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ON THE CONSTRUCTION OF SYSTEMS AND DESIGNS USEFUL IN THE THEORY OF RANDOM SEARCH[1]

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ABSTRACT

This paper presents some methods of constructing systems and designs useful in random search theory. A brief introduction to basic concepts useful in simple search and relation of search systems with important combinatorial configurations is given in Section 1. Section 2 gives some new pairwise balanced (PWB) designs. Construction of the systems is discussed in Sections 3 and 4. In Section 5, a criteria for comparing search systems is given together with some methods for constructing systems optimal in this sense. The results are given without proof because proofs will appear elsewhere [1].

1. INTRODUCTION

Problems of search occur in almost every field, such as medical diagnosis, the parameter of a probability distribution, search for an unknown or hidden object, etc. The *general search problem* is to determine an unknown object $x \in S$, the *basic set* of all possible objects of search, by selecting one after the other certain *test functions* $f_1, f_2, ..$ from a family F defined on S and observing

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their values at x till we have enough information to find x. When S is finite consisting of, say, n elements $a_1 a_2,..,a_n$; $f_i \in F$ depending on $a_j \in S$ $(i = 1,2,\ldots,m)$ $(j = 1,2,\ldots,n)$ and taking only q distinct values $0,1,..,q-1$ such that at each stage the choice of a function depends to some extent on chance, the search is called *simple random search*. Rényi [2] introduced important notions of simple search theory. To a simple search problem, we can associate an $m \times n$ matrix $M = ((f_i(a_j)))$ corresponding to the value of $f_i \in F$ at the point $a_j \in S$. The rows of M correspond to the functions and the columns to the elements of S.

A system F of functions on S is called *separating* on S iff the columns of M are all different. A separating system F on S is called *minimal* if no proper subset of F is a separating system on S. For every $f \in F$ let the set of values of f be the finite set $Y = \{y_1,y_2,\ldots,y_r\}$ such that $f_j(a) = y_\ell$ at $b_{j\ell}$ different points of S, $f_j \in F$, $j = 1,2,\ldots,m$; $\ell = 1,2,\ldots,r$, $\Sigma_{\ell=1}^{r} b_{j\ell} = n$. Associating with each element of S a probability of $1/n$, we have $P(f_j(x) = y_\ell) = b_{j\ell}/n = p_{j\ell}$, entropy of f_j is $H(f_j) = \Sigma_{\ell=1}^{r} p_{j\ell}\cdot \ell og(1/p_{j\ell})$. Then for any separating system F on S, $\Sigma_{j=1}^{r} H(f_j) \geq \ell og n$, and an *optimal separating system* is one for which the equality holds.

For a choice of k distinct elements of S, say, $a_{i_1},a_{i_2},\ldots,a_{i_k}$; let $R_k(a_{i_1},a_{i_2},\ldots,a_{i_k}) = |\{f\colon f(a_{i_1})=f(a_{i_2})=\ldots=f(a_{i_k}),\ f \in F\}|$. System F is *weakly homogeneous of order* k if $R_k(a_{i_1},a_{i_2},\ldots,a_{i_k}) = R_k$, independent of the choice of k elements. Further, let us choose a sequence of k elements $y_{\ell_1},y_{\ell_2},\ldots,y_{\ell_k}$ of Y also and let $R_k(y_{\ell_1},y_{\ell_2},\ldots,y_{\ell_k};\ a_{i_1},\ \ a_{i_k}) =$ $|\{f\colon f(a_{i_j})=y_{\ell_j};\ j=1,2,\ldots,k;\ f \in F\}|$. Then the system is *strongly homogeneous of order* k if, $R_k(y_{\ell_1},y_{\ell_2},\ldots,y_{\ell_k};\ a_{i_1},ai_2,\ldots,a_{i_k}) = R_k(y_{\ell_1},y_{\ell_2},\ldots,y_{\ell_k})$ independent of the choice of k elements of S.

A system F of m functions defined on S of n elements will be denoted by F(m,n) and its $m \times n$ matrix by M. Strongly and weakly homogeneous systems will thus be denoted by SHS(m,n) and WHS(m,n) with matrices M_S and M_W respectively.

Following result is an easy consequence of above definitions and of partially balanced arrays ([3]):

A partially balanced array of N columns, m rows, s symbols, strength t and parameters $\lambda(y_1,y_2,\ldots,y_t)$ is equivalent to transposed matrix of a SHS(N,m) of order t in s symbols with parameters $R_t(y_1,y_2,\ldots,y_t) = \lambda(y_1,y_2,\ldots,y_t)$.

PWB designs ([4]) are found to be closely related to search systems. Transposed matrix of SHS(m,n) of order 2 in 2 symbols gives incidence matrix of PWB design. However, converse is not ture, in fact there are examples when the system is not even WHS. In Section 3, a stronger result is given:

Incidence matrix of a PWB design is related to the matrix of a SHS of order 2 in 2 symbols iff each treatment has equal number of replications. (These PWB designs have been termed as Symmetrical Unequal Block (SUB) arrangements by [5].)

2. Some New PWB Designs

2.1. PWB designs based on finite geometrical structures:

Using the results on finite geometrical structures ([6]) following PWB designs can be obtained by considering points not lying on the structure as treatments and the lines as blocks.

(a) PWB((q+1)(q+1-n); $k_1 = q+1$, $k_2 = q+1-n'$; $\lambda = 1$), from an $\{(n-1)q+n, n\}$ - *arc* in PG(2,q) with $q = 0 (\mathit{mod}\, n)$, $n' = q/n$.

(b) PWB($v = q^2(q^2-q+1)$; $k_1 = q^2$, $k_2 = q(q-1)$; $\lambda = 1$) from $x^{q+1} + y^{q+1} + z^{q+1} = 0$ in $PG(2,q^2)$, $q = p^m$.

(c) PWB($v = q(q^2+1)$; $k_1 = q+1$, $k_2 = q$, $k_3 = q-1$; $\lambda = 1$) from elliptric quadric in $PG(3,q)$ and lines as blocks. Also PWB($v = q(q^2+1)$; $k_1 = q(q+1)$, $k_2 = q^2$; $\lambda = q+1$) is obtainable by taking planes as blocks.

2.2. PWB designs based on finite Baer subplanes:

A subplane of a projective (affine) plane P is a subset Q of points p and lines L such that Q is itself a projective (affine) plane, relative to incidence relation given in P. A Baer sub-plane is a sub-plane such that

(i) $\forall$ $p' \in P$ $\exists$ a unique line $L \in Q \ni p'$ is incident with L

(ii) $\forall$ $L' \in P$ $\exists$ a unique point $p \in Q \ni L'$ is incident with p.

A proper sub-plane $Q(2,m)$ of $P(2,s)$, $s = p^n$ is a Baer sub-plane iff $s = m^2$ ([7]).

Considering points not in Q as treatments and lines as blocks following PWB designs are obtained.

(a) PWB($v = m(m^3-1)$; $k_1 = s-m$, $k_2 = s$; $\lambda = 1$) from Baer projective sub-plane.

(b) PWB($v = m^2(m^2-1)$; $k_1 = s-m$, $k_1 = s-1$, $k_3 = s$; $\lambda = 1$) from Baer affine sub-plane.

3. Strongly Homogeneous Systems of Order 2 in 2 Symbols

Consider system $F(m,n)$ and its matrix M having entries, say, 0 and 1. For i-th and j-th columns of M let

λ_{ij} = Number of rows having 1 in both the columns

ϕ_{ij} = Number of rows having 0 in both the columns

γ_{ij} = Number of rows having 1 in i-th and 0 in j-th column

δ_{ij} = Number of rows having 0 in i-th and 1 in j-th column.

We then have 4 $n \times n$ matrices.

$$\Lambda = ((\lambda_{ij})), \quad \Phi = ((\phi_{ij})), \quad \Gamma = ((\gamma_{ij})) \text{ and } \Delta = ((\delta_{ij})).$$

Matrices Λ and Φ are symmetric and $\Gamma = \Delta'$.

$\Lambda + \Phi + \Gamma + \Delta = mJ_n$, where J_n is $n \times n$ matrix with all entries 1.

For a SHS(m,n) of order 2 we shall have

(i) $\Lambda = \lambda J_n, \quad \Phi = \phi J_n, \quad \Gamma = \gamma J_n, \quad \Delta = \delta J_n$

(ii) $\Gamma = \Delta, \quad \Phi = (m-\lambda-2\delta)J_n$.

Thus a SHS(m,n) of order 2 in 2 symbols is specified by only two parameters λ and γ. If we denote M' by N, then

(iii) $\underset{n\times m}{N}\, \underset{\sim}{1}_m = r\, \underset{\sim}{1}_n$ i.e. each column of M has same number of 1's, where $r = \lambda+\gamma$ and $\underset{\sim}{1}_n$ is $n \times 1$ vector having each element as 1.

(iv) $\underset{n\times n}{C} = NN' = \gamma I_n + \lambda J_n$.

(v) $\overline{N} = J - N$ corresponds to SHS(m,n) of order 2 in 2 symbols with parameters $m - \lambda - 2\gamma$ and γ.

(vi) $m \times n_1$ submatrix M_1 of $\underset{m\times n}{M}$ will correspond to SHS(m,n) of same order and with same parameters for $n_1 \leq n$. This means that deleting columns from the matrix of SHS does not affect the property of strong homogeneity.

Above properties of SHS(m,n) of order 2 in 2 symbols lead us to following theorem.

THEOREM 3.1. A necessary and sufficient condition for an $m \times n$ matrix M with entries 0 and 1 to be the matrix of a SHS(m,n) of order 2 in 2 symbols with parameters λ and γ is that

(i) $\underset{\sim}{j}'M = (\lambda+\gamma)\underset{\sim}{j} = r\,\underset{\sim}{j}$ i.e. each column of M have same number of 1's

(ii) $C = M'M = \gamma I + \lambda J$ i.e. diagonal entries of C are r and every off-diagonal entry is λ.

COROLLARY 3.1. A necessary condition for SHS(m,n) of order 2 in 2 symbols is $m \geq n$.

In view of this theorem, we conclude that incidence matrix of a SUB arrangement is closely related to a SHS(b,v) of order 2 in 2 symbols with parameters λ and $\gamma = (r-\lambda)$. Remembering that BIB designs are a special case of these with $k_i = k$ this includes all BIB designs. Therefore, for constructing SHS(m,n) of order 2 in 2 symbols, the whole force of BIB designs is available together with other SUB arrangements.

4. WEAKLY HOMOGENEOUS SYSTEMS OF ORDER 2 IN 2 SYMBOLS

In the notations of Section 3, for a WHS(m,n) of order 2 in 2 symbols, we have

$$\Lambda + \Phi + \Gamma + \Delta = mJ, \quad \Lambda + \Phi = \alpha J$$

where $\alpha_{ij} = \lambda_{ij}+\phi_{ij}$ = Number of rows having i-th and j-th columns same and for WHS(m,n) of order 2 $\alpha_{ij} = \alpha$ for all i and j. These properties give following theorems.

THEOREM 4.1. Let A be a $n \times m$ matrix with entries 0 or 1 such that

$$AA' = \begin{bmatrix} r_1 & \lambda \dots \lambda \\ \lambda & r_2 \dots \lambda \\ \vdots & \vdots \quad \vdots \\ \lambda & \lambda \dots r_n \end{bmatrix},$$

then $A' = M$ represents a WHS(m,n) of order 2 only if $r_1 = r_2 = \dots = r_n$. Furthermore, in this case WHS is a SHS.

THEOREM 4.2. Let A be a $n \times m$ matrix with entries 1 and -1, then a necessary condition for $A' = M$ to represent WHS(m,n) with parameter α is that $AA' = (m-\beta)I_n + \beta J_n$, where $\beta = 2\alpha - m$.

THEOREM 4.3. Let A be a $n \times m$ matrix with entires 1 and -1 such that

(i) $AA' = (m-\beta)I + \beta J$

(ii) $-\frac{1}{3} \le \frac{\beta}{m} \le 1$

(iii) m and β are both even (or odd),

then $A' = M$ represents a WHS(m,n) with parameter $\alpha = (\beta+m)/2$.

WHS(v,v) can be obtained from the 2 and 3-class association schemes. In this direction following results are obtained.

THEOREM 4.4. A sufficient condition for the $v \times v$ association matrix B_1 in a 2-class association scheme to represent a WHS(v,v) with parameter α is that

$$\alpha = p_{11}^1 + p_{22}^1 = 2 + p_{11}^2 + p_{22}^2.$$

THEOREM 4.5. Association matrix B_1 in a 3-class association scheme will represent a WHS(v,v) with parameter α if

$$\alpha = \sum_{i=1}^{3} p'_{ii} + 2p'_{23} = 2 + \sum_{i=1}^{3} p^2_{ii} + 2p^2_{23} = 2 + \sum_{i=1}^{3} p^3_{ii} + 2p^3_{23}$$

THEOREM 4.6. Consider a 2-class association scheme in v objects with $v \times v$ association matrices B_0, B_1, B_2. Then $v \times v$ matrix $N = \Sigma_{i=0}^{2} C_i B_i$ with entries 1 and -1 will represent a WHS(v,v) with parameter $\alpha \neq v$ if either

(i) $\underline{c} = (1,1,-1), \quad p'_{12} - p^2_{12} = 1, \quad \alpha = V - 2p'_{12}$

or

(ii) $\underline{c} = (1,-1,1), \quad p'_{12} - p^2_{12} = -1, \quad \alpha = V - 2p^2_{12}.$

It can be noted that N and -N represent same WHS.

THEOREM 4.7. Consider a 3-class association scheme in v objects with $v \times v$ association matrices B_i, $i = 0,1,2,3$. Then the $v \times v$ matrix $N = \Sigma_{i=0}^{3} C_i B_i$ with entries 1 and -1 will represent a WHS(v,v) with parameter $\alpha \neq v$ if one of the following holds.

(a) $\underline{c} = (1,1,1,-1), \quad p^1_{13} + p^1_{23} = p^2_{13} + p^2_{23} = 1 + p^3_{13} + p^3_{23},$
$\alpha = v - 2(p^1_{13}+p^1_{23})$

(b) $\underline{c} = (1,1,-1,1), \quad p^1_{12} + p^1_{23} = 1 + p^2_{12} + p^2_{23} = p^3_{12} + p^3_{23},$
$\alpha = v - 2(p^1_{12}+p^1_{23})$

(c) $\underline{c} = (1,-1,1,1), \quad 1 + p^1_{12} + p^1_{13} = p^2_{12} + p^2_{13} = p^3_{12} + p^3_{13},$
$\alpha = v - 2(p^2_{13}+p^2_{12})$

(d) $\underline{c} = (1,1,-1,-1), \quad p^1_{12} + p^1_{13} = 1 + p^2_{12} + p^2_{13} = 1 + p^3_{12} + p^3_{13},$
$\alpha = v - 2(p^1_{12}+p^1_{13})$

(e) $\underline{c} = (1,-1,1,-1)$, $1 + p^1_{12} + p^1_{23} = p^2_{12} + p^2_{23} = 1 + p^3_{12} + p^3_{23}$,

$\alpha = v - 2(p^2_{12}+p^2_{23})$

(f) $\underline{c} = (1,-1,-1,1)$, $1 + p^1_{13} + p^1_{23} = 1 + p^2_{13} + p^2_{23} = p^3_{13} + p^3_{23}$,

$\alpha = v - 2(p^3_{13}+p^3_{23})$

(g) $\underline{c} = (1,-1,-1,-1)$, $\alpha = v - 2$.

In view of the theorems 4.4 to 4.7, we have to look for the association schemes which satisfy conditions of any one theorem. Among the 2-class association schemes condition hold good for the following.

(a) Triangular Association schemes with $v = 15$, $v = 6$.

(b) $L_g(n)$ schemes with $v = n^2$ for $n = 2g$.

(c) $NL_g(n)$ schemes with $v = n^2$, $n = 2g$.

(d) SLB schemes with $k = 2r-1$ and $k = 2r+1$.

(e) Pseudo Geometric Association schemes with $v = \frac{k}{t}[(r-1)(k-1)+t]$, for $t = k-r+1$ or $t = k-r-1$.

5. Comparison of Simple Random Search Systems

For a $WHS(R_1,n)$ of order 2 in 2 symbols 0 and 1 with parameter $\alpha = R_2$ following result is given in [2].

$$P_1(n,N,F,x) \geq 1 - (n-1)\left(\frac{R_2}{R_1}\right)^n ,$$

where $P_1(n,N,F,x)$ is probability of determining $x \in S$ uniquely in N steps using family F of functions. Maximizing this probability means minimization of R_2/R_1 for fixed n. But $R_2/R_1 \geq (n-2)/2(n-1)$, (for proof see [2]). Define a WHS to be optimal WHS for fixed n if R_2 and R_1 are such that this bound is attained.

For a $SHS(R_1,n)$ of order 2 in symbols 0 and 1, with parameter λ and γ we have $R_2/R_1 = R_1 - 2\gamma$. Thus we look for these SUB arrangements for which

$$\frac{R_2}{R_1} = \frac{R_1-2\gamma}{R_1} = \frac{n-2}{2(n-1)} \quad \text{or} \quad R_1 = \frac{4\gamma(n-1)}{n} \quad \text{or} \quad b = \frac{4(r-\lambda)(v-1)}{v} .$$

THEOREM 5.1. Only PWB design with above parameters is the one for which all blocks are of same size.

THEOREM 5.2. BIB design D with parameters $b = 4t-2$, $v = 2t$, $r = 2t-1$, $k = t$, $\lambda = t-1$ gives the optimal $WHS(4t-2,2t)$. In fact this WHS is SHS.

THEOREM 5.3. Existence of a Hadamard matrix of order $4t$ imply existence of an optimal WHS (in fact SHS) of order 2 on a set S with $n = 2t$ elements. In other words, for every set S of $n = 2t$ elements, there exists a SHS of order 2 on S such that the lowest bound for $P_1(n,N,F,x)$ is maximised if Hadamard matrix of order $4t$ exists.

THEOREM 5.4. Existence of Hadamard matrix of order $4t$ imply the existence of an optimal $WHS(4t-1,4t)$ of order 2 in 2 symbols with parameter $\alpha = 2t-1$.

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