Translation prepared under ONR Contract N00014-67-A-0321-0002.

ELEMENTS OF THE GENERAL THEORY OF RANDOM STREAMS OF EVENTS

by Yu. K. Belayev

Translated by M. R. Leadbetter

Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 703

August, 1970

Elements of the General Theory of Random Streams of Events

by

Yu. K. Belayev

Translated by M. R. Leadbetter

This report is a translation of the second of three appendices by Yu. K. Belayev to the Russian edition of the book *Stationary & Related Stochastic Process* by Harald Cramér & M. R. Leadbetter. References to "the main text" refer to this book. Since this is the second appendix, the equation numbers all begin with 2, and these have been left unaltered in this translation.

1. Elements of the general theory of random streams, by Yu. K. Belayev. (Appendix No. 2 to the Russian edition of *Stationary & Related Stochastic Processes*, by Harald Cramér & M. R. Leadbetter, published by MIR, Moscow, 1969.)

From the contents of this book it is seen that the main ideas and results for problems of "crossing" type are formulated in a natural way in terms of the theory of random streams. The theory of random streams, or, as they are also called, point processes, developed principally in connection with various problems in the theory of mass service, of counters and so on. One may see this in more detail with the relevant references in the book of Cox and Smith [1]. In a related direction, there is a series of works concerning generalizations of Palm streams (see Slivnyak [1], Matthes [1], Kerstan and Matthes [1], Ambartsumyan [1], Mecke [1]).

Translation prepared under ONR Contract N00014-67-A-0321-0002.

A short note by Leadbetter [4] gives a simplified proof of results of Khintchine [1] concerning relations between intensities and parameters of stationary streams. A long article of Beutler and Leneman [1], connected with the work of McFadden [2], concerns various approaches to the definition of a stationary stream on the line. A number of general results are also proved. In particular it is shown that if $L_k(t)$ denotes the random time from an instant t to the k-th following event of a stationary stream, then $S_n(x) = \Sigma_{k=1}^{n} P\{L_k(t) \le x\}$ is a convex non-decreasing function for $x \in (0,\infty)$.

We note that in the works of Zitek [1] and Fieger [1], integrals based on functions of intervals on the line are used for generalizing Palm-Khintchine functions, the construction being a particular case of that used below in the definition of a parametric measure.

We restrict ourselves to these short bibliographic references. We are now interested in such generalizations of the main ideas of the theory of random streams which are linked in a direct way with problems of "crossing" type. Further examples are given in the following paragraph. Some of the results given in this paragraph appeared in a communication given by the author on the occasion of a summer school on the theory of Probability in Palang in June 1967 and also in seminars at Moscow University and the Mathematical Institute of the Academy of Sciences (see Belayev [9]).

Let $(T, \mathcal{M}_T)$ be a measurable space of values of the parameter $t \in T$. $\mathcal{M}_T$ is a σ-algebra of measurable sets, (Ω, A_Ω, P) an outcome probability space of elementary events $\omega \in \Omega$. In questions of "crossing" type, the main objects of study are subsets of points of the parameter space T occurring in a random way. We introduce the following definitions.

Definition 2.1. By a random (point) set $S_\omega = \{s_\alpha\}$ of points $s_\alpha = s_\alpha(\omega) \in T$ is meant such a set for which, for any $\Delta \in \mathcal{M}_T$, the number $\eta(\Delta)$ of points in

$S_\omega \cap \Delta$ is a random variable. The random variables $\eta(\Delta)$, $\Delta \in M_T$ take integer values $0, 1, \ldots, \infty.$, and possess the additivity property

$$\eta\left(\bigcup_1^\infty \Delta i\right) = \sum_1^\infty \eta(\Delta i).$$

if $\Delta_i \cap \Delta_j = \phi$, $i \neq j$. The system of random variables $\eta(\Delta)$ is called the random stream arising from the random sets S_ω.

In this book random streams of arrivals, departures and crossings are studied, arising from random sets on $[T, M_T]$ where $T = (-\infty, \infty)$ and M_T is the class of Borel sets. For example $N_u(s,t)$ is the number of points of the random set $S_\omega = \{x: \xi(x) = u\} \cap (s,t)$. Using the fact that the sum, difference and limit of (probability one convergent) random variables, are all random variables, one may show that the number $N_u(\Delta)$ of points of the set $\{s: \xi(s) = u\} \cap \Delta$ is a random variable for any Borel set Δ.

If $\eta(\Delta)$ is a random stream on $[T, M_T]$ then corresponding to each set $\Delta \in M_T$, one may look at the function

$$(2.1) \qquad \mu(\Delta) = E\eta(\Delta).$$

This function is a measure since $\mu(\Delta) > 0$, $\mu(\bigcup_1^\infty \Delta i) = \Sigma_1^\infty \mu(\Delta i)$, $\Delta_i \cap \Delta_j = \phi$, $\Delta_i \in M_T$. The reader will find the main ideas of measure theory used below in Halmos [1]. If it is assumed that $\mu(\Delta)$ is a σ-finite measure, absolutely continuous with respect to the measure $\nu(\Delta)$, then by the Radon-Nikodym theorem

$$(2.2) \qquad \mu(\Delta) = \int_\Delta \mu(t) d\nu(t).$$

Definition 2.2. The measure $\mu(\Delta)$ is called the *principal measure* of the stream $\eta(\Delta)$. If $\mu(\Delta)$ is σ-finite then the stream $\eta(\Delta)$ is called *finite*. A function $\mu(t)$ satisfying the equality (2.2) is called the *intensity function of the* *stream relative to the measure* $\nu(\Delta)$. The value of $\mu(t)$ at the point t is called the *intensity of the stream at* t.

The direct calculation of the principal measure or intinsity function is often accompanied by considerable difficulties. In this connection, the introduction of the idea of parametric measure of the stream, which generalizes Khintchine's [1] idea of the parameter, turns out to be useful.

By a *subdivision* of the set $\Delta \in M_T$ we will mean a collection of a not more than countable number of sets $d(\Delta) = \{\Delta_\alpha\}$ such that $\Delta = \bigcup_\alpha \Delta_\alpha$, $\Delta_\alpha \in M_T$, $\Delta_\alpha \cap \Delta_\beta = \phi$, $\alpha \neq \beta$. Let $D(\Delta)$ be the class of all possible subdivisions of the set $\Delta \in M_T$

$$(2.3) \qquad \lambda(\Delta) = \sup_{d(\Delta)\in D(\Delta)} \sum_{\Delta_\alpha \in d(\Delta)} P\{\eta(\Delta_\alpha) > 0\} .$$

It may be shown that $\lambda(\Delta)$ is a measure on M_T and $\lambda(\Delta) \leq \mu(\Delta)$ where $\mu(\Delta)$ is the principal measure defined by (2.1). Thus if the stream is finite and if $\mu(\Delta)$ is absolutely continuous with respect to $\nu(\Delta)$, then

$$(2.4) \qquad \lambda(\Delta) = \int_\Delta \lambda(t)d\nu(t).$$

Definition 2.3. The measure $\lambda(\Delta)$ determined by equation (2.3) is called the *parametric measure* of the stream $\eta(\Delta)$. The function $\lambda(t)$ satisfying (2.4) is called the *function of parameters of the random stream* $\eta(\Delta)$, relative to the measure $\nu(\Delta)$. The value $\lambda(t)$ at the point t is called the *parameter* of the stream at t.

We note that in most problems a unique choice of $\lambda(t)$ or $\mu(t)$ -- determined exactly up to sets of zero ν-measure, is proveded by continuity conditions.

The existence of the parameter $\lambda(t) \equiv \lambda$ for stationary streams on the line $T = (-\infty,\infty)$ was shown by Khintchine [1] in which he considered the function of parameters relative to Lebesgue measure on $T = (-\infty,\infty)$. Korolyuk showed (see 3.8 of the main text) that under very weak restrictions, for stationary process on the line the intensity and parameter coincide. A theorem obtained below generalizes this result. However it is necessary to impose certain conditions on the principal σ-field M_T, for its formulation.

We shall say that *there is a fundamental system* $C = \{\Delta_{n,k}\}$ *of dissecting sets* in the space $[T,M_T]$ if

I. $\Delta_{n,k} \in M_T$, $T = \bigcup_k \Delta_{n,k}$.

II. $\Delta_{n,k} \cap \Delta_{n,\ell} = \phi$, $k \neq \ell$

III. $\Delta_{n,k} = \bigcup_{i \in I_{n,k}} \Delta_{n+1,i}$, $I_{n,k} \subset [0,1\ldots]$.

IV. For any $t_1 \neq t_2$ there exists $n = n\,(t_1,t_2)$ such that $t_1 \in \Delta_{n,i}$, $t_2 \in \Delta_{n,j}$, $i \neq j$.

V. The minimal σ-field containing C is M_T.

In the case of m-dimensional Euclidean space, the space $T = R^m$ and the σ-field M_T of Borel sets, a fundamental system C may be formed by means of successive dissections of R^m into a countable number of non-intersecting m-dimensional, right-angled parallelepipeds $\Delta_{n,k}$ (with corresponding inclusion and exclusion of adjacent sides), the lengths of the edges tending to zero as $n \to \infty$.

THEOREM 2.1. Let there exist a function of parameters $\lambda(t)$, relative to a measure $\nu(\Delta)$, for the random stream $\eta(\Delta)$ arising from random sets S_ω on $[T,M_T]$, and let M_T have a fundamental system C of dissecting sets.

Then the intensity $\mu(t)$ relative to $\nu(\Delta)$ exists and

$$\mu(t) = \lambda(t) \tag{2.5}$$

except on a set $\Delta_0 \in M_T$, $\nu(\Delta_0) = 0$.

PROOF: Let $\Delta \in C$. By properties II and III of the fundamental system there exists a choice of indices $I_{n+k} = I_{n+k}(\Delta)$ such that $\Delta = \bigcup_{i\in I_{n+k}} \Delta_{n+k,i}$ for any $k \geq 0$.

We define random variables on the sets $\Delta_{n,k}$

$$\tilde{\eta}_n(\Delta_{n,k}) = 1 \quad \text{if} \quad \eta(\Delta_{n,k}) > 0$$
$$= 0 \quad \text{if} \quad \eta(\Delta_{n,k}) = 0.$$

We extend the definition of $\tilde{\eta}_n$ to sets $\bigcup_{i\in I} \Delta_{n,i}$ by putting

$$\tilde{\eta}_n(\bigcup_{i\in I} \Delta_{n,i}) = \sum_{i\in I} \tilde{\eta}_n(\Delta_{n,i}). \tag{2.6}$$

For sets $\Delta = \bigcup_{i\in I_{n+k}} \Delta_{n+k,i}$ we have

$$\tilde{\eta}_n(\Delta) = \tilde{\eta}_n(\bigcup_{i\in I_n} \Delta_{n,i}) \leq \tilde{\eta}_{n+1}(\bigcup_{i\in I_{n+1}} \Delta_{n+1,i}) \leq \ldots \leq \eta(\Delta). \tag{2.7}$$

From property (iv) it follows that with probability one

$$\lim_{n\to\infty} \tilde{\eta}_n(\Delta) = \eta(\Delta). \tag{2.8}$$

By monotone convergence, using (2.7) and (2.8) we find

$$\lim_{n\to\infty} E\eta_n(\Delta) = E\eta(\Delta) = \mu(\Delta). \tag{2.9}$$

On the other hand, from the definition $\tilde{\eta}_n(\Delta)$ and (2.6) we obtain

$$E\tilde{\eta}_{n+k}(\Delta) = \sum_{i\in I_{n+k}} P\{\eta(\Delta_{n+k,i}) > 0\}.$$

Thus from (2.3) and (2.9) it follows that

$$\lambda(\Delta) = \mu(\Delta) \tag{2.10}$$

for any $\Delta \in C$. Using property (v) we see that the measures $\mu(\Delta)$ and $\lambda(\Delta)$ agree on M_T. Hence $\mu(\Delta)$ is also absolutely continuous with respect to $\nu(\Delta)$ and $\mu(t) = \nu(t)$ a.e. (ν).

From this result, we obtain a corollary:

COROLLARY 2.1. In order that there exists an intensity function $\mu(t)$ relative to a measure $\nu(\Delta)$, for the random stream $\eta(\Delta)$ arising from the random sets S_ω, on $[T, M_T]$, it is necessary and sufficient that there exists the function of parameters $\lambda(t)$ with respect to $\nu(\Delta)$. If this holds, then (2.5) holds everywhere except on a set Δ_0, $\nu(\Delta_0) = 0$.

A number of authors have studied the connection between the property of being "ordinary" and the strict positivity of the interval between successive events of stationary streams. The lemma of Dobrushin cited in 3.8 (see also Slivnyak [1] and Vasiliev [1]) will serve as an example of such a family of results. In connection with the generality of the space T, on sets of which the random stream is defined, we introduce a generalization of the ideas of an ordinary stream and of the positivity of the distance between successive events of the stream.

Definition 2.4. The random stream $\eta(\Delta)$ defined on $[T, M_T]$ is called *ordinary* with respect to the fundamental system $C = \{\Delta_{n,k}\}$ if

$$(2.11) \qquad \lim_{n \uparrow \infty} \sup_k \frac{P\{\eta(\Delta_{n,k}) > 1\}}{P\{\eta(\Delta_{n,k}) > 0\}} = 0$$

where if $P\{\eta(\Delta_{n,k}) > 0\} = 0$ we take $P\{\eta(\Delta_{n,k}) > 1\} / P\{\eta(\Delta_{n,k}) > 0\} = 0$.

Definition 2.5. The random stream $\eta(\Delta)$ defined on the space $[T, M_T]$ is called *regular* with respect to the sequence of sets $\{\Delta_k\}$, $\Delta_k \subset \Delta_{k+1} \ldots \bigcup_k \Delta_k = T$ if

$P\{\eta(\Delta_k) < \infty\} = 1$ for all k and with probability one there exist sets $\Delta_{i,\omega} \in M_T$, $\Delta_{i,\omega} \subset \Delta_k$ such that $\Delta_{i,\omega} \cap \Delta_{j,\omega} = \phi$ $(i \neq j)$, $\eta(\Delta_{i,\omega}) = 1$, $\eta(\Delta_k) = \Sigma\eta(\Delta_{i,\omega})$.

The following two theorems show that the ideas of ordinary and regular streams are closely related to each other.

THEOREM 2.2. If the random stream $\eta(\Delta)$ is ordinary, relative to the fundamental system $C = \{\Delta_{n,k}\}$ and for sets $\Delta_k = \cup_{i\in I_k} \Delta_{n_k,i}$, $\Delta_k \subset \Delta_{k+1}$, $\cup_k \Delta_k = T$, the parametric measure $\lambda(\Delta_k)$ is finite, then $\eta(\Delta)$ is regular relative to $\{\Delta_k\}$.

PROOF: The assertion of the theorem follows from

$$P\left\{\bigcup_{i\in I_{k,\ell}} [\eta(\Delta_{n_k+\ell,i}) > 1]\right\} \leq \sum_{i\in I_{k,\ell}} P\{\eta(\Delta_{n_k+\ell,i}) > 1\}$$

$$\leq \sup_i \frac{P\{\eta(\Delta_{n_k+\ell,i}) > 1\}}{P\{\eta(\Delta_{n_k+\ell,i}) > 0\}} \cdot \sum_{i\in I_{k,\ell}} P\{\eta(\Delta_{n_k+\ell,i}) > 0\}$$

$$\leq \sup_i \frac{P\{\eta(\Delta_{n_k+\ell,i}) > 1\}}{P\{\eta(\Delta_{n_k+\ell,i}) > 0\}} \circ \lambda(\Delta_k) .$$

$\to 0$ as $\ell \to \infty$ where $\Delta_k = \cup_{i\in I_{k,\ell}} \Delta_{n_k+\ell,i}$ in conformity with Property III of a fundamental system. Analogously to the conclusion of Theorem 2.1 we show that $\mu(\Delta_k) = \lambda(\Delta_k) < \infty$, completing the proof of the theorem.

For the formulation of a second theorem, generalizing the lemma of Dobrushin proved in 3.8, we introduce the following ideas. The fundamental system $C = \{\Delta_{n,k}\}$ is called *homogeneous* if $P\{\eta(\Delta_{n,k}) > 0\} = p_n(0)$, $P\{\eta(\Delta_{n,k}) > 1\} = p_n(1)$ for any n, i.e. these probabilities are independent of k.

THEOREM 2.3. Let $\eta(\Delta)$ be regular with respect to the sequence of sets $\Delta_k = \bigcup_{i\in I_k} \Delta_{n_k,i}$, $\Delta_k \subset \Delta_{k+1}$, $\bigcup_k \Delta_k = T$ formed from the sets of a homogeneous fundamental system $C = \{\Delta_{n,k}\}$ and $\mu(\Delta_k) < \infty$. Then $\eta(\Delta)$ is ordinary with respect to C.

PROOF: It follows from Property III of a fundamental system that $\Delta_k = \bigcup_{i\in I_{k,\ell}} \Delta_{n_k+\ell,i}$. Denote by $N_{k,\ell}$ the number of indices in the set $I_{k,\ell}$. The number is finite since otherwise it would follow from the fact that the system is homogeneous that $\eta(\Delta_k) = \infty$. Further,

$$(2.12) \qquad N_{k,\ell}\, P\{\eta(\Delta_{n_k+\ell,i}) > 0\}$$

$$\le N_{k,\ell}\, [P\{\eta(\Delta_{n_k+\ell,i}) > 0\} + P\{\eta(\Delta_{n_k+\ell,i}) > 1\}]$$

$$= N_{k,\ell}\, P_{n_k+\ell}(0)\left[1 + \frac{P_{n_k+\ell}(1)}{P_{n_k+\ell}(0)}\right]$$

$$\le \mu(\Delta).$$

If $\ell \to \infty$, then by regularity the left hand side tends to $\lambda(\Delta_k) = \mu(\Delta_k)$ which is possible only if

$$(2.13) \qquad \lim_{n\to\infty} \frac{p_n(1)}{p_n(0)} = 0.$$

Equality (2.13) is equivalent to (2.11) in the conditions for a fundamental system to be ordinary, and the proof is complete.

Thus the ideas of an ordinary and regular stream are close to each other. However they do not coincide, as may easily be shown by considering a Poisson stream with random parameter λ, $P(\lambda \le x) = 1-x^{-\alpha}$, $1 > \alpha > 0$, $x > 1$. This stream is regular, but not ordinary. We note that for it $\mu(\Delta) = \infty$.

For studying possible applications to random streams arising from random fields and streams of crossings of hypersurfaces, we give the general ideas of stationarity.

A stationary stream on the line may be considered as a stream for which the joint distributions of the variables $\eta(\Delta_i)$ and $\eta(\Delta_i+t)$, $i = 1,\dots,m$, coincide for any displacement t and any Borel sets Δ_i. Here $\Delta_i+t = \{s+t: s\in\Delta_i\}$. We notice that the displacements form a group of transformations of the space $T = (-\infty,\infty)$. In accordance with this we suppose that the initial space T is a group. We do not restrict ourselves to the case of commutative groups, but define left and right translations, putting, respectively,

$$g\Delta = \{gt: t\in\Delta\}, \qquad \Delta g = \{tg: t\in\Delta\}.$$

We shall suppose that the translations preserve measurability, i.e. if $\Delta\in M_T$ then $g\Delta\in M_T$ and $\Delta g\in M_T$. Briefly we write $T(M_T) = M_T$ to denote this property. A typical example is the group of parallel translations of m-dimensional Euclidean space R^m.

Definition 2.6. The random stream $\eta(\Delta)$ defined on $[T,M_T]$ where T is a group will be called *left (right)* homogeneous if $TM_T = M_T$ and for any $g\in T$ and any $\Delta_i\in M_T$ $(i = 1,2,\dots,m)$, the joint distribution of $\eta(g\Delta_i)[\eta(\Delta_i g)]$, $i = 1,\dots,m$ coincide with those of $\eta(\Delta_i)$.

In the theory of stationary streams, one of the central results is the theorem of Khintchine [1] concerning the existence of the parameter (cf 3.8). The parametric measure of a left (right) homogeneous stream is a left (right) invariant measure (Haar measure). In this connection it seems natural to obtain a general analogue of Khintchine's Theorem from the uniqueness (up to a multiplicative constant) of the invariant measure. This in its turn is connected with properties of regular invariant measures (cf Halmos [1]).

We impose a number of restrictions on the group T. We shall let T be a locally compact Hansdorff topological group, and M_T the σ-algebra generated by the compact sets. The latter condition will be briefly written $M_T = M_T(C)$. The principal assumption made is that for the (right) homogeneous stream, its left (right)-invariant measure is regular. This assumption is satisfied by a wide variety of spaces. For example, if T is separable, then each compact set is a G_δ. Consequently, the concepts of Borel and Baire measures coalesee. As is known (cf. Halmos [1]) Baire measures are regular. Thus for separable spaces T, the parametric measure $\lambda(\Delta)$ is regular. Two regular invariant measures differ only by a constant multiplier. Let $\nu(\Delta)$ be the regular invariant measure. Then

$$(2.14) \qquad \lambda(\Delta) = \lambda \cdot \nu(\Delta).$$

By combining (2.14) and (2.4) we obtain the following result.

THEOREM 2.4. Let T be a locally compact Hausdorff topological group for which $T(M_T) = M_T(C) = M_T$. Let $\eta(\Delta)$ be a left (right) homogeneous stream on $[T, M_T]$ and $\lambda(\Delta)$ the corresponding regular left (right) invariant parametric measure. Then there exists the function of parameters of the stream $\eta(\Delta)$ relative to the left (right) invariant measure $\nu(\Delta)$, exactly equal to the constant λ satisfying (2.14).

This theorem generalizes the result of Khintchine [1] mentioned above concerning the existence of a parameter for a stream on the line $T = (-\infty, \infty)$ when M_T is the class of Borel sets. Another example of a homogeneous stream is that considered by Belayev [8] and consisting of exits from the circle $T = \{(x_1, x_2): x_1^2 + x_2^2 = r^2\}$ by a two-cimensional normal process. This is homogeneous with respect to translations on the circle T and the invariant measure $\nu(\Delta)$ is the length of the arc Δ on T.

In §11.1 of the main text, special attention is paid to questions of definition of conditional probability relative to zero probability conditions such as a "crossing of a level u at a time t", and so on. The following general construction avoids the necessity of the passage to the limit. Let $\eta(\Delta)$ be a random stream on $[T,\mathcal{M}_T]$, derived from the random sets S_ω. To each $s \in S_\omega$ we adjoin an event A_s or its complement $\overline{A}_s$ depending on which of these two events occur. For example let there be a neighbourhood Δ_t of each point $t \in T$. Then as A_s one may take the event that, for $s \in S_\omega$, the set $\Delta_s \cap [S_\omega \backslash s]$ is empty. The complementary event $\overline{A}_s$ occurs if there are points of S_ω different from s in the neighbourhood Δ_s. We consider the random subset $S_\omega(A) \subset S_\omega$ consisting exactly of those points $s \in S_\omega$ for which A_s takes place. We suppose, of course, that the set $S_\omega(A)$ is a random set in the sense of Definition 2.1. It gives rise to a random stream $\eta(\Delta \circ A) \leq \eta(\Delta)$ which may be regarded as a "thinned" substream of the stream $\eta(\Delta)$. We suppose that there exists a function of parameters $\lambda(t)$ for the stream $\eta(\Delta)$, relative to a measure $\nu(\Delta)$. Since the parametric measure of the stream $\eta(\Delta,A)$ is $\lambda(\Delta,A) \leq \lambda(\Delta)$, where $\lambda(\Delta)$ is the parametric measure relative to $\nu(\Delta)$, there exists the function of parameters $\lambda(t,\Delta)$ of the stream $\eta(\Delta,A)$ relative to $\nu(\Delta)$ and moreover we may take $\lambda(t,A) \leq \lambda(t)$.

Definition 2.7. The *conditional probability* of the event A relative to the condition that an event of the stream $\eta(\Delta)$ occurs at t, is defined by the relation

$$(2.15) \qquad \tilde{P}\{A \mid t \in S_\omega\} = \lambda(t,A)/\lambda(t) \ .$$

As in Chapter 11, the conditional probabilities (2.15) have a natural statistical interpretation for homogeneous ergodic streams. It is exactly such a wide definition of conditional probability which made it possible to examine asymptotic properties of envelopes, about which more detail will be given in §3.*

The calculation of the high moments $E\{\eta(\Delta)\}^k$ and the covariance $E\{\eta(\Delta_1)\eta(\Delta_2)\}$ etc. ate conveniently carried out by means of multi-dimensional analogues of the principal and parametric measures. An account is given below of the general construction given by Belayev [7,9].

Let $\eta(\Delta)$ be a random stream on the space $[T,M_T]$, arising from a random set S_ω. We consider the direct product of k spaces T, formed by collections of points $(t_1,t_2,\ldots,t_k)$, $t_i \in T$, $i - 1,2,\ldots,k$. We denote this product by $T^k = T\times T\times\ldots\times T$. In the space T^k we introduce the σ-algebra $M_T{}^k = M_T\times M_T\times\ldots\times M_T$, generated by rectangle sets of the form $\Delta_1\times\Delta_2\times\ldots\times\Delta_k$, where $\Delta_i \in M_T$. We denote by $S_\omega{*}^k$ the set in T^k of points obtained by all possible choices of k different points from the random set S_ω. This means that all possible choices of k points are taken without repetition. Thus for any point $(s_{\alpha_1},\ldots,s_{\alpha_k}) \in S_\omega{*}^k$ we have $s_{\alpha_i} \neq s_{\alpha_j}$, $i \neq j$, $s_{\alpha_i} \in S_\omega$, $i,j = 1,\ldots,k$. The set $S_\omega{*}^k$ introduced is random, and gives rise to random stream $\eta{*}^k(\Delta^k)$, $\Delta^k \in M_T{}^k$, which we shall call the k-th stream of the stream $\eta(\Delta)$.

Using the idea of a k-th stream, it is easy to introduce the multi-dimensional analogue of the intensity parameter. Let $\nu(\Delta)$ be a measure on $[T,M_T]$ and $\nu^k(\Delta^k)$ the k-fold product measure of ν on the space $[T^k,M_T{}^k]$.

Definition 2.8. The *k-th principal measure* of the random stream $\eta(\Delta)$ arising from a random set S_ω on the space $[T,M_T]$ is defined to be the principal measure of its k-th stream $\eta{*}^k(\Delta_k)$ given on $[T^k,M_T{}^k]$ by the formula

$$(2.16) \qquad \mu_k(\Delta^k) = E[\eta{*}^k(\Delta^k)].$$

If the measure $\mu_k(\Delta^k)$ is σ-finite and absolutely continuous with respect to $\nu^k(\Delta^k)$, then by the k-th intensity function of the stream $\eta(\Delta)$ relative to

* (The third appendix.)

the measure $\nu(\Delta)$, we mean the intensity function $\mu_k(t_1,\ldots,t_k)$ of the k-th stream $\eta^{*k}(\Delta_k)$ relative to the measure $\nu^k(\Delta_k)$. The value $\mu_k(t_1,\ldots,t_k)$ is called the k-th intensity of the stream $\eta(\Delta)$ relative to the measure $\nu(\Delta)$, corresponding to the points $t_1,\ldots,t_k$. The k-th intensity function is characterized by the relation.

$$(2.17) \qquad \mu_k(\Delta^k) = \int_{\Delta^k \subset T^k} \mu_k(t_1,\ldots,t_k)\, d\nu(t_1) \ldots d\nu(t_k).$$

Definition 2.9. The *k-th parametric measure* of the random stream $\eta(\Delta)$ arising from the random set S_ω on the space $[T,\mathcal{M}_T]$ is the parametric measure $\lambda_k(\Delta^k)$ of the k-th stream $\eta^{*k}(\Delta^k)$ on $[T^k,\mathcal{M}_T{}^k]$. If $\lambda_k(\Delta^k)$ is σ-finite and absolutely continuous relative to the measure $\nu^k(\Delta^k)$, then by the *function of* k *parameters* of the stream $\eta(\Delta)$ relative to the measure $\nu(\Delta)$ we mean the function of parameters $\lambda_k(t_1,\ldots,t_k)$ of the stream $\eta^{*k}(\Delta_k)$ relative to the measure $\nu^k(\Delta^k)$. The value of $\lambda_k(t_1,\ldots,t_k)$ is called the k-parameter of the stream $\eta(\Delta)$ relative to the measure $\nu(\Delta)$ corresponding to the points $t_1\ldots t_k$. The function of k parameters is characterized by the relation

$$(2.18) \qquad \lambda_k(\Delta^k) = \int_{\Delta^k} \lambda_k(t_1,\ldots,t_k)\, d\nu(t_1) \ldots d\nu(t_k).$$

We note that from the existence of a fundamental dissection of the space $[T,\mathcal{M}_T]$, there follows the existence of a fundamental dissection of $[T^k,\mathcal{M}_T{}^k]$. Hence from Theorem 2.1, reformulated to the k stream $\eta^{*k}(\Delta^k)$, we obtain

COROLLARY 2.2. In order that the random stream $\eta(\Delta)$ on $[T,\mathcal{M}_T]$ should possess a k-th intensity function $\mu_k(t_1,\ldots,t_k)$ relative to the measure $\nu(\Delta)$, it is necessary and sufficient that there exist the function of k-parameters $\lambda_k(t_1,\ldots,t_k)$ relative to $\nu(\Delta)$. Then

(2.19) $\lambda_k(t_1,\ldots,t_k) = \mu_k(t_1,\ldots,t_k)$

up to a set $\Delta_0^{\ k} \in \mathcal{M}_T^{\ k}$, $\nu^k(\Delta_0^{\ k}) = 0$.

In a number of problems of interest in the supplement, we may calculate the high moments of the number of events of the stream $\eta(\Delta)$ falling in an interval Δ, and also the mixed moment $E\{\eta(\Delta_1)\eta(\Delta_2)\}$, $\Delta_1 \cap \Delta_2 = \phi$. The following theorem connects the moments with the k-th principal measure of the stream.

THEOREM 2.5. Let $k = k_1+\ldots+k_\ell$ where $k_i = 1,2,\ldots$ are integers and let $\mu_k(\Delta^k)$ be the k-th principal measure of the stream $\eta(\Delta)$ on $[T,\mathcal{M}_T]$. Then for the rectangle sets

$$\tilde{\Delta}^k = \underbrace{\Delta_1 \times \ldots \times \Delta_1}_{k_1 \text{ times}} \times \ldots \times \underbrace{(\Delta_\ell \times \ldots \times \Delta_\ell)}_{k_\ell \text{ times}}$$

where $\Delta_i \cap \Delta_j = \phi$, $\Delta_i \in \mathcal{M}_T$, we have the identity

$$\mu_k(\tilde{\Delta}^k) = E\left\{\prod_{i=1}^{\ell}\prod_{j=1}^{k_i}[\eta(\Delta_i) - j + 1]\right\}. \tag{2.20}$$

PROOF: We denote the coordinaties of the random points of the set S_ω by $s_{i,1}, \ldots, s_{i,\eta_i}$, $i = 1,\ldots,\ell$, $\Delta_i \in \mathcal{M}_T$, $\eta_i = \eta(\Delta_i)$, $s_{ji} \neq s_{rp}$ when $j \neq r$ or $i \neq p$. The number of points of the set $S_\omega{}^{*k}$ occurring in $\tilde{\Delta}^k$ is equal to the number of all possible ordered choices without repetition, k_i from each set $S_\omega \cap \Delta_i = \{s_{i,1},\ldots,s_{i,\eta_i}\}$, $i = 1,\ldots,\ell$. The number of different choices from the set $S_\omega \cap \Delta_i$ of size k_i without repetition is

$$\prod_{j=1}^{k_i}[\eta(\Delta_i) - j + 1].$$

Since the sets Δ_i are mutually disjoint, the points of the set $S_\omega{}^{*k} \cap \tilde{\Delta}^k$ are obtained from any combination of choices $(s_{i,i_1},\ldots,s_{i,i_{k_\ell}})$ $i = 1,2,\ldots,\ell$

from the sets $S_\omega \cap \Delta_i$,

$$(s_{1,i}, \ldots, s_{1,i_{k_1}}, \ldots, s_{\ell,\ell_1}, \ldots, s_{\ell,\ell_{k_\ell}}) \in \tilde{\Delta}^k \cap S_\omega{*}^k.$$

The total number of points of the set $\tilde{\Delta}^k \cap S_\omega{*}^k$ is

$$\prod_{i=1}^{\ell} \prod_{j=1}^{k_i} (\eta(\Delta_i) - j + 1).$$

The identity (2.20) follows from (2.16), i.e. the fact that the k-th principal measure $\mu_k(\tilde{\Delta}^k)$ is the expectation of the number of points of $S_\omega{*}^k$ occurring in Δ^k.

We formulate two important cases of Theorem 2.5 as a corollary.

COROLLARY 2.3. If in the conditions of the previous theorem, $k_i = 0$, $i \neq 1$, $k = k_1$, $\Delta = \Delta_1$, then

$$(2.21) \qquad \mu_k \underbrace{(\Delta\times\Delta\times\ldots\times\Delta)}_{k\text{-times}} = E\eta(\Delta)[\eta(\Delta)-1] \ldots [\eta(\Delta) - k + 1].$$

If in the conditions of the previous theorem $k_i = 1$, $i = 1,\ldots,\ell$, $k = \ell$ then

$$(2.22) \qquad \mu_k(\Delta_1\times\ldots\times\Delta_\ell) = E\eta(\Delta_1) \ldots \eta(\Delta_\ell).$$

As on the one hand functions of k parameters are generally easier to determine than k-th intensity functions, and on the other hand these functions coincide, we give the following result from (2.19), (2.21), (2.22).

COROLLARY 2.4. Let the function of k parameters $\lambda_k(t_1,\ldots,t_k)$ relative to the measure $\nu(\Delta)$, exist for the random stream $\eta(\Delta)$ on $[T,M_T]$. Then the k-th order factorial moment of $\eta(\Delta)$ and the mixed moment for $\eta(\Delta_1) \ldots \eta(\Delta_\ell)$

are given in terms of the function of k-parameters by the relations

(2.23) $$J_k(\Delta) = E\eta(\Delta)[\eta(\Delta) - 1] \dots [\eta(\Delta) - k + 1]$$

$$= \int_{t_i\in\Delta, t_i\neq t_j} \lambda_k(t_1,\dots,t_k)d\nu(t_1) \dots d\nu(t_k)$$

and

(2.24) $$C_\ell(\Delta_1,\dots,\Delta_\ell) = E\eta(\Delta_1) \dots \eta(\Delta_\ell)$$

$$= \int_{\substack{t_i\in\Delta_i,\ \Delta_i\cap\Delta_j=\phi \\ i\neq j,\ i=1,\dots,\ell}} \lambda_\ell(t_1,\dots,t_\ell)d\nu(t_1)\dots d\nu(t_\ell)$$

We illustrate the above results by examples.

For the first example, consider a stream of renewals (cf. Cox and Smith [1]), $\eta(\Delta)$ defined on $T = (0,\infty)$ by random sets $S_\omega = \{t_i\}$, $0 = t_0 < t_i < t_{i+1}$, $\tau_i = t_i - t_{i-1}$, $i = 1,2,\dots,$ where τ_i are mutually independent random variables, $P(\tau_i \le x) = F(x)$, $F(0+) = 0$. Let $H(t) = E\eta(0,t)$ when the k-th principal measure for an element of volume in the neighbourhood of the point $(t_1,\dots,t_k) \in R^k$, where $t_1 < t_2 < \dots < t_k$, is

(2.25) $$\mu(dt_1,\dots,dt_k) = dH(t_1)dH(t_2-t_1) \dots dH(t_k-t_{k-1}).$$

If there exists the probability density $f(t) = F'(t)$, then there exists $h(t) = H'(t)$. In this case the k-th intensity function relative to Lebesgue measure has the form

$$\mu_k(t_1, \dots, t_k) = h(t_1)\, h(t_2-t_1) \dots h(t_k-t_{k-1}).$$

Corresponding to (2.23) and (2.25) we obtain the k-th factorial moment of the number of renewals on $(0,t)$ as

$$E\eta(0,t)[\eta(0,t) - 1] \ldots [\eta(0,t) - k + 1] = k!\, H_k{*}(t)$$

where

$$H_k{*}(t) = \int_0^t H_{k-1}{*}(t-s)dH(s), \quad H_1{*}(t) = H(t).$$

The second example concerns the intersection of two levels by the sample functions of a process $\xi(t)$. We suppose that on $T = R'_1 \cup R'_2$, $R'_i = (-\infty,\infty)$ is given a random set $S_\omega = \{t_{1,i}\} \cup \{t_{2,i}\}$ where $t_{i,j} \in R'_i$, $i = 1,2$. R'_i may be interpreted as two parallel horizontal lines passing through the points $(0,u_i)$. Denote the 1-intensity functions by $\lambda^i(t_i)$ $t_i \in R'_i$ and the 2-intensity functions by $\lambda_2{}^{ij}(t_i,t_j)$, $t_i \in R'_i$, $t_j \in R'_j$. We note that these intensities coincide with the functions of parameters. In calculating factorial moments of the second order, we take into account that $T \times T$ is formed from four planes $R'_i \times R'_j$. Hence from (2.3), the second factorial moment of the number of points $0 < t_{i,j} < t$, $i = 1,2$ is

$$J_{(2)}(0,t) = \sum_{i,j=1,2} \int_0^t \int \lambda_2{}^{ij}(t_1,t_2)\, dt_1\, dt_2.$$

It is useful to keep in mind that the factorial moments may be used for an approximation to the probability of occurrence of at least one event of the stream $\eta = \eta(\Delta)$. From calculation of the first three factorial moments we have

$$\max[0, E\eta - \tfrac{1}{2} E\eta(\eta-1)] \le P\{\eta > 0\}$$

$$\le \min[1, E\eta, E\eta - \tfrac{1}{2} E\eta(\eta-1) + \tfrac{1}{6} E\eta(\eta-1)(\eta-2)].$$

Analogous inequalities may be written down by calculating factorial moments of higher order.

We consider a particular question about the calculation of the k-th intensity for the "summed" stream

$$\eta_\Sigma(\Delta) = \sum_{i=1}^{m} \eta_i(\Delta)$$

when the single streams $\eta_i(\Delta)$ are mutually independent and have k-th intensities $\mu_{k,i}(t_1,\ldots,t_k)$ relative to the measure $\nu(\Delta)$. The one-intensity and two-intensity of $\eta_\Sigma(\Delta)$ exist and are given by

$$\mu_{1,\Sigma}(t) = \sum_{i=1}^{m} \mu_{1,i}(t)$$

$$\mu_{2,\Sigma}(t_1,t_2) = \sum_{i=1}^{m} \mu_{2,i}(t_1,t_2) + \sum_{i\neq j} \mu_{1,i}(t_1)\mu_{1,j}(t_2) .$$

Finally in the case of the k-intensity we have

$$\mu_{k,\Sigma}(t_1,\ldots,t_k) = \sum_{\substack{k_1+\ldots+k_m=k \\ k_i \geq 0}} \mu_{k_1,1}(t_{1,1},\ldots,t_{1,k_1}) \ldots \mu_{k_m,m}(t_{m,1},\ldots,t_{m,k_m})$$

where the summations are over all possible partitions of the set $(t_1,\ldots t_k)$ into disjoint subsets $(t_{1,1},\ldots,t_{1,k_1}) \ldots (t_{m,1},\ldots,t_{m,k_m})$. If the subset corresponding to the i-th stream turns out to be empty for such a partition, $(k_i=0)$, then $\mu_{0,i} = 1$.

References

The following references which are quoted in this appendix are extracted with the same numbers from the complete reference list for the (translated) book.

AMBARTSUMYAN, R.V. [1] "On a relation for stationary point processes", *DAN Arm. SSR* 42 1966, 141-147.

BELAYEV, Yu. K. [7] "On bursts and shines of random fields", *DAN SSSR* 176 3(1967), 495-497.

----- [9] "On properties of random streams" *Teor. Ver. i Prim.* 13 3(1968), 578-580.

BEUTLER, F.J. and LENEMAN, O.A.Z. [1] "The theory of stationary point processes" *Acta Math.,* 116 3-4(1966), 159-197.

COX, D.R. and SMITH, W.L. [1] *Renewal Theory,* Sov. Radio, M., 1967.

FIEGER, W. [1] "A generalization of Palm's formulae for a general process of calls" *Math. Scand.* 16 2(1965), 121-147.

HALMOS, P.R. [1] *Measure Theory,* Van Nostrand, 1950.

KERSTAN, J. and MATTHES, K. [1] "Stationary random point series II", *Jahr. D.M.V.* 66 (1964), 106-118.

KHINTCHINE, A. Ya. [1] *Mathematical Methods in the Theory of Queueing,* Griffin, London, 1960.

LEADBETTER, M.R. [4] "On three basic results in the theory of stationary point processes", *Proc. Amer. Math. Soc.,* 19 1(1968), 115-117.

McFADDEN, J.A. [2] "On the lengths of intervals in a stationary point process", *J. Roy. Statist. Soc (B),* 24, 1962, 364-382.

MATTHES, K. [1] "Stationary random point series I", *Jahr. D.M.V.* 66 (1963) 66-79.

MECKE, J. [1] "Stationary random measures on locally compact Abelian groups", *Z. Wahr.* 9, 1(1967) 36-58.

SLIVNYAK, I.M. "Some properties of ordinary stationary streams of events", *Teor. Ver. i Prim.,* 7 (1962) 347-352.

VASILIEV, P.I. [1] "On ordinary stationary streams",
Uchen. Zapic. Kishinev. Gos. Univ., 82 (1965) 44-48.

ZITEK, F. "On the theory of ordinary streams",
Czech. Mat. Journ. 8, (1958) 448-459.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Department of Statistics University of North Carolina Chapel Hill, North Carolina	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Elements of the general theory of random stream of events

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

Malcolm R. Leadbetter

6. REPORT DATE August, 1970	7a. TOTAL NO. OF PAGES 21	7b. NO. OF REFS 16

8a. CONTRACT OR GRANT NO. N0014-67-A-0321-0002 b. PROJECT NO. NR042-214/1-6-69(436) c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Institute of Statistics Mimeo Series No. 703 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch Office of Naval Research Washington, D.C. 20360

13. ABSTRACT

This report is a translation of the second of three appendices by Yu. K. Belayev to the Russian edition of the book *Stationary and related stochastic processes* by Harald Cramér and M.R. Leadbetter. The contents concern the general theory of point processes with special emphasis on methods useful in dealing with "crossing problems" for stochastic processes.

DD FORM 1 NOV 65 1473

Security Classification

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Stochastic processes						
Point processes						
Random streams						
Zero crossings						

UNCLASSIFIED

Security Classification