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ON COVARIANCE OPERATORS

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INTRODUCTION

Let H be a real and separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and Borel σ-field Γ. A "covariance operator" is any operator mapping H into H that is linear, bounded, non-negative, self-adjoint, and trace-class. If μ is a probability measure on (H,Γ) and $\int_H ||\underset{\sim}{x}||^2 d\mu(\underset{\sim}{x}) < \infty$, then μ has a mean element $\underset{\sim}{m}$ and covariance operator R defined by

$$\langle \underset{\sim}{m},\underset{\sim}{u}\rangle = \int_H \langle \underset{\sim}{x},\underset{\sim}{u}\rangle \, d\mu(\underset{\sim}{x})$$

$$\langle R\underset{\sim}{u},\underset{\sim}{v}\rangle = \int_H \langle \underset{\sim}{x}-\underset{\sim}{m},\underset{\sim}{u}\rangle \langle \underset{\sim}{x}-\underset{\sim}{m},\underset{\sim}{v}\rangle \, d\mu(\underset{\sim}{x}),$$

for all $\underset{\sim}{u}$, $\underset{\sim}{v}$ in H [1].

A probability measure μ on (H,Γ) is said to be Gaussian if the probability distribution on $B[R^1]$ (≡ Borel sets of the real line) induced from μ by a bounded linear functional is Gaussian, for *all* bounded linear functionals on H. Thus, if $P^{\underset{\sim}{u}}$ is defined by

$$P^{\underset{\sim}{u}}\{r: r\epsilon A\} = \mu\{x: \langle \underset{\sim}{x},\underset{\sim}{u}\rangle \epsilon A\}, \quad \text{for} \quad A \in B[R^1],$$

then μ is Gaussian if and only if $P^{\underset{\sim}{u}}$ is Gaussian for all $\underset{\sim}{u}$ in H. Mourier [1] has shown that each Gaussian measure μ satisfies $\int_H ||\underset{\sim}{x}||^2 d\mu(\underset{\sim}{x}) < \infty$, and that for each covariance operator R in H and element $\underset{\sim}{m}$ of H there exists a Gaussian measure having R as covariance operator and $\underset{\sim}{m}$ as mean element.

In applications, one usually has an underlying probability space (Ω,β,P) and a β/Γ measurable mapping of Ω into H, say X. X then induces from P a measure μ_X on (H,Γ) in the usual way. An example of a β/Γ measurable map is a stochastic process (S_t), $t\in T$, where T is a compact interval, the sample functions of (S_t) belong almost surely to $L_2[T]$ (Lebesgue measure), and (S_t) is $B[T]\times\beta/B[R^1]$ measurable. The fact that S is β/Γ measurable follows easily from the fact that Γ is the smallest σ-field such that all bounded linear functionals on H are $\Gamma/B[R^1]$ measurable, and that $\langle S,\underset{\sim}{u}\rangle$ is $\beta/B[R^1]$ measurable for all $\underset{\sim}{u}$ in H [2].

Covariance operators thus arise naturally in problems involving Gaussian measures; typical are those involving mean-square continuous and measurable Gaussian processes on a compact interval. Examples of problems whose solutions can be given in terms of covariance and cross-covariance operators are the following: Equivalence of probability measures (see, e.g., [3]); mean-square estimation [4]; and the computation of mutual information [5], [6]. Equivalence of probability measures has applications in the detection of signals in noise [7].

In the problems cited above, a major aspect of the solution usually involves determination of the range of the square root of one or two covariance operators. For example, to show that two Gaussian probability measures on (H,Γ) are equivalent, one must show that the ranges of the square roots of the two covariance operators are identical. This can be a difficult problem. When H is $L_2[T]$, the integral operator representation of a covariance operator will often have kernel that is the Green's function of a linear differential operator in $L_2[T]$, and the range of the covariance operator can then be easily determined (see, e.g., [8], pp. 978-979). However, the range of the square root operator can be vastly more difficult to determine. Thus, methods for specifying the range of self-adjoint, non-negative bounded linear operators are of some interest, and in this paper we obtain a number of such results. They are first stated for general bounded linear operators, and in most cases the simplification introduced when the operators are self-adjoint and non-negative are obvious. Some of these results are then further developed for the case when the operators are square roots of covariance operators. Finally, these results are applied to determine the range space of the square roots of several ubiquitous covariance operators in $L_2[T]$.

Relations Between Ranges of Operators

Suppose that R_1 and R_2 are bounded linear operators mapping H into H, that P_1 is the projection operator mapping H onto $\overline{\text{range}(R_1^*)}$ (the closure of the range of the adjoint of R_1), and that $N(R_1)$ is the null space of R_1. Several useful results on covariance and cross-covariance operators are a direct consequence of the following result.

Proposition 1. Range $(R_1) \subset$ range (R_2) if and only if there exists a bounded linear operator $G: H \to H$ such that $R_1 = R_2 G$, $G = GP_1 = P_2 G P_1$.

Proof: Sufficiency is clear. To prove necessity, suppose range $(R_1) \subset$ range (R_2); then for all $\underset{\sim}{x}$ in H there exists $\underset{\sim}{u}_{\underset{\sim}{x}}$ in H such that $R_1\underset{\sim}{x} = R_2\underset{\sim}{u}_{\underset{\sim}{x}}$. Define a mapping G on H by $G\underset{\sim}{x} = GP_1\underset{\sim}{x} = P_2\underset{\sim}{u}_{\underset{\sim}{x}}$.

To see that G is single-valued, suppose that $G\underset{\sim}{x} = \underset{\sim}{z}$ and $G\underset{\sim}{x} = \underset{\sim}{y}$. Then $\underset{\sim}{z} = P_2\underset{\sim}{z}+\underset{\sim}{z}'$, $\underset{\sim}{y} = P_2\underset{\sim}{y}+\underset{\sim}{y}'$, where $\underset{\sim}{z}'$ and $\underset{\sim}{y}'$ are orthogonal to $\overline{\text{range}(R_2^*)}$. Hence $R_1\underset{\sim}{x} = R_2P_2\underset{\sim}{z} = R_2P_2\underset{\sim}{y}$, by the definition of $G \Rightarrow R_2P_2(\underset{\sim}{z}-\underset{\sim}{y}) = \underset{\sim}{0}$ $\Rightarrow \underset{\sim}{z}-\underset{\sim}{y} \in N(R_2)$. If both $\underset{\sim}{z}$ and $\underset{\sim}{y}$ are in $N(R_2)$, then $P_2\underset{\sim}{z} = P_2\underset{\sim}{y} = \underset{\sim}{0}$, and by the definition of G, $\underset{\sim}{z} = \underset{\sim}{y} = \underset{\sim}{0}$. If either $\underset{\sim}{z}$ or $\underset{\sim}{y}$ is not in $N(R_2)$, then both must belong to $\overline{\text{range }(R_2^*)}$, and then $\underset{\sim}{z} = \underset{\sim}{y}$ follows from $N(R_2) = \overline{\text{range }(R_2^*)}^{\perp}$.

G is clearly linear and defined everywhere in H.

Suppose next that $\underset{\sim}{x}_n \to \underset{\sim}{x}$, $G\underset{\sim}{x}_n \to \underset{\sim}{y}$. By the definition of G, $R_1\underset{\sim}{x}_n \to R_1\underset{\sim}{x} \Rightarrow R_2G\underset{\sim}{x}_n \to R_1\underset{\sim}{x}$. Since $G\underset{\sim}{x}_n \to \underset{\sim}{y}$, one has $R_2G\underset{\sim}{x}_n \to R_2\underset{\sim}{y}$; thus $R_1\underset{\sim}{x} = R_2\underset{\sim}{y}$. Hence $G\underset{\sim}{x} = \underset{\sim}{y}$, and G is closed.

G is thus a closed linear operator, defined everywhere in H; by the closed-graph theorem, G is bounded.

Corollary 1: Suppose R_1 and R_2 are bounded linear operators in H. Then

(a) range $(R_1) \subset$ range (R_2) $\iff$ there exists a scalar $k < \infty$ such that $\langle R_1R_1^*\underset{\sim}{u},\underset{\sim}{u}\rangle \le k\langle R_2R_2^*\underset{\sim}{u},\underset{\sim}{u}\rangle$, all $\underset{\sim}{u}$ in H;

(b) range $(R_1) \subset$ range (R_2) $\iff$ $R_1R_1^* = R_2QR_2^*$, Q a bounded linear operator;

(c) range $(R_2) \subset$ range $([R_1R_1^* + R_2R_2^*]^{\frac{1}{2}})$;

(d) range $([R_1R_1^* + R_2R_2^*]^{\frac{1}{2}}) \subset$ range (R_2) $\iff$ range $(R_1) \subset$ range (R_2);

(e) For $\underset{\sim}{u}$ in H, $\underset{\sim}{u} \in$ range (R_2) $\iff$ there exists a scalar $k < \infty$ such that $\langle \underset{\sim}{u},\underset{\sim}{x}\rangle^2 \le k\langle R_2R_2^*\underset{\sim}{x},\underset{\sim}{x}\rangle$, all $\underset{\sim}{x}$ in H;

(f) range (R_1) = range (R_2) $\iff$ there exists a bounded linear operator G with bounded inverse such that $(R_1R_1^*)^{\frac{1}{2}} = (R_2R_2^*)^{\frac{1}{2}}G$;

(g) range (R_1) = range (R_2) $\iff$ there exists a bounded linear operator Q having bounded inverse and such that $R_1R_1^* = (R_2R_2^*)^{\frac{1}{2}}Q(R_2R_2^*)^{\frac{1}{2}}$.

PROOF: (a) $||R_1^*\underset{\sim}{u}|| \le k||R_2^*\underset{\sim}{u}||$, all $\underset{\sim}{u} \in H \Rightarrow$ for any $\underset{\sim}{v} \in H$, $|\langle \underset{\sim}{v},R_1^*\underset{\sim}{u}\rangle| \le k||\underset{\sim}{v}||\ ||R_2^*\underset{\sim}{u}||$, all $\underset{\sim}{u}$. Define a linear functional $f_{\underset{\sim}{v}}$ on range (R_2^*) by $f_{\underset{\sim}{v}}(R_2^*\underset{\sim}{u}) = \langle \underset{\sim}{v},R_1^*\underset{\sim}{u}\rangle$; from above, $f_{\underset{\sim}{v}}$ is bounded on range (R_2^*) and can thus be extended by continuity to a bounded linear functional on $\overline{\text{range }(R_2^*)}$. Hence, by Riesz' theorem, there exists an element $\underset{\sim}{z}$ in $\overline{\text{range }(R_2^*)}$ such that $\langle \underset{\sim}{v},R_1^*\underset{\sim}{u}\rangle = \langle \underset{\sim}{z},R_2^*\underset{\sim}{u}\rangle$, all $\underset{\sim}{u}$ in H, or $R_1\underset{\sim}{v} = R_2\underset{\sim}{z}$. Hence range $(R_1) \subset$ range (R_2). The converse follows from Proposition 1.

(b) - (e) Clear; for (e), use the operator R_1 defined by $R_1\underset{\sim}{x} = \langle \underset{\sim}{x},\underset{\sim}{u}\rangle\ \underset{\sim}{u}$.

(f) From (a), range (R_1) = range $([R_1R_1^*]^{\frac{1}{2}})$. Suppose range (R_1) = range (R_2). Then, there will exist bounded linear operators G_1, G_2 such that $(R_1R_1^*)^{\frac{1}{2}} = (R_2R_2^*)^{\frac{1}{2}}G_1$ and $(R_2R_2^*)^{\frac{1}{2}} = (R_1R_1^*)^{\frac{1}{2}}G_2$,

$PG_1P = G_1$, $PG_2P = G_2$, P the projection operator mapping H onto $\overline{\text{range } (R_1)}$. Let $G_1' = G_1 + P^{\perp}$, with $P^{\perp}$ the projection operator with range $N(R_1^*)$. Then $(R_1R_1^*)^{\frac{1}{2}} = (R_2R_2^*)^{\frac{1}{2}}G_1'$, since $N(R_1^*) = N([R_1R_1^*]^{\frac{1}{2}})$. If $G_1'\underset{\sim}{x} = \underset{\sim}{0}$, then both $PG_1P\underset{\sim}{x} = \underset{\sim}{0}$ and $P^{\perp}\underset{\sim}{x} = \underset{\sim}{0}$, since it is impossible otherwise for $PG_1P\underset{\sim}{x} = -P^{\perp}\underset{\sim}{x}$. Now, if $P^{\perp}\underset{\sim}{x} = \underset{\sim}{0}$, then $P\underset{\sim}{x} = \underset{\sim}{x}$ and $PG_1Px \neq 0$ if $\underset{\sim}{x} \neq \underset{\sim}{0}$, since $(R_1R_1^*)^{\frac{1}{2}}\underset{\sim}{x} \neq \underset{\sim}{0}$. Hence $G_1'^{-1}$ exists. To see that $G_1'^{-1}$ is bounded, one notes that $G_2'G_1' = G_1'G_2' = I$, where $G_2' \equiv G_2+P^{\perp}$.

(g) Clear.

COROLLARY 2: Suppose R_i, $i = 1,\dots,N$ is linear and bounded. Let $R_0 \equiv \sum_{i=1}^{N} R_iR_i^*$. Then

(a) $(R_iR_i^*)^{\frac{1}{2}} = R_iA_i^*$, where A_i is partially isometric, A_i isometric on $\overline{\text{range } (R_i^*)}$ and $A_i\underset{\sim}{u} = 0$ for $\underset{\sim}{u} \in N(R_i)$;

(b) $(R_iR_i^*)^{\frac{1}{2}} = R_0^{\frac{1}{2}}G_i$, G_i bounded, with $\sum_{i=1}^{N} G_iG_i^*$ isometric on $\overline{\text{range } (R_0^{\frac{1}{2}})}$ and identically zero on $N(R_0^{\frac{1}{2}})$;

(c) Range $(R_0^{\frac{1}{2}}) = \sum_{i=1}^{N}$ range $[(R_iR_i^*)^{\frac{1}{2}}] = \sum_{i=1}^{N}$ range (R_i), where $\sum_i A_i$ is the linear manifold generated by $\bigcup_i A_i$.

PROOF: (a) From (a) of Corollary 1, $(R_iR_i^*)^{\frac{1}{2}} = R_iA_i^*$ for A_i bounded, and A_i must be partially isometric, isometric on $\overline{\text{range } (R_i^*)}$, because $||(R_iR_i^*)^{\frac{1}{2}}\underset{\sim}{x}|| = ||A_iR_i^*\underset{\sim}{x}|| = ||R_i^*\underset{\sim}{x}||$, and one can define $A_i\underset{\sim}{u} = \underset{\sim}{0}$ for $\underset{\sim}{u} \in N(R_i)$, by Proposition 1.

(b) $(R_iR_i^*) \leq R_0$ for $i = 1,\dots,N$, so that $(R_iR_i^*)^{\frac{1}{2}} = R_0^{\frac{1}{2}}G_i$, G_i bounded, $G_i = G_iP_0$, P_0 the projection operator with range equal to $\overline{\text{range } (R_0)}$. Thus

$$R_0 = R_0^{\frac{1}{2}} \left(\sum_{i=1}^{N} G_iG_i^* \right) R_0^{\frac{1}{2}}$$

$$= R_0^{\frac{1}{2}} P_0 \left(\sum_{i=1}^{N} G_i G_i^*\right) P_0 R_0^{\frac{1}{2}}$$

so that $\sum_{i=1}^{N} G_i G_i^*$ is isometric on $\overline{\text{range } (R_0^{\frac{1}{2}})}$, and zero on $N(R_0^{\frac{1}{2}})$.

(c) It is obvious that range $(R_0^{\frac{1}{2}}) \supset \sum_{i=1}^{N}$ range $(R_i R_i^*)^{\frac{1}{2}}$, since range $(R_0^{\frac{1}{2}}) \supset$ range $(R_i R_i^*)^{\frac{1}{2}}$, $i = 1,\ldots,N$, and range $(R_0^{\frac{1}{2}})$ is a linear manifold. From (b), $(R_i R_i^*)^{\frac{1}{2}} = R_0^{\frac{1}{2}} G_i$, $\sum_{i=1}^{N} G_i G_i^*$ isometric on $\overline{\text{range } (R_0^{\frac{1}{2}})}$, zero on $N(R_0^{\frac{1}{2}})$. Suppose that $R_0^{\frac{1}{2}} \underset{\sim}{y} = \underset{\sim}{x}$; then

$$\sum_{i=1}^{N} (R_i R_i^*)^{\frac{1}{2}} G_i^* y = R_0^{\frac{1}{2}} \sum_{i=1}^{N} G_i G_i^* \underset{\sim}{y} = \underset{\sim}{x},$$

so that range $(R_0^{\frac{1}{2}}) \subset \sum_{i=1}^{N}$ range $(R_i R_i^*)^{\frac{1}{2}}$. By (a) of Proposition 1, range $(R_i R_i^*)^{\frac{1}{2}}$ = range (R_i).

COROLLARY 3: Suppose that R_1 and R_2 are linear, bounded, self-adjoint, and non-negative. Then

(a) range $([R_1 + R_2]^{\frac{1}{2}}) \supset$ range $(R_2^{\frac{1}{2}})$; range $(R_2^{\frac{1}{2}}) \supset$ range $([R_1 + R_2]^{\frac{1}{2}})$ if and only if range $(R_1^{\frac{1}{2}}) \subset$ range $(R_2^{\frac{1}{2}})$.

(b) Suppose H is $L_2[T]$, T a compact interval, and R_1 and R_2 are integral operators with kernels defined by

$$R_i(t,s) = \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \hat{R}_i(\lambda)d\lambda ,$$

with

$$\int_{-\infty}^{\infty} |R_i(\tau)|\, d\tau < \infty;$$

then range $(R_1^{\frac{1}{2}}) \subset$ range $(R_2^{\frac{1}{2}})$ if there exists a scalar $k < \infty$ such that $\hat{R}_1(\lambda) \le k\hat{R}_2(\lambda)$ a.e. $d\lambda$ on $(-\infty,\infty)$.

PROOF: Clear; for (b), use Parseval's theorem and convolution.

Applications to Specific Square Roots

Let H be $L_2[0,T]$ for T finite; we proceed to specify the range of the square root for the covariance operators with the following kernels:

(1) $R_1(t,s) = \min(t,s)$;

(2) $R_2(t-s) = T - \max(t,s)$;

(3) $R_3(t,s) = T - |t-s|$;

(4) $R_4(t,s) = e^{-\alpha|t-s|}, \quad \alpha > 0$;

(5) $R_5(t,s) = \int_{-\infty}^{\infty} e^{-i\lambda(t-s)} \hat{R}_5(\lambda)\, d\lambda$,

where $\hat{R}_5$ is a rational spectral density function with denominator of degree exactly two more than the degree of the numerator.

The integral operators with these kernels are encountered quite often in communication theory problems. For example, R_5 could have kernel of the form $e^{-\alpha|t-s|} \cos\omega_0(t-s)$ for $\alpha > 0$, $\omega_0 > 0$. The ranges of the covariance operators themselves are easy to find, since each kernel is the Green's function of a linear differential operator in $L_2[T]$ with constant coefficients; the ranges of R_1, R_3 and R_4 are given, for example, in [8]. However, for the square roots, only the range of $(R_1^{\frac{1}{2}})$ appears to be known (e.g. [3], [9]); it is included here for completeness.

(a) Range of $R_1^{\frac{1}{2}}$.

$$[R_1\underset{\sim}{x}](t) = \int_0^T \min(t,s)\underset{\sim}{x}_s ds = \int_0^t \int_u^T \underset{\sim}{x}_v dv du = [LL^*\underset{\sim}{x}](t),$$

where

$$[L\underset{\sim}{u}](t) = \int_0^t \underset{\sim}{u}_s ds, \quad [L^*\underset{\sim}{u}](t) = \int_t^T \underset{\sim}{u}_s\, ds.$$

Hence $R_1 = LL^*$, so that by Corollary 2 of Proposition 1, $R_1^{\frac{1}{2}} = LA$, A isometric on $\overline{\text{range}\ (L^*)}$. Since $\overline{\text{range}\ (L^*)} = H$, A is unitary. Hence $\text{range}(R_1^{\frac{1}{2}})$ consists of all elements of $L_2[0,T]$

that are equal almost everywhere to an absolutely continuous function that vanishes at the origin and which has derivative belonging to $L_2[0,T]$.

(b) Range of $R_2^{\frac{1}{2}}$. Using integration by parts, $R_2 = L^*L$; using the same procedure as in (a), $R_2^{\frac{1}{2}} = L^*B$, B unitary. Hence range $(R_2^{\frac{1}{2}})$ consists of all elements of $L_2[0,T]$ that are equal almost everywhere to an absolutely continuous function that vanishes at T and has derivative belonging to $L_2[0,T]$.

(c) Range of $R_3^{\frac{1}{2}}$. $R_3 = R_1 + R_2$; hence by part (c) of Corollary 2, range $(R_3^{\frac{1}{2}})$ consists of all elements of $L_2[0,T]$ that are equal almost everywhere to an absolutely continuous function having derivative belonging to $L_2[0,T]$.

(d) Range of $R_4^{\frac{1}{2}}$. Rosanov has shown [10] that the zero-mean Gaussian measures having R_3 and R_4 as covariance operators are equivalent. This implies that $R_3 = R_4 T R_4$, T invertible ([3], Theorem 5.1) so that by (g) of Corollary 1, range $(R_3^{\frac{1}{2}})$ = range $(R_4^{\frac{1}{2}})$.

(e) From (b) of Corollary 3, range $(R_5^{\frac{1}{2}})$ = range $(R_4^{\frac{1}{2}})$.

Applications to Discrimination of Gaussian Measures

Suppose that μ_i is a Gaussian measure with covariance operator R_i and null mean element, and that μ_i' is a Gaussian measure with covariance operator R_i and mean element $\underset{\sim}{m}_i$. Then μ_i and μ_i' are equivalent if and only if $\underset{\sim}{m}_i \in$ range $(R_i^{\frac{1}{2}})$, and are orthogonal otherwise [3]. The results of the preceding section thus give necessary and sufficient conditions on $\underset{\sim}{m}_i$ in order that μ_i and μ_i' be equivalent, for the specific covariance operators treated there.

Further, if μ_i and μ_j are equivalent, then range $(R_i^{\frac{1}{2}})$ = range $(R_j^{\frac{1}{2}})$ [3]. The preceding results show that μ_1 (Wiener measure) is orthogonal to μ_i for $i = 2,\ldots,5$; and that μ_2 is orthogonal to μ_i for $i = 1,3,4,5$.

Finally, suppose that $R_0 = \sum_{i=1}^{N} R_i$, where the R_i are unspecified covariance operators; let μ_i, $i = 1,\ldots,N$ be the Gaussian measure with null mean element and covariance operator R_i, μ_0 the Gaussian measure with null mean and covariance operator R_0. Then one sees from Corollary 3 that $\mu_0 \perp \mu_i$ when range $(R_i^{\frac{1}{2}}) \neq$ range $(R_j^{\frac{1}{2}})$ for $j \in \{1,.,N\}$. Moreover, if range $(R_i^{\frac{1}{2}})$ = range $(R_0^{\frac{1}{2}})$, then for μ_0 to be equivalent to μ_i it is necessary that range $(R_j^{\frac{1}{2}}) \neq$ range $(R_i^{\frac{1}{2}})$ for $j \neq i$. Hence, if μ_0 and μ_i are equivalent, then μ_0 and μ_j are orthogonal for $j \neq i$.

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