* *This research was partially supported by the National Science Foundation under Grant GU-2059.*

1 *Department of Mathematics and Department of Statistics.*

2 *Department of Statistics.*

GAUSSIAN STOCHASTIC PROCESSES AND GAUSSIAN MEASURES*

by

Balram S. Rajput[1] Stamatis Cambanis[2]

Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 705

August, 1970

GAUSSIAN STOCHASTIC PROCESSES AND GAUSSIAN MEASURES*

Balram S. Rajput
Department of Mathematics and Department of Statistics
University of North Carolina
Chapel Hill, North Carolina 27514

and

Stamatis Cambanis
Department of Statistics
University of North Carolina
Chapel Hill, North Carolina 27514

*This research was partially supported by the National Science Foundation under Grant GU-2059.

1. INTRODUCTION

Gaussian stochastic processes are used in connection with problems such as estimation, detection, mutual information, etc. These problems are often effectively formulated in terms of Gaussian measures on appropriate Banach or Hilbert spaces of functions. Even though both concepts, the Gaussian stochastic process and the Gaussian measure, have been extensively studied, it seems that the connection between them has not been adequately explained. Two important questions arising in this context are the following:

(Q_1) Given a Gaussian stochastic process with sample paths in a Banach function space, is there a Gaussian measure on the Banach space which is induced by the given stochastic process?

(Q_2) Given a Gaussian measure on a Banach function space, is there a Gaussian stochastic process with sample paths in the Banach space which induces the given measure?

The purpose of this paper is to explore questions Q_1 and Q_2 in the commonly encountered spaces $C[0,1]$ and $L_2[0,1]$. Section 2 provides affirmative answers to both questions Q_1 and Q_2 for the space $C[0,1]$, with no restrictive assumptions. The space $L_2[0,1]$ is considered in Section 3. Under some mild conditions, a stochastic process induces in a natural way a probability measure on $L_2[0,1]$. If in addition the process is Gaussian, a necessary and sufficient condition for the induced measure to be Gaussian is derived in Theorem 3.1 and Corollary 3.1. The question Q_2 is answered in the affirmative in Theorem 3.3. Section 4 includes some generalizations and extensions of the results of Sections 2 and 3. The most important result is Theorem 4.1 which provides an affirmative answer to question Q_1 for $L_2(T)$, T any Borel measurable subset of the real line, under conditions considerably weaker than those of Theorem 3.1.

It should be mentioned that, throughout the literature, whenever the need arises to have a Gaussian measure induced on an appropriate function space by a Gaussian process, then, either (i) it is assumed that the Gaussian process is mean square continuous and that the index set is a compact interval [5, 7, 9], and thus unnecessary assumptions are made on the process, or (ii) the term "Gaussian process" is used to mean a generalized Gaussian process, i.e., a Gaussian measure, or a measurable map which induces a Gaussian measure [1, 2] and thus the problem of inducing the Gaussian measure from a Gaussian process is not considered.

Throughout this paper real Banach spaces, and real stochastic processes are considered. The basic notation, definitions and properties that are consistently used in subsequent sections are given in the following.

Let $\mathcal{X}$ be a real, separable Banach space of functions on $[0,1]$ and denote by $\mathcal{B}(\mathcal{X})$ the smallest σ-algebra of subsets of $\mathcal{X}$ which contains all the open subsets of $\mathcal{X}$. Let $X(t,\omega)$, $t \in [0,1]$, be a real stochastic process defined on the probability space (Ω,F,P) and such that $X(\cdot,\omega) \in \mathcal{X}$ almost surely (a.s.). If the map $T: (\Omega,F) \to (\mathcal{X},\mathcal{B}(\mathcal{X}))$, defined by

$$T\omega = X(\cdot,\omega) \tag{1.1}$$

is measurable, then the probability measure μ_X induced by $X(t,\omega)$, $t \in [0,1]$, on $(\mathcal{X},\mathcal{B}(\mathcal{X}))$ is defined by

$$\mu_X(B) = P\{T^{-1}(B)\} = P\{\omega \in \Omega: X(\cdot,\omega) \in B\} \tag{1.2}$$

for all $B \in \mathcal{B}(\mathcal{X})$.

A stochastic process $(\Omega,F,P; X(t,\omega), t \in [0,1])$ is said to be Gaussian if for every finite n and $t_1,\dots,t_n \in [0,1]$, the random variables $X(t_1,\omega),\dots,X(t_n,\omega)$ are jointly Gaussian.

Some definitions and properties related to Gaussian measures are summarized here. Let $\mathcal{X}$ be any separable Banach space and $\mathcal{B}(\mathcal{X})$ the

σ-algebra generated by the open subsets of X. It is well known that $\mathcal{B}(X)$ is also the smallest σ-algebra of subsets of X with respect to which all continuous functionals on X are measurable. A probability measure μ on $(X,\mathcal{B}(X))$ is said to be Gaussian if every continuous linear functional F on X is a Gaussian random variable. It is easily seen that μ is Gaussian if and only if any finite number of continuous linear functionals $\{F_1,\ldots,F_n\}$ on X are jointly Gaussian random variables. Let now $X = H$, a separable Hilbert space with inner product and norm denoted respectively by $\langle\cdot,\cdot\rangle$ and $||\cdot||$. If μ is a Gaussian measure on $(H,\mathcal{B}(H))$, its mean and its covariance operator are defined [8] as the element $u_o \in H$ and the bounded, linear, nonnegative, self-adjoint and trace class operator S on H which satisfy

$$E[\langle u,v\rangle] = \langle u_o,v\rangle \text{ , for all } v \in H \tag{1.3}$$

$$E[\langle u - u_o,v\rangle\langle u - u_o,w\rangle] = \langle Sv,w\rangle \text{ , for all } v,w \in H \tag{1.4}$$

The support of a Gaussian measure μ on $(H,\mathcal{B}(H))$ is the set $u_o + \overline{sp}\{\phi_n\}$, where $\overline{sp}\{\phi_n\}$ is the closure in H of the linear manifold $sp\{\phi_n\}$ generated by the eigenfunctions $\{\phi_n\}$ of the covariance operator S which correspond to its nonzero eigenvalues $\{\lambda_n\}$ [4]. Also, if μ is a Gaussian measure on $(H,\mathcal{B}(H))$ then [8]

$$E[||u||^2] = \int_H ||u||^2 d\mu(u) = \sum_n \lambda_n < +\infty \tag{1.5}$$

Note that if a (not necessarily Gaussian) probability measure μ on $(H,\mathcal{B}(H))$ satisfies (1.5) then its mean u_o and its covariance operator S are well defined by (1.3) and (1.4) respectively.

2. GAUSSIAN STOCHASTIC PROCESSES AND GAUSSIAN MEASURES ON C[0,1]

In this section it is shown that both questions raised in the introduction have an affirmative answer for the space $X = C[0,1]$ of all real valued continuous functions on $[0,1]$ with the suprimum norm.

First, question Q_1 is considered. Let $(\Omega,F,P: X(t,\omega),\ t \in [0,1])$ be a Gaussian stochastic process with a.s. continuous sample paths. It is shown in Proposition 2.1(i) that the map T defined by (1.1) is measurable, and in Theorem 2.1 that the measure μ_X induced by $X(t,\omega)$, $t \in [0,1]$, on $(X = C[0,1],\ B(X))$ is Gaussian.

Proposition 2.1. Let (Ω,F,P) be a probability space.

(i) If $(\Omega,F,P;\ X(t,\omega),\ t \in [0,1])$ is a stochastic process with a.s. continuous sample paths, then the map T defined by (1.1) is measurable.

(ii) Conversely, if a map $K: (\Omega,F) \to (X = C[0,1],\ B(X))$ is measurable, then $X(t,\omega) = (K\omega)(t)$, $t \in [0,1]$, is a stochastic process defined on the probability space (Ω,F,P).

Proof. (i) It suffices to show that the inverse image under T of a closed sphere in X with center $v \in X$ and radius $\varepsilon > 0$ belongs to F. Let $B = \{u \in X: ||u - v|| \le \varepsilon\}$. Then

$$T^{-1}(B) = \{\omega \in \Omega: \sup_{t\in[0,1]} |X(t,\omega) - v(t)| \le \varepsilon\}$$

$$= \bigcap_r \{\omega \in \Omega: v(r) - \varepsilon \le X(t,\omega) \le v(r) + \varepsilon\}$$

where the intersection is taken over all rationals in $[0,1]$. Since $X(t,\omega)$ is a stochastic process, $\{\omega \in \Omega: a \le X(t,\omega) \le b\} \in F$ for every $t \in [0,1]$ and $a \le b$. Hence $T^{-1}(B) \in F$.

(ii) For every fixed $t \in [0,1]$ we have $\{\omega \in \Omega: X(t,\omega) \le a\} = K^{-1}\{u \in X: u(t) \le a\}$. Since $F_t(u) = u(t)$, $u \in X$, is a continuous linear functional on X, $\{u \in X: u(t) \le a\} \in B(X)$ (see Section 1), and by the measurability of K, $\{\omega \in \Omega: X(t,\omega) \le a\} \in F$. ||

Theorem 2.1. Let $(\Omega,F,P;\ X(t,\omega),\ t \in [0,1])$ be a Gaussian stochastic process with a.s. continuous sample paths. Then the probability measure

μ_X induced by $X(t,\omega)$ on $(X = C[0,1], \mathcal{B}(X))$ is Gaussian.

Proof. In order to show that μ_X is Gaussian it suffices to show that every continuous linear functional on X is a Gaussian random variable on the probability space $(X,\mathcal{B}(X),\mu_X)$. Let F be a continuous linear functional on X. Then there exists a real valued function ϕ of bounded variation on $[0,1]$ such that for all $u \in X$,

$$F(u) = \int_0^1 u(t)d\phi(t)$$

where the integral is Riemann-Stieltjes. Then by (1.2)

$$\mu_X\{u \in X: F(u) \leq a\} = P\{\omega \in \Omega: \int_0^1 X(t,\omega)d\phi(t) \leq a\}.$$

Hence, in order to show that $F(u)$ is a Gaussian random variable on the probability space $(X,\mathcal{B}(X),\mu_X)$, it suffices to show that

$$\xi(\omega) = \int_0^1 X(t,\omega)d\phi(t) \qquad \text{a.s.}$$

is a Gaussian random variable on (Ω,F,P). It is known that for continuous functions the Riemann-Stieltjes integral can be approximated as follows: Consider the partitions Δ_n, $n = 1,2,\ldots$ of $[0,1]$ defined by the points $\{t_{k,n} = \frac{k}{n}, k = 0,1,\ldots,n-1,n\}$, and define

$$\xi_n(\omega) = \sum_{k=1}^{n} X(t_k,\omega)[\phi(t_k) - \phi(t_{k-1})]$$

Then $\lim_{n\to+\infty} \xi_n(\omega) = \xi(\omega)$ a.s. and the sequence of random variables $\{\xi_n\}$ is a Gaussian family. Since $\xi(\omega)$ is finite a.s., it follows by Lemma 5.1 that it is Gaussian. ||

We now turn to question Q_2 raised in the introduction and give an affirmative answer in the following theorem:

Theorem 2.2. Given any Gaussian measure μ on $(X = C[0,1],\mathcal{B}(X))$, there exists a Gaussian stochastic process $(\Omega,F,P; X(t,\omega), t \in [0,1])$ with a.s. continuous sample paths which induces μ on $(X,\mathcal{B}(X))$.

Proof. Define the probability space $(\Omega,\mathcal{F},P)$ by $\Omega = X$, $\mathcal{F} = \mathcal{B}(X)$, $P = \mu$ and denote the identity map between Ω and X by I: $I\omega = u$, where $u = \omega$. Clearly $I: (\Omega,\mathcal{F}) \to (X,\mathcal{B}(X))$ is a measurable map. Define $X(t,\omega)$ by

$$X(t,\omega) = (I\omega)(t) = u(t), \quad t \in [0,1], \quad \omega \in \Omega.$$

Then, by Proposition 2.1(ii), $X(t,\omega)$, $t \in [0,1]$, is a stochastic process on $(\Omega,\mathcal{F},P)$. It is clear by (1.1), (1.2) and the definition of $X(t,\omega)$, $t \in [0,1]$, that $\mu_X = \mu$, i.e., $X(t,\omega)$ induces μ on $(X,\mathcal{B}(X))$. In order to show that $X(t,\omega)$, $t \in [0,1]$, is a Gaussian process it suffices to show that for every n and $t_1,\ldots,t_n \in [0,1]$, the random variables $X(t_k,\omega)$, $k = 1,\ldots,n$, defined on $(\Omega,\mathcal{F},P)$ are jointly Gaussian; or equivalently that the random variables $F_{t_k}(u) = (I\omega)(t_k) = u(t_k)$, $k = 1,\ldots,n$, defined on $(X,\mathcal{B}(X),\mu)$ are jointly Gaussian. But this follows from the fact that each F_{t_k} is a continuous linear functional on X and μ is a Gaussian measure on $(X,\mathcal{B}(X))$. ||

3. GAUSSIAN STOCHASTIC PROCESSES AND GAUSSIAN MEASURES ON $L_2[0,1]$

In this section we consider questions Q_1 and Q_2 for the Hilbert space $L_2[0,1]$ of real valued, square integrable functions with respect to the Lebesgue measure.

First, question Q_1 is considered. Let $(\Omega,\mathcal{F},P;\ X(t,\omega),\ t\in[0,1])$ be a Gaussian stochastic process. The problem is under what conditions does $X(t,\omega)$, $t\in[0,1]$, induce a Gaussian measure on $(X = L_2[0,1],\mathcal{B}(X))$. Obvious minimal conditions are that $X(t,\omega)$ be product $(\mathcal{B}[0,1] \times \mathcal{F})$ measurable, where $\mathcal{B}[0,1]$ is the σ-algebra of Borel measurable subsets of $[0,1]$, and that almost all sample paths of $X(t,\omega)$ belong to $L_2[0,1]$. Denote by (I_1) and (I_2) the following sets of conditions:

$$(I_1)\begin{cases}\text{(i) } X(t,\omega): ([0,1]\times F, \mathcal{B}[0,1]\times F) \to (R, \mathcal{B}(R)) \text{ is measurable, where } R \text{ is the real line and } \mathcal{B}(R) \text{ is the class of Borel subsets of } R. \\ \text{(ii) } X(t,\omega) \text{ is Gaussian with mean } m(t), \text{ autocorrelation } r(t,s) \text{ and covariance } R(t,s).\end{cases}$$

$$(I_2)\begin{cases}\text{(i) and (ii) of } (I_1) \text{ and} \\ \text{(iii) } \int_0^1 X^2(t,\omega)dt < +\infty \quad \text{a.s.}\end{cases}$$

An application of Fubini's theorem proves the following proposition:

<u>Proposition 3.1</u>. If the stochastic process $(\Omega,F,P;\ X(t,\omega),\ t \in [0,1])$ satisfies (i) and (iii) of (I_2), then the map T defined by (1.1) is measurable.

Hence, if (i) and (iii) of (I_2) are satisfied $X(t,\omega)$, $t \in [0,1]$, induces by (1.2) a probability measure μ_X on $(X = L_2[0,1], \mathcal{B}(X))$. Now the question under consideration is: If, in addition, $X(t,\omega)$, $t \in [0,1]$, is Gaussian, under what conditions is μ_X Gaussian. A sufficient condition is given in Theorem 3.1 and is shown to be also necessary in Corollary 3.1. For the proof of Theorem 3.1 the following lemma is needed.

<u>Lemma 3.1.</u> Let $(\Omega,F,P;\ X(t,\omega),\ t \in [0,1])$ be a stochastic process satisfying (I_1) and let $f(t)$ be a real valued, Borel measurable function on $[0,1]$. If

$$E\left(\int_0^1 |X(t,\omega)f(t)|dt\right)^2 < +\infty \tag{3.1}$$

then the random variable $\xi(\omega)$ defined by

$$\xi(\omega) = \int_0^1 X(t,\omega)f(t)dt \quad \text{a.s.} \tag{3.2}$$

is Gaussian with mean $\int_0^1 m(t)f(t)dt$ and covariance $\int_0^1\int_0^1 R(t,s)f(t)f(s)dtds$.

Remark 3.1. If in Lemma 3.1 the condition (3.1) is replaced by

$$\text{(II)} \qquad E\left(\int_0^1 X^2(t,\omega)dt\right) = \int_0^1 r(t,t)dt < +\infty$$

or by

$$\text{(III)} \qquad E\left(\int_0^1 |X(t,\omega)|dt\right)^2 < +\infty,$$

then the conclusions of the lemma are true for every $f \in L_2[0,1]$ and for every real, Borel measurable, a.e. uniformly bounded function f on $[0,1]$ respectively. Note that, as follows by Schwartz's inequality, (II) implies (III).

Proof. In view of (I_1) and (3.1) the random variable $\xi(\omega)$ is well defined by (3.2) as an a.s. sample path integral.

Let $H(X)$ be the closure in $L_2(\Omega,F,P)$ of the linear manifold generated by the random variables $X(t,\omega)$, $t \in [0,1]$. Then $H(X) \subseteq L_2(\Omega,F,P)$. Clearly every family of random variables in $H(X)$ is jointly Gaussian. The integral

$$\eta(\omega) = \int_0^1 X(t,\omega)f(t)dt \tag{3.3}$$

exists in the quadratic mean sense in $H(X)$ if and only if

$$\sigma^2 = \int_0^1\int_0^1 r(t,s)f(t)f(s)dtds < +\infty \tag{3.4}$$

[6, p. 33]. If the integral exists, $\eta(\omega)$ is Gaussian with $E(\eta^2) = \sigma^2$. It will be shown that the hypotheses of the Lemma imply (3.4). Note that $r(t,s) = R(t,s) + m(t)m(s)$ implies

$$\sigma^2 \leq \int_0^1\int_0^1 |R(t,s)f(t)f(s)|dtds + \left(\int_0^1 |m(t)f(t)|dt\right)^2 \tag{3.5}$$

It follows by Tonelli's theorem and by (3.1) that

$$\int_0^1\int_0^1 |R(t,s)f(t)f(s)|dtds \leq \int_0^1\int_0^1 E|X(t,\omega)X(s,\omega)||f(t)f(s)|dtds$$
$$= E\left(\int_0^1 |X(t,\omega)f(t)|dt\right)^2 < +\infty \tag{3.6}$$

Also, by Tonelli's theorem and (3.1), we obtain

$$\int_0^1 |m(t)f(t)|dt \leq E(\int_0^1 |X(t,\omega)f(t)|dt) < +\infty \tag{3.7}$$

Equations (3.5), (3.6) and (3.7) imply (3.4).

It is next shown that $\xi(\omega) = \eta(\omega)$ a.s. Tonelli's theorem and (3.1) imply that

$$\int_\Omega \int_0^1 \int_0^1 |X(t,\omega)X(s,\omega)f(t)f(s)|dsdtdP(\omega) < +\infty \tag{3.8}$$

Hence, by Fubini's theorem and (3.4), it follows that

$$E(\xi^2) = \sigma^2 \tag{3.9}$$

By applying a property of the quadratic mean integral (3.3) [6, p. 30] and (3.8) we obtain

$$\begin{aligned} E(\xi\eta) &= \int_0^1 E[\xi(\omega)X(t,\omega)f(t)]dt \\ &= \int_0^1 \{\int_\Omega (\int_0^1 X(s,\omega)f(s)X(t,\omega)f(t)ds)dP(\omega)\}dt = \sigma^2 \end{aligned} \tag{3.10}$$

By (3.9), (3.10) and the fact that $E(\eta^2) = \sigma^2$, it follows that

$$E[|\xi-\eta|^2] = E(\xi^2) - 2E(\xi\eta) + E(\eta^2) = 0$$

Hence $\xi = \eta$ in $L_2(\Omega,F,P)$; i.e., $\xi(\omega) = \eta(\omega)$ a.s. and since η is Gaussian, so is ξ.

Since by (3.1), $\int_0^1 |X(t,\omega)f(t)|dt \in L_1(\Omega,F,P)$, we have $X(t,\omega)f(t) \in L_1([0,1] \times \Omega, \mathcal{B}[0,1] \times F, dt \times P)$ and therefore

$$E(\xi) = \int_0^1 m(t)f(t)dt$$

Since $E(\xi^2) = E(\eta^2) = \sigma^2$, it follows by (3.1) that

$$\mathrm{Var}(\xi) = E(\xi^2) - E^2(\xi) = \int_0^1 \int_0^1 R(t,s)f(t)f(s)dtds \qquad ||$$

<u>Theorem 3.1</u>. If the stochastic process $(\Omega,F,P;X(t,\omega),t \in [0,1])$ satisfies (I_1) and (II), then the probability measure μ_X induced on

$(X = L_2[0,1], \mathcal{B}(X))$ by $X(t,\omega)$ is Gaussian with mean $m \in X$ and covariance operator generated by the kernel $R(t,s)$.

Proof. Note that (II) implies (iii) of (I_2) and by Proposition 3.1 μ_X is well defined by (1.2). In order to show that μ_X is Gaussian it suffices to show that every continuous linear functional on X is a Gaussian random variable on the probability space $(X, \mathcal{B}(X), \mu_X)$. Let F be a continuous linear functional on X. Then there exists a $f \in X$ such that

$$F(u) = \langle u,f\rangle = \int_0^1 u(t)f(t)dt$$

for all $u \in X$. It follows by (1.2) that

$$\mu_X\{u \in X: F(u) \leq a\} = P\{\omega \in \Omega: \int_0^1 X(t,\omega)f(t)dt \leq a\}.$$

Hence the random variables $F(u)$ on $(X, \mathcal{B}(X), \mu_X)$ and $\xi_f(\omega) = \int_0^1 X(t,\omega)f(t)dt$ on $(\Omega, \mathcal{F}, P)$ are identically distributed. Since $\xi(\omega)$ is Gaussian by Remark 3.1, so is $F(u)$.

If $u_o \in X$ is the mean of μ_X, it follows from (1.3) and Remark 3.1 that

$$\int_0^1 u_o(t)f(t)dt = E[\langle u,f\rangle] = E[\xi_f(\omega)] = \int_0^1 m(t)f(t)dt$$

for all $f \in X$. Hence $u_o(t) = m(t)$ a.e. [Leb.] on $[0,1]$; i.e., $m = u_o$ in X.

If S is the covariance operator of μ_X and $\xi_g(\omega) = \int_0^1 X(t,\omega)g(t)dt$, $g \in X$, it follows by (1.4) and the fact that the random variables $\langle u,f\rangle$, $\langle u,g\rangle$ and $\xi_f(\omega)$, $\xi_g(\omega)$ are identically distributed that

$$\begin{aligned}\langle Sf,g\rangle &= E[\langle u - u_o,f\rangle\langle u - u_o,g\rangle] \\ &= E[\xi_f(\omega)\xi_g(\omega)] - \langle u_o,f\rangle\langle u_o,g\rangle\end{aligned}$$

for all $f,g \in X$. As in (3.8) we obtain

$$\int_\Omega \int_0^1 \int_0^1 |X(t,\omega)X(s,\omega)f(t)g(s)|dsdtdP(\omega) < +\infty$$ and by Fubini's Theorem

we have

$$\langle Sf,g\rangle = \int_0^1 \int_0^1 r(t,s)f(t)g(s)dtds - \langle m,f\rangle\langle m,g\rangle$$
$$= \int_0^1 \int_0^1 R(t,s)f(t)g(s)dtds = \langle Rf,g\rangle$$

for all $f,g \in X$, where R is the trace class (because of (II)) operator on X with kernel $R(t,s)$. Hence $S = R$. ||

It should be noted that condition (II) is satisfied if the stochastic process $X(t,\omega)$, $t \in [0,1]$, has either one of the following properties: mean square continuity, wide sense stationarity, uniformly bounded autocorrelation function.

Theorem 3.1 shows that condition (II) is sufficient for the induced measure μ_X to be Gaussian. It is shown in the following corollary that it is also necessary.

Corollary 3.1. If the stochastic process $(\Omega,F,P;\ X(t,\omega),\ t \in [0,1])$ satisfies (I_2), then the probability measure μ_X induced on $(X = L_2[0,1], B(X))$ by $X(t,\omega)$ is Gaussian if and only if (II) is satisfied.

Proof. The "if" part is shown in Theorem 3.1. The "only if" part follows from (1.5) and the fact that the random variables $||u||^2$ on $(X,B(X),\mu_X)$ and $||X(\cdot,\omega)||^2$ on (Ω,F,P) are identically distributed. ||

One may wonder whether μ_X can be shown to be a Gaussian measure if the stochastic process $X(t,\omega)$, $t \in [0,1]$, satisfies conditions (I_2) and (III), which are in general weaker than (I_1) and (II). The answer, in the affirmative, is provided by the following theorem.

Theorem 3.2. If the stochastic process $(\Omega,F,P;\ X(t,\omega),\ t \in [0,1])$ satisfies (I_2) and (III), then the probability measure μ_X induced on

$(\mathcal{X} = L_2[0,1], \mathcal{B}(\mathcal{X}))$ by $X(t,\omega)$ is Gaussian with mean $m(t)$ and covariance operator S with kernel $R(t,s)$ and (II) is satisfied.

<u>Proof</u>. Let $\{\phi_n(t)\}$ be a complete orthonormal set in $L_2[0,1]$ such that each $\phi_n(t)$ is continuous on $[0,1]$ for all n. Then $X(\cdot,\omega) \in L_2[0,1]$ implies

$$X(t,\omega) = \sum_n \xi_n(\omega)\phi_n(t) \tag{3.11}$$

in $L_2[0,1]$ a.s., where

$$\xi_n(\omega) = \int_0^1 X(t,\omega)\phi_n(t)dt \quad \text{a.s.}$$

It follows by Remark 3.1 and the proof of Theorem 3.1 that the random variables $\{\xi_n(\omega)\}$ are jointly Gaussian.

In order to show that μ_X is Gaussian it suffices to show that every continuous linear functional on $\mathcal{X}$ is Gaussian; i.e., that the random variable $F(u) = \langle u,f\rangle$ on $(\mathcal{X},\mathcal{B}(\mathcal{X}),\mu_X)$ is Gaussian for every $f \in \mathcal{X}$. It is seen in the proof of Theorem 3.1 that the random variable $F(u)$ is identically distributed with the random variable $\xi(\omega)$ on $(\Omega,\mathcal{F},P)$ given by $\xi(\omega) = \int_0^1 X(t,\omega)f(t)dt$ a.s. It follows from (3.11) that

$$\xi(\omega) = \sum_n \langle f,\phi_n\rangle \xi_n(\omega) \quad \text{a.s.}$$

By Lemma 5.1 it follows that $\xi(\omega)$ is Gaussian.

The validity of (II) follows from Corollary 3.1 and the claims about the mean and covariance of μ_X from Theorem 3.1. ||

<u>Remark 3.2</u>. It follows from Theorem 3.2 that if condition (I_2) is satisfied, then (II) and (III) are equivalent.

The affirmative answer to question Q_2 is given in the following theorem.

<u>Theorem 3.3.</u> Given any Gaussian measure μ on $(\mathcal{X} = L_2[0,1],\mathcal{B}(\mathcal{X}))$ there exists a Gaussian stochastic process $(\Omega,\mathcal{F},P;\ X(t,\omega),\ t \in [0,1])$

which satisfies (I_1) and (II) and induces μ on $(X,\mathcal{B}(X))$.

In order to prove Theorem 3.3. the following lemma is needed.

Lemma 3.2. Let μ be a Gaussian measure on $(X = L_2[0,1],\mathcal{B}(X))$, let u_o and S be the mean and the covariance operator of μ respectively, and let $\{\phi_n\}_{n=1}^{\infty}$ be the eigenfunctions of S corresponding to its nonzero eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$.

(1) There exists a set $A_1 \in \mathcal{B}[0,1]$ with $\text{Leb}(A_1) = 0$ such that

(i) The series $\sum_{n=1}^{\infty} <\phi_n, u - u_o>\phi_n(t)$ converges in $L_2(X,\mathcal{B}(X),\mu)$ for all $t \in A_1^c$ (A_1^c denotes the complement of A_1); and

(ii) for all $t \in A_1^c$ there exists a set $N_t \in \mathcal{B}(X)$ such that $\mu(N_t) = 0$ and the series $\sum_{n=1}^{\infty} <\phi_n, u - u_o>\phi_n(t)$ converges for all $u \in N_t^c$. Define $Y(t,u)$ equal to the limit of the series on $A_1^c \times X$ and equal to zero on $A_1 \times X$.

(2) The series $\sum_{n=1}^{\infty} <\phi_n, u - u_o>\phi_n(t)$ converges in $L_2([0,1] \times X, \mathcal{B}[0,1] \times \mathcal{B}(X), \text{Leb} \times \mu)$. Denote the limit by $Y'(t,u)$.

(3) There exists a set $A \in \mathcal{B}[0,1]$ such that $\text{Leb}(A) = 0$ and $Y(t,u) = Y'(t,u)$ a.e. $[\text{Leb} \times \mu]$ on $A^c \times X$.

Proof of Lemma 3.2. Proof of (1). Let $f(t) \sum_{n=1}^{\infty} \lambda_n \phi_n^2(t)$. Since all the functions in this series are nonnegative, real valued and measurable, it follows that $f(t)$ is an extended, real valued, measurable function. Since, by (1.5),

$$\int_0^1 f(t)dt = \sum_{n=1}^{\infty} \lambda_n \int_0^1 \phi_n^2(t)dt = \sum_{n=1}^{\infty} \lambda_n < +\infty,$$

we conclude that there exists a set $A_1 \in \mathcal{B}[0,1]$ such that $\text{Leb}(A_1) = 0$ and $f(t) < +\infty$ on A_1^c. Let $t \in A_1^c$ be fixed. Then the random variables $\{\psi_n(t,u) = <\phi_n, u - u_o>\phi_n(t)\}_{n=1}^{\infty}$ on $(X,\mathcal{B}(X),\mu)$ are jointly Gaussian with mean zero and

$$E[\psi_n^2(t,u)] = \langle S\phi_n,\phi_n\rangle\phi_n^2(t) = \lambda_n\phi_n^2(t)$$

$$E[\psi_n(t,u)\psi_k(t,u)] = \langle S\phi_n,\phi_k\rangle\phi_n(t)\phi_k(t) = \delta_{nk}\lambda_n\phi_n^2(t),$$

where δ_{nk} is Kronecher's δ. Hence $\{\psi_n(t,u)\}_{n=1}^{\infty}$ is a sequence of independent, zero mean, Gaussian random variables and $\sum_{n=1}^{\infty} E[\psi_n^2(t,u)] = \sum_{n=1}^{\infty}\lambda_n\phi_n^2(t) < +\infty$. This implies (i) and also by [3, p. 197] for all $t \in A_1^c$ the series $\sum_{n=1}^{\infty}\psi_n(t,u)$ converges a.e. $[\mu]$, which proves (ii). Clearly the two limits, in (i) and (ii), are equal a.e. $[\mu]$.

__Proof of (2).__ Let $\mathcal{B}[0,1] \times \mathcal{B}(X)$ be the product σ-algebra of $[0,1] \times X$. Then $\psi_n(t,u)$ is a measurable function. By Tonelli's theorem we have

$$\int_0^1\int_X \psi_n^2(t,u)d\mu(u)dt = \langle S\phi_n,\phi_n\rangle = \lambda_n.$$

Hence $\psi_n \in L_2([0,1] \times X,\ \mathcal{B}[0,1] \times \mathcal{B}(X),\ \text{Leb} \times \mu) \equiv L_2([0,1] \times X)$ and $\psi_n\psi_k \in L_1([0,1] \times X)$ for all n and k. It follows by Fubini's Theorem that

$$\int_0^1\int_X \psi_n(t,u)\psi_k(t,u)d\mu(u)dt = \langle S\phi_n,\phi_k\rangle\langle\phi_n,\phi_k\rangle = \delta_{nk}\lambda_k.$$

Hence $\{\psi_n(t,u)\}_{n=1}^{\infty}$ are orthogonal in $L_2([0,1] \times X)$ and by (1.5),

$$\sum_{n=1}^{\infty}\int_0^1\int_X \psi_n^2(t,u)d\mu(u)dt = \sum_{n=1}^{\infty}\lambda_n < +\infty.$$

It follows that the series $\sum_{n=1}^{\infty}\psi_n(t,u)$ converges in $L_2([0,1] \times X)$.

__Proof of (3).__ Since by (2), $\sum_{n=1}^{\infty}\psi_n(t,u) = Y'(t,u)$ in $L_2([0,1] \times X)$, it follows that there exists a subsequence such that $\lim_{k\to+\infty}\sum_{n=1}^{N_k}\psi_n(t,u) = Y'(t,u)$ ε [Leb $\times\ \mu$] on $[0,1] \times X$. Let $B = \{(t,u) \in [0,1] \times X\colon \sum_{n=1}^{N_k}\psi_n(t,u)$ does not converge to $Y'(t,u)\}$. Then $(\text{Leb} \times \mu)(B) = 0$ and by Fubini's theorem,

$$0 = (\text{Leb} \times \mu)(B) = \int_0^1\int_X \chi_B(t,u)d\mu(u)dt$$

$$= \int_0^1\left(\int_{B_t} d\mu(u)\right)dt = \int_0^1 \mu(B_t)dt$$

where $B_t = \{u \in X: (t,u) \in B\} \in \mathcal{B}(X)$. Hence $\mu(B_t) = 0$ a.e. [Leb.] on $[0,1]$ and thus there exists a set $A_2 \in \mathcal{B}[0,1]$ such that $\mathrm{Leb}(A_2) = 0$ and $\mu(B_t) = 0$ for all $t \in A_2^c$. It follows that for every $t \in A_2^c$ we have $\mu(B_t) = 0$ and $\lim_{k\to+\infty} \sum_{n=1}^{N_k} \psi_n(t,u) = Y'(t,u)$ a.e. $[\mu]$.

Let $A = A_1 \cup A_2$. Then for every $t \in A^c$ there exists $B_t \in \mathcal{B}(X)$ such that $\mu(B_t) = 0$ and

$$\lim_{k\to+\infty} \sum_{n=1}^{N_k} \psi_n(t,u) = Y'(t,u) \quad \text{for all} \quad u \in B_t^c.$$

By part (1.ii) it follows that for every $t \in A^c$ there exists $N_t \in \mathcal{B}(X)$ such that $\mu(N_t) = 0$ and

$$\lim_{N\to+\infty} \sum_{n=1}^{N} \psi_n(t,u) = Y(t,u) \quad \text{for all} \quad u \in N_t^c.$$

Since $\{\sum_{n=1}^{N_k} \psi_n(t,u)\}_{k=1}^{\infty}$ is a subsequence of $\{\sum_{n=1}^{N} \psi_n(t,u)\}_{N=1}^{\infty}$, it follows that for every $t \in A^c$, $N_t \subseteq B_t$. Hence for all $t \in A^c$ and $u \in B_t^c$:

$$Y(t,u) = Y'(t,u) = \sum_{n=1}^{\infty} \psi_n(t,u).$$

It now follows from $[0,1] \times X = \{A \times X\} \cup \{(A^c \times X) \cap B\} \cup \{(A^c \times X) \cap B^c\}$, $\mathrm{Leb}(A) = 0$ and $(\mathrm{Leb} \times \mu)(B) = 0$, that $(\mathrm{Leb} \times \mu)\{(A^c \times X) \cap B^c\} = (\mathrm{Leb} \times \mu)\{[0,1] \times X\}$. For all $(t,u) \in \{(A^c \times X) \cap B^c\}$ we have $t \in A^c$ and $u \in B_t^c$ and thus $Y(t,u) = Y'(t,u)$. It follows that $Y(t,u) = Y'(t,u)$ a.e. $[\mathrm{Leb} \times \mu]$ on $[0,1] \times X$. $||$

<u>Proof of Theorem 3.3</u>. Define the probability space (Ω, F, P) by $\Omega = X$, $F = \mathcal{B}(X)$, $P = \mu$ and denote the identity map between Ω and X by I: $I\omega = u$, where $u = \omega$. Define $X(t,\omega)$ by

$$X(t,\omega) = \begin{cases} 0 & \text{on } A \times \Omega \\ u_0(t) + Y(t, I^{-1}(\omega)) & \text{on } A^c \times \Omega, \end{cases} \tag{3.12}$$

where A and Y are the same as in Lemma 3.2. Since $u_o(t)$ and $Y(t,u)$ are $\mathcal{B}[0,1] \times \mathcal{B}(X)$ measurable, it follows that $X(t,\omega)$ is $\mathcal{B}[0,1] \times F$ measurable and hence $(\Omega,F,P;\ X(t,\omega),\ t \in [0,1])$ is a product measurable stochastic process. It also follows by parts (1) and (3) of Lemma 3.2 that for all $t \in A^c$,

$$X(t,\omega) = u_o(t) + \sum_{n=1}^{\infty} \langle \phi_n,\ I^{-1}(\omega) - u_o \rangle\ \phi_n(t) \tag{3.13}$$

a.e. [P] and also in $L_2(\Omega,F,P)$. Since the random variables $\{\langle\phi_n, u - u_o\rangle\}_{n=1}^{\infty}$ on $(X,\mathcal{B}(X),\mu)$ are jointly Gaussian, so are the random variables $\{\langle\phi_n, I^{-1}(\omega) - u_o\rangle\}_{n=1}^{\infty}$ on (Ω,F,P). Hence the convergence in $L_2(\Omega,F,P)$ of the series (3.13), along with (3.12), imply that $X(t,\omega)$, $t \in [0,1]$ is a Gaussian stochastic process.

Now $u_o(t) \in L_2[0,1]$ implies $u_o(t) \in L_2([0,1] \times \Omega)$. Also $Y(t,u) \in L_2([0,1] \times X)$ implies $Y(t,I^{-1}(\omega)) \in L_2([0,1] \times \Omega)$. It follows that $X(t,\omega) \in L_2([0,1] \times \Omega)$ and by Fubini's theorem we obtain

$$+\infty > \int_\Omega \int_0^1 X^2(t,\omega)\,dt\,dP(\omega) = E\left(\int_0^1 X^2(t,\omega)\,dt\right) ;$$

i.e., $X(t,\omega)$ satisfies (II). Thus $X(t,\omega)$, $t \in [0,1]$, satisfies both (I_1) and (II) and by Theorem 3.1 it induces a Gaussian measure μ_X on $(X,\mathcal{B}(X))$. It will be shown that $\mu_X = \mu$. For this it suffices to show $\mu_X(B) = \mu(B)$ for any open sphere with radius $\varepsilon > 0$ and center any $w \in X$. Let $B = \{u \in X: ||u - w|| < \varepsilon\}$. Then we have by (1.2), (3.12) and (3.13)

$$\begin{aligned} \mu_X(B) &= P\{\omega \in \Omega: ||X(\cdot,\omega) - w(\cdot)|| < \varepsilon\} \\ &= \mu\{u \in X: ||u_o - \sum_{n=1}^{\infty} \langle\phi_n, u - u_o\rangle\phi_n - w|| < \varepsilon\} \end{aligned}$$

Since the support of μ is $\sup(\mu) = u_o + \overline{sp}\{\phi_n\}$, we have

$$\mu_X(B) = \mu\{v \in \overline{sp}\{\phi_n\}: ||u_o + \sum_{n=1}^{\infty} <\phi_n, v>\phi_n - w|| < \varepsilon\}$$
$$= \mu\{v \in \overline{sp}\{\phi_n\}: ||u_o + v - w|| < \varepsilon\}$$
$$= \mu\{u \in \sup(\mu): ||u - w|| < \varepsilon\}$$
$$= \mu\{u \in X: ||u - w|| < \varepsilon\} = \mu(B)$$

which completes the proof. ||

4. SOME GENERALIZATIONS AND EXTENSIONS

In this section the following generalizations and extensions of the results presented in Sections 2 and 3 are considered.

4.1 From [0,1] to any Borel measurable subset T of the real line.

All the results presented in Sections 2 and 3 for the case where the parameter t takes values on $T = [0,1]$ clearly hold for any compact subset T of the real line. Moreover it is easily seen that all the results of Section 3 also hold for every Borel measurable subset T of the real line. The only use of the compactness of the interval $[0,1]$ is made in the proof of Theorem 3.2, in concluding that the continuous, orthonormal and complete functions $\phi_n(t)$ in $L_2[0,1]$ are uniformly bounded. However, what is essential in the proof is clearly the uniform boundedness on T of each function in a complete orthonormal set in $L_2(T,\mathcal{B}(T),\text{Leb})$, and this can be always satisfied by an appropriate choice of a complete orthonormal set even if T is a set of infinite Lebesgue measure.

4.2 On the induction of a Gaussian measure on an appropriate L_2 space by a Gaussian stochastic process.

The two questions raised in the introduction have been answered in the affirmative for the function spaces $C[0,1]$ and $L_2[0,1]$ in Sections 2 and 3 respectively. It should be remarked that the only case where an affirmative answer is obtained under restrictive assumptions, and by no means in general, is the important case of inducing a Gaussian measure on $L_2[0,1]$ from a Gaussian process $X(t,\omega)$, $t \in [0,1]$: the restrictive assumption being condition (II). We now turn our attention to this case and ask whether assumption (II) is essential to inducing a Gaussian measure or not. It turns out that the need for assumption (II) is solely due to the specific way in which we attempted to provide an answer, and that an affirmative answer can indeed be given in the general case with no restrictive assumptions whatever. Specifically, it is presently shown that every product measurable, Gaussian stochastic process $X(t,\omega)$, $t \in T$, where T is any Borel measurable set on the real line, induces a Gaussian measure on an appropriate Hilbert space of square integrable functions on T.

Let (Ω,F,P) be a probability space. In this section by (I_1), (I_2) and (II) we mean conditions (I_1), (I_2) and (II) with the interval $[0,1]$ replaced by a Borel measurable set T of the real line. It follows by Theorem 3.1 and Section 4.1 that a stochastic process $X(t,\omega)$, $t \in T$, satisfying (I_1) induces a Gaussian measure on $L_2(T,\mathcal{B}(T),\text{Leb.})$ if (II) is satisfied; i.e., if $\int_T r(t,t)dt < +\infty$.

Consider a measure ν on $(T,\mathcal{B}(T))$ such that

(IV) $$\int_T r(t,t)d\nu(t) < +\infty$$

That such measures ν exist follows from the following particular choice: Define ν_o on $(T,\mathcal{B}(T))$ by $[\frac{d\nu_o}{d\,\text{Leb}}](t) = f(t)g(t)$, where $g(t) \geq 0$, $g \in L_1(T,\mathcal{B}(T),\text{Leb.})$ and

$$f(t) = \begin{cases} r(t,t) & \text{for} \quad 0 \leq r(t,t) < 1 \\ \frac{1}{r(t,t)} & \text{for} \quad 1 \leq r(t,t) \end{cases}$$

It is clear that ν_o satisfies (IV) and is a finite measure. The following theorem can be proved in the same way as Theorem 3.1.

Theorem 4.1. If the stochastic process $(\Omega,F,P;\ X(t,\omega),\ t \in T)$ satisfies (I_1), then for every measure ν on $(T,\mathcal{B}(T))$ satisfying (IV), $X(t,\omega)$, $t \in T$, induces a Gaussian measure μ_X on $(X = L_2(T,\mathcal{B}(T),\nu),\ \mathcal{B}(X))$ with mean $m \in X$ and covariance operator S generated by the kernel $R(t,s)$.

Hence, even though a product measurable Gaussian process does not necessarily induce a Gaussian measure on $(H_{Leb} = L_2(T,\mathcal{B}(T),Leb.),\ \mathcal{B}(H_{Leb}))$, the necessary and sufficient condition for the latter being (II), it always induces a Gaussian measure on every $(H_\nu = L_2(T,\mathcal{B}(T),\nu),\mathcal{B}(H_\nu))$ with ν satisfying (IV). For instance, a wide sense stationary process $X(t,\omega)$, $t \in (-\infty,+\infty) \equiv R$, satisfying (I_1) does not induce a measure on $L_2(R,\mathcal{B}(R),Leb.)$, but it induces a Gaussian measure on $L_2(R,\mathcal{B}(R),\nu)$ for every finite measure ν. Also a harmonizable Gaussian process $X(t,\omega)$, $t \in R$, does not necessarily induce a Gaussian measure on $L_2(R,\mathcal{B}(R),Leb.)$, but it induces a Gaussian measure on every $L_2(R,\mathcal{B}(R),\nu)$ for ν a finite measure. For a more concrete example consider the Wiener process $W(t,\omega)$, $t \in [0,+\infty) \equiv R^+$; then $r(t,s) = \min(t,s)$ and even though $W(t,\omega)$ does not induce a measure on $L_2(R^+,\mathcal{B}(R^+),Leb.)$ it induces a Gaussian measure for example on $L_2(R^+,\mathcal{B}(R^+),\nu)$, where ν is defined by $[\frac{d\nu}{d\ Leb}](t) = e^{-t}$.

Remark 4.1. Denote by N the set of measures ν on $(T,\mathcal{B}(T))$ which satisfy (IV). A meaningful choice of ν in N assigns positive measure to all Lebesgue measurable subsets, with positive Lebesgue measure, of the set $T_o = \{t \in T: r(t,t) \neq 0\}$. That such measures ν exist is demonstrated by the measure ν_o. The construction of ν_o makes also clear that there exist $\nu \in N$ which are absolutely continuous with respect to the Lebesgue measure with, moreover, $[\frac{d\nu}{d\ Leb.}](t) \neq 0$ a.e. [Leb.] on T_o; i.e., there exist $\nu \in N$ which are equivalent to the Lebesgue measure on $(T_o,\mathcal{B}(T_o))$.

Remark 4.2. Theorem 4.1 states that for every $\nu \in N$, a product measurable, Gaussian stochastic process $X(t,\omega)$, $t \in T$, induces a Gaussian measure μ_ν on $H_\nu = L_2(T,\mathcal{B}(T),\nu)$. Let S_ν be the covariance operator of μ_ν and $\{\phi_{\nu,n}\}_n$ its eigenfunctions corresponding to its nonzero eigenvalues. The support of μ_ν is $\sup(\mu_\nu) = m + \overline{sp}\{\phi_{\nu,n}\}_n = m + \overline{R}(S_\nu)$, where $\overline{R}(S_\nu)$ is the closure in H_ν of the range $R(S_\nu)$ of the operator S_ν. Hence the Hilbert spaces (H_ν) and the induced Gaussian measures (μ_ν), as well as the supports $(m + \overline{R}(S_\nu))$, depend on the choice of ν in N. Some interesting questions arise in this connection. First, whether the equivalence or singularity of the Gaussian measures induced by two product measurable, Gaussian stochastic processes depend on ν in N. This problem will not be considered here. Secondly, how does $\sup(\mu_\nu)$, or $\overline{R}(S_\nu)$, depend on $\nu \in N$. Note that $X(\cdot,\omega) - m(\cdot) \in \overline{R}(S_\nu)$ a.s. for all $\nu \in N$, and therefore it would be interesting to know whether there exists a minimal $\overline{R}(S_\nu)$ for some $\nu \in N$. Even though an affirmative answer to the latter question does not seem in general plausible, the following remarks can be made.

(i) Let $\nu_i \in N$, $i = 1,2$, be such that ν_i << Leb. and let $[\frac{d\nu_i}{d\,\mathrm{Leb.}}](t) = \phi_i(t)$. If $\frac{\phi_2(t)}{\phi_1(t)} \leq C_{12} < +\infty$ on T_o, then it is easily seen that $H_{\nu_1} \subset H_{\nu_2}$ and $R(S_{\nu_2}) \subset R(S_{\nu_1})$. However, an inclusion relationship between $\overline{R}(S_{\nu_1})$ and $\overline{R}(S_{\nu_2})$ does not seem to hold in general, except if S_{ν_1} and S_{ν_2} are strictly positive definite operators, in which case $\overline{R}(S_{\nu_1}) \subset \overline{R}(S_{\nu_2})$.

(ii) If Leb $\in N$, then for every $\nu \in N$ such that ν << Leb. and $\phi(t) \leq C < +\infty$ on T_o we have $H_{\mathrm{Leb.}} \subset H$ and $R(S\ ) \subset R(S_{\mathrm{Leb}})$.

However, there may exist $\nu \in N$ such that $\overline{R}(S_\nu) \subset \overline{R}(S_{Leb.})$. For instance consider the Wiener process $W(t,\omega)$, $t \in [0,1]$; then $r(t,t) = t$, Leb. $\in N$, and if ν is defined by $[\frac{d\nu}{d\,Leb}](t) = \frac{1}{t^p}$, $1 \leq p < 2$, then $\nu \in N$ and by (i), $\overline{R}(S_\nu) = H_\nu \subsetneqq H_{Leb} = \overline{R}(S_{Leb.})$.

(iii) If Leb $\notin N$ then (i) is still applicable. However a measure $\nu \in N$ corresponding to a minimal $\overline{R}(S_\nu)$ does not seem in general to exist. For instance consider the Wiener process $W(t,\omega)$, $t \in [0,+\infty) \equiv R^+$ and the measures ν_k, $k = 1,2,\ldots$, defined on $(R^+,\mathcal{B}(R^+))$ by $[\frac{d\nu_k}{d\,Leb}](t) = \phi_k(t)$ equal to 1 on $[0,k)$ and to $e^{-\frac{(t-k)^2}{2k^2}}$ on $[k,+\infty)$. Then Leb. $\notin N$, $\nu_k \in N$, $\overline{R}(S_{\nu_k}) = H_{\nu_k}$, and by (i), $\overline{R}(S_{\nu_{k+1}}) \subsetneqq \overline{R}(S_{\nu_k})$ for all k. Note that $\phi_k(t) \underset{k}{\to} 1$ for all $t \in [0,+\infty)$ and hence there is no $\nu \in N$ such that $H_\nu \in H_{\nu_k}$ for all k.

4.3. <u>Gaussian stochastic processes and Gaussian measures on $L_p[0,1]$.</u>

It is shown in Theorem 3.2 that if the stochastic process $X(t,\omega)$, $t \in [0,1]$, satisfies (I_2) and (III) then it induces a Gaussian measure on $(H_2 = L_2[0,1],\mathcal{B}(H_2))$. Note that (III) implies only integrability and not square integrability of almost all sample paths of $X(t,\omega)$. Hence the question arises as to whether a Gaussian measure is induced on $(H_1 = L_1[0,1],\mathcal{B}(H_1))$ by $X(t,\omega)$ if (I_1) and (III) are satisfied. The answer to this question is shown to be affirmative in Theorem 4.2.

This naturally raises the question of inducing a Gaussian measure on $L_p[0,1]$, $1 \leq p < +\infty$. If the stochastic process $(\Omega,F,P;\, X(t,\omega),\, t \in [0,1])$ is product $(\mathcal{B}[0,1] \times F)$ measurable and if $\int_0^1 |X(t,\omega)|^p dt < +\infty$ a.s. for $1 \leq p < +\infty$, then the map T defined by (1.1) with $X = L_p[0,1]$ is measurable and $X(t,\omega)$, $t \in [0,1]$, induces on $(X,\mathcal{B}(X))$ a probability measure μ_X

defined by (1.2). The following theorem can be proved as Theorem 3.1.

Theorem 4.2. If the stochastic process $(\Omega,F,P;\ X(t,\omega),\ t \in [0,1])$ satisfies (I_1) and (III) for $p = 1$ or

(V) $$E(\int_0^1 |X(t,\omega)|^p\, dt)^{2/p} < +\infty$$

for $1 < p < +\infty$, then the probability measure μ_X induced on $(H_p = L_p[0,1], B(H_p))$ by $X(t,\omega)$ is Gaussian.

Theorem 4.2 answers question Q_1 for the spaces $L_p[0,1]$, $1 \leq p < +\infty$. Theorem 3.1 is thus obtained from Theorem 4.2 for $p = 2$. Theorem 4.2 continuous to hold if the set $[0,1]$ is replaced by a Borel measurable set T on the real line.

The study of question Q_2 in the general $L_p[0,1]$ space, $1 \leq p < +\infty$, appears to be considerably more complicated than in the case $p = 2$. The results reported recently in [7] and [10] seem to provide the appropriate structure to approach this question.

5. APPENDIX

We prove the following lemma which is used in the proofs of Theorems 2.1 and 3.2.

Lemma 5.1. If the real random variables $\{\xi_n,\ n = 1,2,\ldots\}$ are jointly Gaussian and $\lim_{n\to+\infty} \xi_n = \xi$ a.s., where ξ is an a.s. finite random variable, then ξ is Gaussian.

Proof. Since the sequence of random variables ξ_n converges a.s. to the a.s. finite random variable ξ, it follows that the sequence of distribution functions of the ξ_n's converges completely to the distribution function of ξ. Hence if we denote by $f_n(t) = e^{iu_n t - \frac{1}{2}\sigma_n^2 t^2}$ the characteristic function of ξ_n, $n = 1,2,\ldots$, and by $f(t)$ the characteristic function of ξ, we have

$$\lim_{n\to+\infty} f_n(t) = f(t) \quad \text{for all} \quad t \in (-\infty,+\infty) \tag{5.1}$$

Eq. (5.1) implies that

$$\arg f_n(1) = u_n \xrightarrow[n\to+\infty]{} u_\infty = \arg f(1) \tag{5.2}$$

and that

$$|f_n(\sqrt{2})| = e^{-\sigma_n^2} \xrightarrow[n\to+\infty]{} |f(\sqrt{2})| < +\infty.$$

Note that the limit $|f(\sqrt{2})|$ is strictly positive; since if $|f(\sqrt{2})| = 0$ then $\sigma_n^2 \xrightarrow[n\to+\infty]{} +\infty$ and $|f(t)| = \lim_{n\to+\infty} |f_n(t)| = \begin{cases} 1 & \text{for} \quad t = 0 \\ 0 & \text{for} \quad t \neq 0 \end{cases}$, which contradicts the fact that $f(t)$ is continuous in t. Hence

$$\sigma_n^2 \xrightarrow[n\to+\infty]{} \sigma_\infty^2 < +\infty . \tag{5.3}$$

It follows from (5.1), (5.2) and (5.3) that

$$f(t) = e^{iu_\infty t - \frac{1}{2}\sigma_n^2 t^2} \quad \text{for all} \quad t \in (-\infty,+\infty)$$

and thus ξ is Gaussian. ||

REFERENCES

[1] C. R. Baker, Mutual information for Gaussian processes, SIAM J. Appl. Math., 19 (1970), 451-458.

[2] I. M. Gel'fand and A. M. Yaglom, Calculation of the amount of information about a random function, contained in another such function, Amer. Math. Soc. Transl. (2), 12 (1959), 199-246.

[3] P. R. Halmos, "Measure Theory," Van Nostrand, Princeton, N. J., 1950.

[4] K. Ito, The topological support of Gauss measure on Hilbert space, Nagoya Math. J., 38 (1970), 181-183.

[5] G. Kallianpur and H. Oodaira, The equivalence and singularity of Gaussian measures, in M. Rosenblatt (ed.), "Proc. of Symp. on Time Series Analysis," 279-291, Wiley, New York, 1962.

[6] K. Karhunen, Über lineare Methoden in der Wahrscheinlichkeitsrechnung, Ann. Acad. Scient. Fennicae, Ser. AI, No 37 (1947), 1-79.

[7] J. Kuelbs, Gaussian measures on Banach space, J. Funct. Anal., 5 (1970), 354-367.

[8] E. Mourier, Éléments aléatoires dans un espace de Banach, Ann. d' Inst. H. Poincaré, 13 (1953), 161-244.

[9] W. L. Root, Singular Gaussian measures in detection theory, in M. Rosenblatt (ed.), "Proc. of Symp. on Time Series Analysis," 292-315, Wiley, New York, 1962.

[10] H. Sato, Gaussian measure on Banach space and abstract Wiener measure, Nagoya Math. J., 36 (1969), 65-81.