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ON ABSTRACT WIENER MEASURE

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§1 Introduction. An abstract Wiener measure is a σ-extension in a Banach space $(X, ||\cdot||_X)$ of the canonical Gaussian cylinder measure $\mu_{\mathcal{X}}$ of a real separable Hilbert space $\mathcal{X}$ such that the norm $||\cdot||_X$ is measurable on $\mathcal{X}$ and $\mathcal{X}$ is dense in X in the norm $||\cdot||_X$. Gross showed [4, p. 39] that there exists an abstract Wiener measure on any real separable Banach space. Sato showed [6, p. 73] that if μ is a Gaussian measure on a real, separable or reflexive Banach space $(X, ||\cdot||_X)$ then there exists a separable closed subspace $\tilde{X}$ of X such that $\mu(\tilde{X}) = 1$ and $\mu_{\tilde{X}} = \mu/\tilde{X}$ is a σ-extension of the canonical Gaussian cylinder measure $\mu_{\mathcal{X}}$ of a real separable Hilbert space $\mathcal{X}$ such that the norm $||\cdot||_{\tilde{X}} \equiv ||\cdot||_{X/\tilde{X}}$ is continuous on $\mathcal{X}$ and $\mathcal{X}$ is dense in $\tilde{X}$ in the norm $||\cdot||_{\tilde{X}}$. The main purpose of this note is to prove that $||\cdot||_{\tilde{X}}$ is measurable (and not merely continuous) on $\mathcal{X}$. Utilizing this and Sato's result mentioned above, we obtain that a Gaussian measure μ on a real, separable or reflexive Banach space X has a restriction $\mu_{\tilde{X}}$ on a closed, separable subspace $\tilde{X}$ of X, which is an abstract Wiener measure. Using this result and a result of Gross [4, p. 40], we show that the subspace $\tilde{X}$ of X is the support of μ. Specializing this result to real Hilbert spaces, we prove that the support of a Gaussian measure on a real Hilbert space is the closure of the linear span of the eigenvectors corresponding to nonzero eigenvalues of the covariance

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operator of μ. This result was recently reported in the literature by Ito [5]. We also obtain a necessary and sufficient condition for a Gaussian measure on a real, separable or reflexive Banach space to be an abstract Wiener measure. As a corollary of this result, we get a necessary and sufficient condition for a real, separable or reflexive Banach space X to be the support of a Gaussian measure μ defined on X.

In Section 2, first we will establish the notation and terminology and then we will make a few remarks about some of the results mentioned above.

§2 Definitions and preliminaries.

The basic definitions, notation and properties that are used consistently are given below. Although most of the following facts can be found in [4] and [6], we summarize them here for the sake of completeness and ready reference.

Let X be a real Banach space, X^* its conjugate space. A cylinder subset of X is defined by

$$\{x \in X: (\xi_1(x),\ldots,\xi_n(x)) \in D\}$$

where $\xi_1,\ldots,\xi_n \in X^*$ and D is a Borel subset of the n-dimensional Euclidean space R^n. The class $\mathcal{U}_X$ of all cylinder sets forms an algebra, and $\overline{\mathcal{U}}_X$ denotes the σ-algebra generated by $\mathcal{U}_X$. A probability measure μ on the measurable space $(X,\overline{\mathcal{U}}_X)$ is called Gaussian if for every $\xi \in X^*$, $\xi(x)$ is a Gaussian random variable with mean zero on the probability space $(X,\overline{\mathcal{U}}_X,\mu)$. If μ is finitely additive probability measure on $(X,\mathcal{U}_X)$ such that

$$\mu\{x \in X: \xi(x) \leq \alpha\} = \frac{1}{\sqrt{2\pi\, v(\xi)}}\int_{-\infty}^{\alpha}\exp[-\frac{1}{2}\frac{t^2}{v(\xi)}]dt$$

for every $\xi \in X^*$ and real α, then μ is called <u>Gaussian cylinder measure</u>.

Let X be a real Hilbert space. <u>The canonical Gaussian cylinder measure</u> μ_X on X is a finitely additive nonnegative set function on $(X, \mathcal{U}_X)$ such that

$$\mu_X\{x \in X: \xi(x) \leq \alpha\} = \frac{1}{\sqrt{2\pi}\,||\xi||_{X^*}} \int_{-\infty}^{\alpha} \exp[-\frac{t^2}{2\,||\xi||^2_{X^*}}]dt,$$

for any $\xi \in X^*$ and any real α, where $||\cdot||_{X^*}$ denotes the norm in X^*. <u>The canonical normal weak distribution</u> F on X is an equivalent class of linear maps from X^* to the space of random variables over a probability space $(\Omega, \mathcal{B}, P)$ (depending on F) such that for every $\xi \in X^*$ the distribution of $F(\xi)$ is Gaussian with mean zero and variance $||\xi||^2_{X^*}$. Two maps F_1, F_2 are equivalent if for every finite number of elements $\xi_1, \ldots, \xi_n \in X^*$ the joint distribution of $F_j(\xi_1), \ldots, F_j(\xi_n)$ is the same for $j = 1,2$. A norm $||\cdot||$ on X is called <u>measurable</u> if for every $\varepsilon > 0$ there exists a finite dimensional projection P_o (depending on ε) such that for every finite dimensional projection P orthogonal to P_o we have

$$\mu_X\{x \in X: ||Px|| > \varepsilon\} < \varepsilon.$$

The two concepts - canonical Gaussian cylinder measure and canonical normal weak distribution - are indeed equivalent [4, p. 32]. Since we will have occassion to use both, we have included them here.

Let $||\cdot||$ be a continuous norm on a real separable Hilbert space X, and X be the Banach space obtained by the completion of X in the norm $||\cdot||$. Through the natural embedding X^* can be considered as a subspace of X^*; therefore μ_X induces a Gaussian cylinder measure on $(X, \mathcal{U}_X)$ as follows:

$$\mu\{x \in X: (\xi_1(x),\dots,\xi_n(x)) \in D\}$$
$$= \mu_X\{x \in X: (\xi_1(x),\dots,\xi_n(x)) \in D\}, \quad (2.1)$$

where $\xi_1,\dots,\xi_n \in X^*$ and D is a Borel set in R^n. If μ has a σ-extension to $(X,\mathcal{U}_X)$ then μ is called <u>the σ-extension of μ_X on the Banach space X</u> and the norm $||\cdot||$ is called <u>admissible</u> on X.

It is important to note that in the definition of an abstract Wiener measure the measurability of the norm of the Banach space in question is fundamental. Indeed the relationship between the underlying Hilbert space and the measurable norm is an abstraction [4, p. 31] of the relationship between the Hilbert space of all absolutely continuous functions in the Wiener space $W \subset C[0,1]$ with square integrable derivative and the supnorm in $C[0,1]$. Therefore the condition of measurability of the norm in the definition of an abstract Wiener measure can not be replaced by some other weaker condition like continuity. Judging from the introduction of [6], Sato seems to imply that his result mentioned in Section 1 showed that every Gaussian measure on a real, separable or reflexive Banach space is an abstract Wiener measure when restricted to a suitable subspace. However, in view of the fact that the measurability of the norm in the definition of an abstract Wiener measure is essential, his result simply shows that a Gaussian measure on a real, separable or reflexive Banach admits a restriction on a suitable subspace, which is a σ-extension of the canonical Gaussian cylinder measure of a Hilbert space and the norm of the Banach space is admissible. We should also remark that although Sato's result appears to be very significant in that it clears a new way for the investigation of a Gaussian measure in a Banach space, yet the strength of his result is significantly diminished due to the fact that the norm of the Banach space under consideration was not shown measurable on the underlying Hilbert space. Our result removes this deficiency and gives more insight in studying a Gaussian measure on a Banach space. Moreover, our result allows us to use important theorems about measurable norms obtained by Gross [2, 3, 4] in studying a Gaussian measure on a Banach space. That results of [2, 3, 4] will yield useful results about Gaussian measures in Banach spaces is shown in Section 4.

§3 Gaussian measure and abstract Wiener measure.

In this section, we show that every Gaussian measure μ on a separable, or reflexive Banach space X admits a restriction $\mu_{\tilde{X}}$ on a separable closed subspace $\tilde{X}$ of X such that $\mu_{\tilde{X}}$ is an abstract Wiener measure.

We will need two preliminary lemmas.

Lemma 3.1. *Let $\{\xi_j: j = 1,\ldots,n\}$ be a finite family of jointly Gaussian real random variables with mean zero defined on the probability space $(\Omega,\mathcal{B},P)$. Let ξ_j, $j = 1,\ldots,k$, $1 \le k \le n$, be linearly independent and non-degenerate and $\xi_i = \sum_{j=1}^{k} a_{ij}\xi_j$, $i = k+1,\ldots,n$, where a_{ij}'s belong to the set R of real numbers. Let ψ be defined from Ω to the n-Euclidean space R^n by $\psi(\omega) = (\xi_1(\omega),\ldots,\xi_k(\omega), \sum_{j=1}^{k} a_{k+1\,j}\xi_j(\omega),\ldots,\sum_{j=1}^{k} a_{nj}\xi_j(\omega))$. Then for any Borel measurable convex subset E of R^n symmetric about the origin and for any $\underline{a} = (a_1,\ldots,a_n) \in \psi(\Omega)$, we have*

$$P\{\omega: \underline{\xi}_n(\omega) \in E\} \ge P\{\omega: \underline{\xi}_n(\omega) \in \underline{a} + E\} \tag{3.1}$$

where $\underline{\xi}_n(\omega) = (\xi_1(\omega),\ldots,\xi_n(\omega))$.

Proof. Without loss of generality we can assume that $\xi_1,\ldots,\xi_k$ are defined on the canonical probability space $(R^k,\mathcal{B}(R^k),P)$ with $\xi_j(x_1,\ldots,x_k) = x_j$, $j = 1,\ldots,k$, where $\mathcal{B}(R^k)$ is the class of Borel subsets of the k-dimensional

Euclidean space R^k. In this setting the map ψ: $R^k \to R^n$ becomes $\psi(x_1,\dots,x_k) = (x_1,\dots,x_k,\sum_{j=1}^{k} a_{k+1\ j}x_j,\dots,\sum_{j=1}^{k} a_{nj}x_j)$. It is clear that ψ is linear and one to one. Since $E \subseteq R^n$ is convex, symmetric about zero and $\psi(R^k)$ is a subspace of R^n, it follows that $F = E \cap \psi(R^k)$ is also convex and symmetric about zero. Now

$$P\{\underline{\xi}_n \in E\} = P\{\underline{\xi}_n \in F\} = P\{\underline{\xi}_k \in \psi^{-1}(F)\} \qquad (3.2)$$

where $\underline{\xi}_k = (\xi_1,\dots,\xi_k)$. Since ψ^{-1}: $\psi(R^k) \to R^k$ is linear, it follows that $\psi^{-1}(F)$ is convex and symmetric about zero. The vector $\underline{\xi}_k$ satisfies all the hypotheses of Corollary 2 of [1, p. 172], therefore we have

$$P\{\underline{\xi}_k \in \psi^{-1}(F)\} \geq P\{\underline{\xi}_k \in \underline{b} + \psi^{-1}(F)\} \qquad (3.3)$$

where $\underline{b} = \psi^{-1}(\underline{a})$. It is easy to verify that $\psi^{-1}(\underline{a}) + \psi^{-1}(F) = \psi^{-1}(\underline{a} + F)$ and $\psi(R^k) \cap (\underline{a} + E) = \underline{a} + \psi(R^k) \cap E = \underline{a} + F$. Using these facts, (3.2) and (3.3), we conclude inequality (3.1) as follows:

$$\begin{aligned} P\{\underline{\xi}_n \in E\} &\geq P\{\underline{\xi}_k \in \psi^{-1}(\underline{a} + F)\} \\ &= P\{\underline{\xi}_n \in \underline{a} + F\} \\ &= P\{\underline{\xi}_n \in \underline{a} + E\}. \end{aligned}$$

It must be noted that inequality (3.1) is trivially true when all ξ_j, $j = 1,\dots,n$, are degenerate at zero. Therefore the conclusion of Lemma 3.1 holds in both cases, namely when all ξ_j, $j = 1,\dots,n$, are degenate at zero or at least one of the ξ_j's $j = 1,\dots,n$, is non-degenerate.

Lemma 3.2. Let μ be a Gaussian measure on a real separable Banach space $(X,||\cdot||_X)$; then for every $\epsilon > 0$

$$\mu\{x \in X: ||x||_X < \epsilon\} > 0. \qquad (3.4)$$

<u>Proof</u>. Since in a separable Banach space the σ-algebra generated by norm open sets coincides with $\mathfrak{U}_X$, equation (3.4) makes sense. Using separability of X, we choose a sequence $\{x_n: n = 1,2,\ldots\}$ of elements of X such that

$$X = \bigcup_{j=1}^{\infty} \Delta(x_j, \epsilon/2)$$

where $\Delta(x_j, \epsilon/2) = \{x \in X: ||x - x_j|| \leq \epsilon/2\}$. Since $1 = \mu(X) \leq \sum_{j=1}^{\infty} \mu\{\Delta(x_j, \epsilon/2)\}$, it follows that there exists some positive integer n_o such that

$$\mu\{\Delta(x_{n_o}, \epsilon/2)\} > 0. \tag{3.5}$$

Using the separability of X once more, we can find a sequence $\{\xi_n: n = 1,2,\ldots\}$ of elements of X^* such that

$$||x||_X = \sup_j |\xi_j(x)|$$

for every $x \in X$. Now

$$\begin{aligned} &\mu\{x \in X: ||x||_X < \epsilon\} \\ &= \mu\{x \in X: \sup_j |\xi_j(x)| < \epsilon\} \\ &\geq \mu\{x \in X: \sup_j |\xi_j(x)| \leq \epsilon/2\} \\ &= \lim_{k\to\infty} \mu\{x \in X: |\xi_j(x)| \leq \epsilon/2,\ j = 1,\ldots,k\}. \end{aligned} \tag{3.6}$$

By Lemma 3.1, we have

$$\begin{aligned} &\mu\{x \in X: |\xi_j(x)| \leq \epsilon/2,\ j = 1,\ldots,k\} \\ &\geq \mu\{x \in X: |\xi_j(x) - \xi_j(x_{n_o})| \leq \epsilon/2,\ j = 1,\ldots,k\} \end{aligned} \tag{3.7}$$

for every positive integer k. From (3.5), (3.6) and (3.7), we obtain

$$\mu\{x \in X: \|x\|_X < \varepsilon\}$$

$$\geq \lim_{k\to\infty} \mu\{x \in X: |\xi_j(x) - \xi_j(x_{n_o})| \leq \varepsilon/2,\ j = 1,\dots,k\}$$

$$= \mu\{x \in X: \sup_j |\xi_j(x) - \xi_j(x_{n_o})| \leq \varepsilon/2\}$$

$$= \mu\{x \in X: \|x - x_{n_o}\|_X \leq \varepsilon/2\}$$

$$= \mu\{\Delta(x_{n_o}, \varepsilon/2)\} > 0.$$

The proof is complete.

Now we are ready to state and prove the main result of this note. As indicated in Section 1, the basic background of this result is developed by Sato in [6]; the only new thing here is to prove the measurability of a certain norm. For the sake of brevity, the notation and terminology of Lemmas 1 through 5 of [6] will be used below without further explanation.

Theorem 3.1. Let μ be a Gaussian measure on a real, separable or reflexive Banach space $(X, \|\cdot\|_X)$. Then there exists a separable closed subspace $\tilde{X}$ of X such that $\mu(\tilde{X}) = 1$ and $\mu/\tilde{X} \equiv \mu_{\tilde{X}}$ is an abstract Wiener measure; i.e. there exists a separable Hilbert space $\mathcal{X}$ such that $\mu_{\tilde{X}}$ is a σ-extension of the canonical Gaussian cylinder measure $\mu_{\mathcal{X}}$ of $\mathcal{X}$, $\|\cdot\|_{\tilde{X}} \equiv \|\cdot\|_{X/\tilde{X}}$ is measurable on $\mathcal{X}$ and $\mathcal{X}$ is dense in $\tilde{X}$ in the norm $\|\cdot\|_{\tilde{X}}$.

Proof. Let $\tilde{X}$ and $\mathcal{X}$ be the same as defined on pages 70 and 71 of [6] respectively. Then in view of Theorem 2 of [6], the proof of our theorem will be complete if we can show that $\|\cdot\|_{\tilde{X}}$ is measurable on $\mathcal{X}$.

It is clear from [6, p. 71] that $\tilde{X}^*$ can be identified with a subset of $\mathcal{X}^*$, and moreover $\tilde{X}^*$ is dense in $\mathcal{X}^*$. By Corollary 1 of [4, p. 38], the identity map on $\tilde{X}^*$ regarded as densely defined map of

X^* into random variables over the probability space $(\tilde{X}, \overline{\mathcal{U}}_{\tilde{X}}, \mu_{\tilde{X}})$ extends to a representative F of the canonical normal distribution over X in a unique manner. Furthermore, the corresponding canonical Gaussian cylinder measure μ_X satisfies (2.1) for any $\xi_1, \ldots, \xi_n \in X^*$, and any Borel set D in R^n.

Using separability of the space $\tilde{X}$, we can choose a sequence $\{\xi_n: n = 1,2,\ldots\}$ of elements of $\tilde{X}^*$ such that

$$||x||_{\tilde{X}} = \sup_j |\xi_j(x)|$$

for every $x \in \tilde{X}$. Since $\tilde{X}^* \subseteq X^*$, it follows that the restriction $\phi_j \equiv \xi_j/X$ belongs to X^* for each j. Define the sequence of pseudonorms $\{||\cdot||_j': j = 1,2,\ldots\}$ on X as follows:

$$||x||_j' = |\phi_j(x)|.$$

Since $\phi_j \in X^*$, the function $f_j(x) = |\phi_j(x)|$ is continuous tame function [4, p. 32] on X for each $j = 1,2,\ldots$. It follows from the definition of F that the random variable $\tilde{f}_j$ corresponding to f_j is $|\xi_j|$ which is defined on the probability space $(\tilde{X}, \overline{\mathcal{U}}_{\tilde{X}}, \mu_{\tilde{X}})$. Since ξ_j is Gaussian random variable, it follows that for every $\varepsilon > 0$

$$\mu_{\tilde{X}}\{x \in \tilde{X}: |\xi_j(x)| < \varepsilon\} > 0. \qquad (3.8)$$

Applying Theorem 1 of [2, p. 406], Corollary 4.5 of [3, p. 383] and (3.7), we have that $||x||_j'$ is measurable pseudonorm for each j. Let $||x||_n = \max_{1 \le j \le n} ||x||_j'$; then $||x||_n$ is a pseudonorm on X and $||x||_n \le ||x||_1' + \cdots + ||x||_n'$. Since finite sum of measurable pseudonorms is a measurable pseudonorm, it follows that $||x||_1' + \cdots + ||x||_n'$ and hence $||x||_n$ is a measurable pseudonorm; moreover the random variable $||\tilde{x}||_n$ corresponding to $||x||_n$ is $\max_{1 \le j \le n} |\xi_j(x)|$.

Since $\lim_{n\to\infty} \max_{1\le j\le n} |\xi_j(x)| = ||x||_{\tilde{X}}$ for every $x \in \tilde{X}$, it follows that the sequence $\{||\tilde{x}||_n: n = 1,2,\ldots\}$ of random variables on $(\tilde{X},\overline{\mathcal{U}}_{\tilde{X}},\mu_{\tilde{X}})$ converges to the random variable $||x||_{\tilde{X}}$ in probability. Since $\tilde{X}$ is separable and $\mu_{\tilde{X}}$ is Gaussian measure, it follows, by Lemma 3.2, that for every $\epsilon > 0$

$$\mu_{\tilde{X}}\{x \in \tilde{X}: ||x||_{\tilde{X}} > \epsilon\} > 0.$$

Thus we have a sequence $\{||x||_n: n = 1,2,\ldots\}$ of nondecreasing measurable pseudonorms on X such that $||x||_{\tilde{X}}$, the limit in probability of the sequence of random variables $\{||\tilde{x}||_n: n = 1,2,\ldots\}$ exists and has the property that

$$\mu_{\tilde{X}}\{x \in \tilde{X}: ||x||_{\tilde{X}} > \epsilon\} > 0.$$

The measurability of $||x||_{\tilde{X}}$ now follows from Corollary 4.4 of [3, p. 383].

§4. The support of a Gaussian measure.

In this section, we obtain the support of a Gaussian measure defined on a real, separable or reflexive Banach space. Sepcifically, we show that if μ is a Gaussian measure on a real, separable or reflexive Banach space X then the support of μ is the separable closed subspace $\tilde{X}$, where $\tilde{X}$ is the same as in Theorem 3.1. Specializing this result to a real Hilbert space, we show that the support of a Gaussian measure μ on a real Hilbert space is the closed subspace spanned by the eigenvectors corresponding to the nonzero eigenvalues of the covariance operator S_μ of μ. This result was first obtained by Ito [5] by a tricky and entirely different method. We also give a necessary and sufficient condition for a Gaussian measure μ on a real, separable or reflexive Banach space to be an abstract Wiener measure.

Theorem 4.1. Let μ be a Gaussian measure on a real, separable or reflexive Banach space X. Then the support of μ is the separable, closed subspace $\tilde{X}$ of X, where $\tilde{X}$ is the same as in Theorem 3.1.

Proof: We already know that $\tilde{X}$ is a closed subspace of $\tilde{X}$ with $\mu_{\tilde{X}}(\tilde{X}) = 1$. The proof will be complete, if we can show that every open subset of $\tilde{X}$ has positive measure. But this follows by Corollary 4 of [4, p. 40]. Hence the proof is complete.

Theorem 4.2. Let μ be a Gaussian measure on a real, reflexive or separable Banach space X. Then a necessary and sufficient condition that μ be an abstract Wiener measure is that for every $\xi \in X^*$ the condition $\int_X \xi^2(x)d\mu(x) = 0$ implies $\xi(x) \equiv 0$.

Proof. If for every $\xi \in X^*$ the condition $\int_X \xi^2(x)d\mu(x) = 0$ implies $\xi(x) \equiv 0$, then, it follows from [6, p. 70], that $\tilde{X} = X$ where $\tilde{X}$ is the same as in Theorem 3.1. Therefore μ is an abstract Wiener measure.

Conversely, suppose μ is an abstract Wiener measure on X, we have to prove that for every $\xi \in X^*$ the condition $\int_X \xi^2(x)d\mu(x) = 0$ implies $\xi(x) \equiv 0$. From Corollary 4 of [4, p. 40], it follows that X is the support of μ. Suppose there exists some $\xi \in X^*$ such that $\int_X \xi^2(x)d\mu(x) = 0$ but $\xi(x) \not\equiv 0$; then, by Theorem 3.1, the support of μ is $\tilde{X}$ which is a proper subspace of X. This is a contradiction. Therefore we have that if $\xi \in X^*$ and $\int_X \xi^2(x)d\mu(x) = 0$ then $\xi(x) \equiv 0$.

The following corollary is an immediate consequence of the above theorem.

Corollary 4.1. Let μ be a Gaussian measure on a real separable, or reflexive Banach space X. Then X is the support of μ if and only if for every $\xi \in X^*$ the condition $\int_X \xi^2(x)d\mu(x) = 0$ implies $\xi(x) \equiv 0$.

Theorem 4.3. (Ito) *Let μ be a Gaussian measure on a real Hilbert space $(X, \langle\cdot\rangle)$. Then the support of μ is $\overline{sp}\{z_n\}$, the closure of the linear span of the eigenvectors $\{z_n: n = 1,2,\dots\}$ corresponding to nonzero eigenvalues of the covariance operator S_μ of μ.*

Proof. Since X is a Hilbert space, it follows, from Theorem 4.1, that the support of μ is $\tilde{X}$, where $\tilde{X}$ is the same as in Theorem 3.1. Thus in order to complete the proof of the theorem, we must show $\overline{sp}\{z_n\} = \tilde{X}$.

Let $\{\xi_\alpha: \alpha \in \Lambda\}$ be the maximal subset of X^* which satisfies (3.1) of [6, p. 68]; and Λ_o and X_α be the same as defined on pages 68 and 70 of [6]. Further let $\xi_\alpha(\cdot) = \langle z_\alpha, \cdot\rangle$, $z_\alpha \in X$ for every $\alpha \in \Lambda$. Since $\langle S_\mu z_\alpha, z_\beta\rangle = \int_X \xi_\alpha(x)\xi_\beta(x)d\mu(x)$, condition 3.1 of [6] becomes

$$\left.\begin{aligned} ||\xi_\alpha||_{X^*} = ||z_\alpha||_X = 1 \\ \langle S_\mu z_\alpha, z_\beta\rangle = \int_X \xi_\alpha(x)\xi_\beta(x)d\mu(x) = 0,\ \alpha \neq \beta,\ \alpha,\beta \in \Lambda. \end{aligned}\right\} \quad (4.1)$$

Define the continuous linear maps $\xi_n: X \to X$ by $\xi_n(\cdot) = \langle z_n, \cdot\rangle$ for each $n = 1,2,\dots$. Then we have

$$||\xi_n||_{X^*} = ||z_n||_X = 1$$

$$\langle S_\mu z_n, z_m\rangle = \int_X \xi_n(x)\xi_n(x)d\mu(x) = \lambda_n\delta_{nm},$$

where δ_{nm} is Kronecher's δ; and it is easy to verify that

$$\{\xi_n: n = 1,2,\dots\} = \{\xi_\alpha: \alpha \in \Lambda_o\} \quad (4.2)$$

Now we shall show $\overline{sp}\{z_n\} = \tilde{X}$. First we prove that $\overline{sp}\{z_n\} \subseteq \tilde{X}$, clearly it is enough to show that each $z_n \in \tilde{X}$. Since, by Lemma 2 of [6], $\tilde{X} = \bigcap_{\alpha\in\Lambda-\Lambda_o} X_\alpha$ where $X_\alpha = \{x \in X: \xi_\alpha(x) = 0\}$; in order to show that a given $z_n \in \tilde{X}$, we must show that $\xi_\alpha(z_n) = 0$ for every $\alpha \in \Lambda-\Lambda_o$. Suppose this is not true, then there exists some $\alpha_o \in \Lambda-\Lambda_o$ such that $\xi_{\alpha_o}(z_n) \neq 0$.

It follows that $\langle z_n, z_{\alpha_o} \rangle \neq 0$; but, by (4.1) and (4.2), $\langle S_\mu z_n, z_{\alpha_o} \rangle = 0$ or $\lambda_n \langle z_n ; z_{\alpha_o} \rangle = 0$. This contradicts the fact that both λ_n and $\langle z_n, z_{\alpha_o} \rangle$ are nonzero. Therefore $z_n \in \tilde{X}$ and hence $\overline{sp}\ \{z_n\} \subseteq \tilde{X}$. In order to prove $\tilde{X} \subseteq \overline{sp}\ \{z_n\}$, we assume the contrary; i.e., we suppose that there exists a $y \in \tilde{X}$ such that $y \notin \overline{sp}\ \{z_n\}$. Write $y = y_1 + y_2$, $y_1, y_2 \in \tilde{X}$, $y_1 \in \overline{sp}\ \{z_n\}$ and $y_2 \in \overline{sp}\ \{z_n\}^\perp$, where $\overline{sp}\ \{z_n\}^\perp$ is the orthogonal subspace to $\overline{sp}\ \{z_n\}$. Let $z = \frac{y_2}{||y_2||}$, then $z \in \overline{sp}\ \{z_n\}^\perp$. Therefore $S_\mu z = 0$ and so $\langle S_\mu z, z_\alpha \rangle = 0$ for every $\alpha \in \Lambda$. Since $||z||_X = 1$, by the definition of the set $\{\xi_\alpha : \alpha \in \Lambda\}$, $z = z_{\alpha_1}$ for some $\alpha_1 \in \Lambda$. Further since $\langle Sz, z \rangle = \int_X \xi_{\alpha_1}^2(x) d\mu(x) = 0$, where $\xi_{\alpha_1}(\cdot) = \langle z, \cdot \rangle$, it follows that $\alpha_1 \in \Lambda - \Lambda_o$. But then $\xi_{\alpha_1}(z) = 1$, which contracts the fact that $z \in \tilde{X}$. Therefore we must have $\tilde{X} \subseteq \overline{sp}\ \{z_n\}$. The proof is complete.

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