defined as follows:

(1) $R_1(t,s) = min(t,s)$

(2) $R_2(t,s) = b - max(t,s)$

(3) $R_3(t,s) = b - |t-s|$

(4) $R_4(t,s) = e^{-\alpha|t-s|} \quad \alpha > 0$

(5) $R_5(t,s) = \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \hat{R}(\lambda)d\lambda$

where $\hat{R}$ is a rational spectral density function with denominator of degree exactly two greater than the degree of the numerator.

Suppose that μ_S is induced by the stochastic process (S_t) defined by

(a) $S_t(\omega) = \int_0^t Y_s(\omega)ds$ (b) $S_t(\omega) = \int_t^b Y_s(\omega)ds$

or

(c) $S_t(\omega) = c(\omega) + \int_0^t Y_s(\omega)ds$, all t in $[0,b]$,

where in (a), (b) and (c) (Y_t) is a measurable stochastic process with sample functions almost surely in $L_2[0,b]$. c is an a.s. finite random variable.

Assume that one of the following two conditions is satisfied: (A) S and N are independent; (B) S is Gaussian with S_t independent of N_v for $v \geq t$ and all t in $[0,b]$. One then has the following results:

(i) For S defined as in (a), N defined by (1), (3), (4), or (5), $\mu_{S+N} \sim \mu_N$.

(ii) For S defined as in (b), N defined by (2) - (5), $\mu_{S+N} \sim \mu_N$.

(iii) For S defined as in (c), N defined by (3) - (5), $\mu_{S+N} \sim \mu_N$.

These results are unchanged if any of the N covariance functions are multiplied by a positive real scalar. (The result given in (i) for the Wiener process (covariance function R_1) has been previously obtained under weaker assumptions [16].)

To obtain the preceding results we note the following [17]:

(1') The integral operator in $L_2[0,b]$ with kernel $R_1(t,s)$ has square root with range containing all absolutely continuous functions on $[0,b]$ that vanish at 0 and have $L_2[0,b]$ derivative.

(2') The integral operator with kernel $R_2(t,s)$ has square root with range containing all absolutely continuous functions on $[0,b]$ that vanish at b and have $L_2[0,b]$ derivative.

(3') - (5') The integral operators with $R_3(t,s)$, $R_4(t,s)$ and $R_5(t,s)$ as kernel have square roots with the same range; this range space contains all absolutely continuous functions on $[0,b]$ with derivatives belonging to $L_2[0,b]$.

The results stated above now follow directly from the corollaries to the two theorems.

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