[1] *Research supported in part by the Office of Naval Research under Contract N00014-67-A-0321-0002; and University of New Mexico.*

[2] *Research supported in part by the National Science Foundation under Grant GU-2059; and Kyushu University, Japan.*

AN ASYMPTOTIC 0-1 BEHAVIOR OF GAUSSIAN PROCESSES

by

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Institute of Statistics Mimeo Series No. 725

December, 1970

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0. **Introduction.** Let $\{X(t), -\infty < t < \infty\}$ be a real separable Gaussian process defined on a probability space (Ω, A, P). We assume $EX(t) \equiv 0$ and $v^2(t) = EX^2(t) > 0$. Denote the correlation function by $\rho(s,t) = EX(x)X(t)/(v(s)v(t))$. This paper is concerned with the probability of the event

$$E_\phi = [\exists t_0(\omega) > a: \ X(t) \le v(t)\phi(t) \ \text{ for all } \ t \ge t_0(\omega)].$$

One of the authors in [3] gives conditions on the correlation function so that $PE_\phi = 1$ or 0 as $I_\phi < \infty$ or $= \infty$, where the quantity $I_\phi = \int_a^\infty \phi(t)^{2/\alpha - 1} exp\{-\phi^2(t)/2\}dt$. He considers the problem as a type of the so-called law of the iterated logarithm which orginated in the study of sums of independent random variables. In this paper, we treat the problem from a different point of view as a type of crossing problem. The resulting simpler proof shows the above 0-1 behavior holds for a larger class of processes, and also makes the intuitive content of the result clearer. The Gaussian processes (or rather the corresponding correlation functions) now included satisfy:

i) There are positive constants Δ, C_1, C_2, T, and α with $0 < \alpha \le 2$, such that $1-C_1h^\alpha \le \rho(t,t+h) \le 1-C_2h^\alpha$ for $0 \le h < \Delta$

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1 Research supported in part by the Office of Naval Research under Contract N00014-67-A-0321-0002.

2 Research supported in part by the National Science Foundation under Grant GU-2059.

and all $t \geq T$; and

ii) $\rho(t,t+s) = O(s^{-\gamma})$ uniformly in t as $s \to \infty$ for some $\gamma > 0$.

Condition ii) replaces the condition that $\rho(t,t+s) = o(1/s)$ in [3]. There is also a slight improvement in condition i).

Section 1 gives the proof for stationary processes making use of a recent result due to Pickands [1].

A well known theorem of Slepian [2] is used in Section 2 to extend the results of Section 1 to nonstationary processes described by conditions i) and ii) above. It has been pointed out in [3] that the result of Section 2 can be made to yield (by a time transformation) the analogous 0-1 behavior for Brownian motion and its indefinite integral.

1. Stationary Case.

Theorem 1.1 Let $\{X(t), -\infty < t < \infty\}$ be a real separable stationary Gaussian process with $EX(t) \equiv 0$ and covariance function r satisfying

1) $r(t) = 1 - C|t|^{\alpha} + o(|t|^{\alpha})$ as $t \to o$ for some $C > o$ and some α with $0 < \alpha \leq 2$; and

2) $r(t) = O(t^{-\gamma})$ as $t \to \infty$ for some $\gamma > 0$.

Then for all functions $\phi(t)$ that are positive and nondecreasing on some interval $[a,\infty)$, it follows that

$$PE_{\phi} \equiv P[\exists\, t_0(\omega) > a: \; X(t) < \phi(t) \text{ for all } t \geq t_0(\omega)] = 1 \text{ or } 0$$

as the integral

$$I_{\phi} \equiv \int_a^{\infty} \phi(t)^{\frac{2}{\alpha}-1} \exp\{-\phi^2(t)/2\}dt \text{ is finite or infinite.}$$

Note that the condition 1) implies the process $X(t)$ has continuous sample functions. The following lemma will be needed for the first half of the proof of this theorem.

Lemma 1.2 If condition 1) of Theorem 1.1 holds and $A(t) = \inf\{(1-r(s))/|s|^{\alpha}: \ 0 \le s \le t\} > 0$, then

$$\lim_{x \to \infty} \frac{P[\max_{0\le s\le t} X(s)>x]}{t\, x^{2/\alpha}\, \Psi(x)} = C^{1/\alpha} H_{\alpha}, \quad \text{where} \quad \Psi(x) = (2\pi)^{-\frac{1}{2}} x^{-1} e^{-x^2/2} \quad \text{and}$$

$$0 < H_{\alpha} \equiv \lim_{T \to \infty} T^{-1} \int_0^{\infty} e^{s}\, P[\sup_{0\le t\le T} Y(t)>s]\, ds < \infty, \quad \text{and}$$

$Y(t)$ is a nonstationary Gaussian process with mean $EY(t) = -|t|^{\alpha}$ and covariance function $r(s,t) = -|s-t|^{\alpha}+|s|^{\alpha}+|t|^{\alpha}$.

Proof. This is Lemma 2.9 of Pickands [1]. In addition to condition 1), Pickands required that $A_1(t) = \inf\{(1-r^2(s))/|s|^{\alpha}: \ 0 \le s \le t\} > 0$. However, checking Pickands' proofs, we note that this requirement can be replaced by only the requirement that $A(t) > 0$.

In other words, these proofs permit $r(s) = -1$ but not $r(s) = 1$ in the interval $0 < s \le t$.

Remark. If it should happen that $A(t) = 0$, then there is a smallest $s_0 > 0$ such that $r(s_0) = 1$ and then both $r(t)$ and $X(t)$ are periodic with period s_0. Since $\max_{(o,t)} X(s) = \max_{(o,\tau)} X(s)$ where $\tau = \min(t,s_0)$, the requirement that $A(t) > 0$ can be eliminated from Lemma 1.2 by replacing the denominator $t\, x^{2/\alpha}\, \Psi(x)$ by $\tau\, x^{2/\alpha}\, \Psi(x)$.

Proof of Theorem 1.1 when $I_\phi < \infty$. Only condition 1) of Theorem 1.1 will be required for this half of the proof. Considering the sequence of intervals $[n, n+1]$ with integer end points, we obtain $\infty > I_\phi \geq \sum_{n=N}^{\infty} \phi(n+1)^{2/\alpha} \Psi(\phi(n+1))$ where the lower limit $a = N$ of the integral I_ϕ is chosen large enough that the integrand of I_ϕ is a decreasing function in the argument ϕ. Define $F_n = [\max_{n\leq s\leq n+1} X(s)\leq\phi(n)]$.

Since by Lemma 1.2, there is a positive constant K such that $PF_n^{\,c} \sim K\phi(n)^{2/\alpha}\,\Psi(\phi(n))$ as $\phi(n) \to \infty$, we obtain $\sum_{n=N}^{\infty} PF_n^{\,c} < \infty$. The Borel Cantelli lemma completes this half of the proof.

Before proceeding to the second half of the proof, we will need the following lemmas.

Lemma 1.3 If condition 1) of Theorem 1.1 holds and $A(t) > 0$, then

$$\lim_{x\to\infty} \frac{P[\max_{0\leq k\leq m} X(kax^{-2/\alpha}) > x]}{t\, x^{2/\alpha}\, \Psi(x)} = C^{1/\alpha}\,\frac{H_\alpha(a)}{a},$$

where $a > 0$, $m = [t/ax^{-2/\alpha}]$ and $[\ ]$ denotes the greatest integer function, and $H_\alpha(a)$ is a certain positive constant.

Proof. This is Lemma 2.5 of Pickands [1]. All the remarks given for Lemma 1.2 also apply here.

Lemma 1.4 If Theorem 1.1 for the case $I_\phi = \infty$ is true under the additional restriction that for large t, $2\log t \leq \phi^2(t) \leq 3\log t$, then it is true without this restriction.

Proof. A similar statement could have been made for the case $I_\phi < \infty$, but it was not needed in the proof. The restriction that $\phi^2(t) \leq 3\log t$ is treated in Lemma 4.1 of [3], and the restriction that $\phi^2(t) \geq 2\log t$ is only a slight modification of Lemma 4.1 [3] when $\alpha < 2$. Suppose $\alpha = 2$. Since $X(t) > \hat{\phi}(t)$ occurs infinitely often (as $t \to \infty$) implies $X(t) > \phi(t)$ occurs i.o. for any $\hat{\phi} \geq \phi$, we need only show that $I_\phi = \infty$ implies $I_{\hat{\phi}} = \infty$ for $\hat{\phi} = max(\phi,u)$ and $u(t) = (2\log t)^{1/2}$.

Letting $A = \{t>a: \phi(t) \leq u(t)\}$ and $B = \{t>a: \phi(t)>u(t)\}$ we write $I_\phi = A_\phi + B_\phi = \infty$, $I_u = A_u + B_u = \infty$, and $I_{\hat{\phi}} = A_{\hat{\phi}} + B_{\hat{\phi}} = A_u + B_\phi$, where for example B_ϕ means $\int_B exp\{-\phi^2(t)/2\}dt$. We may suppose $B_\phi < \infty$ for otherwise the lemma follows immediately. Note that if $t_0 \in B$, then there is a (largest) nonempty interval in B containing t_0. Consequently, B is a union of such (disjoint) intervals I_n whose lengths and left end points will be denoted by Δ_n and t_n. (Unfortunately, it may not be possible to index the t_n's according to their order on the line.) However, we assume $\phi(t)$ crosses $u(t)$ infinitely often as $t \to \infty$, and therefore that the Δ_n's are finite numbers and the sequence $\{t_n\}$ is infinite. If ϕ doesn't cross $u(t)$ i.o., then either $\phi \leq u$ and $I_{\hat{\phi}} = I_u = \infty$, or $\phi > u$ and $I_{\hat{\phi}} = I_\phi = \infty$ for some a. Note that $\phi(t_n+\Delta_n) = u(t_n+\Delta_n)$ since the jumps of ϕ are never downward. Now

$$\infty > B_\phi = \sum_n \int_{I_n} exp\{-\phi^2/2\}dt \geq \sum_n \Delta_n\, exp\{-\phi^2(t_n+\Delta_n)/2\} = \sum_n \frac{\Delta_n}{t_n+\Delta_n},$$

and

$$B_u \leq \sum_n \Delta_n\, exp\{-u^2(t_n)/2\} = \sum_n \Delta_n/t_n = \sum_n \frac{\Delta_n}{t_n+\Delta_n}\left\{\frac{1}{1-\Delta_n/(t_n+\Delta_n)}\right\}.$$

Since $\Delta_n/(t_n+\Delta_n) \to 0$, we have $\sum_n \Delta_n/t_n < \infty$. Finally $B_u < \infty$ implies $A_u = \infty$ which in turn implies $I_{\hat{\phi}} = \infty$.

Lemma 1.5 Let $X(t)$ be Gaussian process with zero mean function and covariance function $r(s,t)$ with $r(t,t) \equiv 1$. Let $E_n = [X(t_{n,\nu}) \le x_{n,\nu};\ \nu = 0,\ldots,m_n]$ with all $t_{n,\nu}$ distinct. Then

$$\left|P(\bigcap_1^n E_k) - \prod_1^n PE_k\right| \le \sum_{1 \le i < j \le n} \sum_{\mu=0}^{m_j} \sum_{\nu=0}^{m_i} |r| \int_0^1 \phi(x_{i,\nu}, x_{j,\mu};\ \lambda r)\, d\lambda,$$

where $\phi(x,y;\ \lambda r)$ is the standard bivariate normal density with correlation coefficient $\lambda r = \lambda r(t_{i,\nu},\ t_{j,\mu})$.

Proof. This type of lemma now appears in many proofs of asymptotic independence for crossing problems. We include the "standard" proof with the necessary differences.

The event $\bigcap E_k$ involves $N = \Pi_1^n(m_k+1)$ random variables $X(t_{n,\nu})$, and the corresponding covariance matrix will be denoted by $\sum_1 = (r_{k\ell})$, where the doubly indexed random variables $X(t_{n,\nu})$ have been renumbered by a single index k (the k in $r_{k\ell}$).

Partition $\sum_1 = \left[\sum_{ij}\right]$ so that each submatrix $\sum_{ij}$ is the covariances of the random variables of E_i with those of E_j. Now the events E_k would be independent if and only if the corresponding covariance matrix were $\sum_0 = \left[\sum_{ij}^0\right]$ with $\sum_{ii}^0 = \sum_{ii}$ but $\sum_{ij}^0 = 0$ matrix for $i \ne j$.

Let $\sum_\lambda = \lambda\sum_1 + (1-\lambda)\sum_0 = (r_{\lambda ij})$ be the covariance matrix for the standardized multivariate normal density $\phi_\lambda(\underset{\sim}{y})$, and

$$F(\lambda) = \int_{-\infty}^{x_{1,0}} \cdot\cdot \int_{-\infty}^{x_{1,m_1}} \cdots \int_{-\infty}^{x_{n,0}} \cdot\cdot \int_{-\infty}^{x_{n,m_n}} \phi_\lambda(\underset{\sim}{y})\, d\underset{\sim}{y},$$

where $d\underset{\sim}{y} = dy_1, dy_2, \ldots, dy_N$. We now have

$$\left|P(\bigcap_1^n E_k) - \prod_1^n PE_k\right| = \left|F(1) - F(0)\right| = \left|\int_0^1 F'(\lambda)\, d\lambda\right| \le \int_0^1 |F'(\lambda)|\, d\lambda.$$

Since $F'(\lambda) = \int \partial\phi_\lambda/\partial\lambda \, d\underset{\sim}{y}$ and $dr_{\lambda k\ell}/d\lambda = 0$ or $r_{k\ell}$ according to whether (k,ℓ) refers to a diagonal $\sum_{kk}$ or not, we obtain by the chain rule for $\partial\phi_\lambda/\partial\lambda$

$$|F'(\lambda)| = \left|\sum_{i<j} \sum{}^{*} r_{k\ell} \int \frac{\partial^2\phi_\lambda}{\partial y_k \partial y_k} d\underset{\sim}{y}\right| \leq \sum_{i<j} \sum{}^{*} |r_{k\ell}| \, \phi(x_k, x_\ell; \lambda r_{k\ell}).$$

The double sum $\sum^*$ extends over all (k,ℓ) that refer to covariances $r_{k\ell}$ of $\sum_{ij}$. Integrating this inequality with respect to λ finishes the proof.

Proof of Theorem 1.1 when $I_\phi = \infty$. Note that condition 2) of Theorem 1.1 eliminates the periodic case discussed following Lemma 1.2. Define a sequence of intervals by $I_n = [n\Delta, n\Delta+\beta]$ for $\Delta > 0$ and $0 < \beta < \Delta$. Let $G_k = \{t_{k,\nu} = k + \frac{\nu}{n_k}: \nu = 0, \ldots, [\beta n_k]\}$ be points in I_k where n_k shall be chosen later.

Let $E_k = [max_{s \in G_k} X(s) \leq \phi(k\Delta+\beta)]$. Now $\infty = I_\phi \leq \sum_{k=N}^{\infty} \Delta\phi(k\Delta)^{2/\alpha} \Psi(\phi(k\Delta))$ implies that $\sum \beta\phi(k\Delta+\beta)^{2/\alpha}\Psi(\phi(k\Delta+\beta)) = \infty$. If we choose $n_k = \phi(k\Delta+\beta)^{2/\alpha}$, then Lemma 1.3 implies there is a positive constant K such that $PE_k^{\,c} \sim K\beta\phi(k\Delta+\beta)^{2/\alpha}\Psi(\phi(k\Delta+\beta))$ as $\phi(k\Delta+\beta) \to \infty$. So we have $\sum PE_k^{\,c} = \infty$.

As in the Borel lemma, the main step is

$$1 - P[E_k^{\,c} \text{ i.o.}] = \lim_{m\to\infty} \prod_m^\infty PE_k + \lim_{m\to\infty} \{P(\bigcap_m^\infty E_k) - \prod_m^\infty PE_k\}$$

The first limit is zero because $\sum PE_k^{\,c} = \infty$, and the second limit will be zero because of the asymptotic independence of the events E_k. Note the separation between I_k's is $\Delta-\beta$. By Lemma 1.5, we have

$$A_{m,n} = \left|P(\bigcap_m^n E_k) - \prod_m^n PE_k\right| \leq \sum_{m\leq i<j\leq n} \sum_{\mu=0}^{[\beta n_j]} \sum_{\nu=0}^{[\beta n_i]} |r| \int_0^1 \phi(\phi(i\Delta+\beta), \phi(j\Delta+\beta); \lambda r) d\lambda,$$

where $r = r(t_{j,\nu}-t_{i,\mu})$. Because $t_{j,\nu}-t_{i,\mu} \geq j\Delta-i\Delta-\beta \geq \Delta-\beta$, and because of condition 2) of Theorem 1.1, Δ can be chosen large enough that $|r(t_{j,\nu}-t_{i,\mu})| \leq M_0(j\Delta-i\Delta-\beta)^{-\gamma}$ for all $j > i \geq m$ and some positive constant M_0 and such that $|r| \leq \delta \equiv \min(\frac{1}{2},\gamma/5)$. In fact, for $M = M_0(1-\beta/\Delta)^{-\gamma}$, we have $|r| \leq M(j\Delta-i\Delta)^{-\gamma}$. Now by Lemma 1.4, we can choose m large enough that $\phi^2(k\Delta+\beta) \geq u^2(k\Delta+\beta) = 2\log(k\Delta+\beta)$, for $k \geq m$. Since $\phi(x,y;\lambda r)$ is a decreasing function in each of the variables x and y, we obtain

$$\begin{aligned}
&\phi(\phi(i\Delta+\beta),\phi(j\Delta+\beta);\lambda r)\\
&\qquad \leq (2\pi)^{-1}(1-\delta^2)^{-\frac{1}{2}} \exp\left\{-\frac{u^2(i\Delta+\beta)-2\lambda r u(i\Delta+\beta)u(j\Delta+\beta)+u^2(j\Delta+\beta)}{2(1-\lambda^2 r^2)}\right\}\\
&\qquad \leq (2\pi)^{-1}(1-\delta^2)^{-\frac{1}{2}} \exp\left\{-\frac{u^2(i\Delta+\beta)-2|r|u(i\Delta+\beta)u(j\Delta+\beta)+u^2(j\Delta+\beta)}{2}\right\}\\
&\qquad \leq (2\pi)^{-1}(1-\delta^2)^{-\frac{1}{2}} \left[\frac{1}{i\Delta+\beta}\right]\left[\frac{1}{j\Delta+\beta}\right]^{1-2|r|}.
\end{aligned}$$

By Lemma 1.4, $n_i = \phi(i\Delta+\beta)^{2/\alpha} \leq (3\log(i\Delta+\beta))^{1/\alpha}$, and since $u(i\Delta+\beta) \leq u(j\Delta+\beta)$ and $|r| < \gamma/5$, we have

$$A_{m,\infty} \leq K \sum_{m\leq i<j<\infty} \frac{\log(j\Delta+\beta)^{2/\alpha}}{(j\Delta-i\Delta)^{\gamma}} \left[\frac{1}{i\Delta+\beta}\right]\left[\frac{1}{j\Delta+\beta}\right]^{1-2\gamma/5}.$$

This double series can be seen to be convergent by letting $k = j-i$ and obtaining

$$A_{m,\infty} \leq K' \sum_{i=m}^{\infty} \sum_{k=1}^{\infty} \frac{(\log(k\Delta+i\Delta+\beta))^{2/\alpha}}{k^{\gamma}} \left[\frac{1}{i}\right] \left[\frac{1}{k+i}\right]^{1-2\gamma/5}$$

$$\le K' \sum_{i=m}^{\infty} \sum_{k=1}^{\infty} \frac{(\ell og(k\Delta+i\Delta+\beta))^{2/\alpha}}{(k+i)^{\gamma/5}} \left[\frac{1}{i}\right]^{1+\gamma/5} \left[\frac{1}{k}\right]^{1+\gamma/5} < \infty .$$

Since this series is convergent, we have $\lim_{m\to\infty} A_{m,\infty} = 0$ where $A_{m,\infty} = |P(\bigcap_m^\infty E_k) - \Pi_m^\infty PE_k|$. This completes the proof.

2. Non stationary case.

We now extend the result for the stationary case to the nonstationary case treated in [3]. Here, let $\{X(t), -\infty<t<\infty\}$ be a real separable Gaussian process with zero mean function.

We assume $v^2(t) = EX^2(t) > 0$ and denote the covariance function of $X(t)/v(t)$ by $\rho(s,t) = EX(s)X(t)/(v(s)v(t))$. We shall assume also $\rho(s,t)$ is continuous in Theorem 2.3. This assumption is not needed in Theorem 2.1 (nor the introduction), since condition 3) below implies $\rho(s,t)$ is continuous.

Theorem 2.1 Suppose the above process $X(t)$ satisfies

3) there are positive constants δ_1, C_1, T_1 and α with $0 < \alpha \le 2$ such that $\rho(t,t+h) \ge 1-C_1h^\alpha$ for $0 < h < \delta_1$ and all $t > T_1$.

Then for all functions $\phi(t)$ that are positive and nondecreasing on some interval $[a,\infty)$ such that

$$I_\phi \equiv \int_a^\infty \phi(t)^{\frac{2}{\alpha}-1} exp\{-\phi^2(t)/2\}dt < \infty,$$

we have

$$P[\exists t_0(\omega) > a: \ X(t) \le v(t)\phi(t) \ \text{ for all } \ t \ge t_0(\omega)] = 1.$$

This theorem is Theorem 1 of [3], but we include the following different proof.

Proof. Let $Y(t)$ be a separable stationary Gaussian process having a covariance function $q(h)$ satisfying $q(h) = 1-C_1^*|h|^\alpha+o(|h|^\alpha)$ as $h \to 0$ and $q(h) \le 1-C_1h^\alpha \le \rho(t,t+h)$ for $0 < h < \delta_1^*$. This second requirement follows for $C_1^* > C_1$. By a well-known result due to Slepian (see Theorem 1 in [2]), the fact that $q(h) \le \rho(t,t+h)$ for all $t \ge T_1$ implies $P[\sup_{(n\Delta,n\Delta+\Delta)} Y(s) \le u] \le P[\sup_{(n\Delta,n\Delta+\Delta)} X(s) \le u]$ for $\Delta < \delta_1^*$ and $n\Delta > T_1$. Now following the "proof of Theorem 1.1 when $I_\phi < \infty$", we have $\sum PG_n^c \le \sum PF_n^c < \infty$, where $F_n = [\sup_{(n\Delta,n\Delta+\Delta)} Y(s) \le \phi(n\Delta)]$ and $G_n = [\sup_{(n\Delta,n\Delta+\Delta)} X(s) \le \phi(n\Delta)]$. The Borel Cantelli lemma applied to the G_n's completes the proof.

We shall need the following lemma for the nonstationary result when $I_\phi = \infty$.

Lemma 2.2. Let $\rho(s,t)$ be a continuous covariance function with $\rho(t,t) \equiv 1$. Suppose $\rho(t,t+s) \to 0$ uniformly in t as $s \to \infty$. Then for every $\Delta > 0$, there is a constant $\delta < 1$ such that $|\rho(t,t+s)| \le \delta < 1$ for all $s \ge \Delta$ and all $t \ge 0$.

Proof. Let $R(s) = \sup_{t\ge0} |\rho(t,t+s)| = \sup_{\tau\ge0} |\rho(\tau s,\tau s+s)|$. Now $R(0) = 1$ and $R(s)$ is continuous since ρ is uniformly continuous. Also $R(s) \to 0$ as $s \to \infty$. The conclusion of Lemma 2.2 fails only if $R(s_0) = 1$ for some $s_0 > 0$. If $|\rho(\tau_0s_0,\tau_0s_0+s_0)| = 1$ for some τ_0, then $r_0(s) = \rho^2(\tau_0s,\tau_0s+s)$ is a periodic covariance function. Since this contradicts the hypothesis that $\rho(t,t+s) \to 0$ as $s \to \infty$, we now suppose that there is a sequence $\{\tau_n\}$ such that $\rho^2(\tau_ns_0,\tau_ns_0+s_0) \to 1$ as $\tau_n \to \infty$. Now $r_n(s) = \rho^2(\tau_ns+\tau_ns+s)$ is a covariance function for each τ_n with spectral distribution functions that we denote by $F_n(\lambda)$. We have $\lim_n \int_{-\infty}^{\infty} (1-\cos s_0\lambda)dF_n(\lambda) = 0$. Consider that

$$1 - r_n(2s_0) = \int_{-\infty}^{\infty} (1-\cos 2s_0\lambda)\,dF_n(\lambda)$$

$$= 2\int_{-\infty}^{\infty} (1+\cos s_0\lambda)(1-\cos s_0\lambda)\,dF_n$$

$$\leq 4\int_{-\infty}^{\infty} (1-\cos s_0\lambda)\,dF_n \to 0 \quad \text{as} \quad n \to \infty.$$

This implies $R(2s_0) = 1$, and consequently the contradictory statement that $R(2^k s_0) = 1$ for all integers $k \geq 0$.

Theorem 2.3 Let the above process $X(t)$ have a continuous correlation function satisfying:

3') There are positive constants δ_2, C_2, T_2 and α' with $0 < \alpha' \leq 2$ such that $\rho(t,t+h) \leq 1-C_2h^{\alpha'}$ for $0 < h < \delta_2$ and all $t > T_2$; and

4) $\rho(t,t+s) = O(s^{-\gamma})$ uniformly in t as $s \to \infty$ for some $\gamma > 0$.

Then for all functions ϕ as in Theorem 2.1 with $I_\phi = \infty$, we have

$$P[X(t) > v(t)\phi(t) \text{ i.o. in } t] = 1.$$

Proof. Let $Y(t)$ be a separable stationary Gaussian process having a covariance function $q(h)$ satisfying $q(h) = 1-C_2^*|h|^{\alpha'}+o(|h|^{\alpha'})$ as $h \to 0$ and $\rho(t,t+h) \leq 1-C_2h^{\alpha'} \leq q(h)$ for $0 < h < \delta_2^*$ $(C_2^* < C_2)$. Applying Slepian's result (see Theorem 1 in [2]) and the "proof of Theorem 1.1 when $I_\phi = \infty$", we obtain $\infty = \sum PE_n^c \leq \sum PH_n^c$ for $\Delta < \delta_2^*$, where $E_k = [\max_{s\in G_k} Y(s) \leq \phi(k\Delta+\beta)]$ and $H_k = [\max_{s\in G_k} X(s) \leq \phi(k\Delta+\beta)]$. Consequently, it only remains to show that the events H_n are asymptotically independent (i.e. in the nonstationary case). Lemma 1.5 yields as in the "proof of Theorem 1.1 when $I_\phi = \infty$"

$$A_{m,n} = \sum_{m\le i<j\le n} \sum_{\mu=0}^{[\beta n_j]} \sum_{\nu=0}^{[\beta n_i]} |\rho| \int_0^1 \phi(\phi(i\Delta+\beta),\ \phi(j\Delta+\beta);\ \lambda\rho)d\lambda,$$

where $\rho = \rho(t_{i,\nu}, t_{j,\mu})$. Since here we cannot choose Δ large, we will choose M_0 large enough that $|\rho| \le M_0(t_{i,\nu}-t_{j,\mu})^{-\gamma}$ for all $t_{i,\nu}$ and $t_{j,\mu}$. Since $t_{i,\nu}-t_{j,\mu} \ge j\Delta-i\Delta-\beta \ge \Delta-\beta$, we have $|\rho| \le M(j-i)^{-\gamma}$ for all $i < j$. Also $|\rho| \le \delta < 1$ for all $t_{i,\nu}$ and $t_{j,\mu}$ for some δ by Lemma 2.2.

Letting $k = j-i$ and using Lemma 1.4, we have

$$A_{m,\infty} \le K' \sum_{i=m}^{\infty} \sum_{k=1}^{\infty} \sum_{\mu} \sum_{\nu} k^{-\gamma} \int_0^1 \phi(u(i\Delta+\beta),\ u(j\Delta+\beta);\ \lambda\rho)d\lambda.$$

Now for all $k >$ some k_0, $|\rho| \le Mk^{-\gamma} < \gamma/5$ and the corresponding portion of the double series dominating $A_{m,\infty}$ is convergent as in the proof for the stationary case. For $1 \le k \le k_0$, we estimate

$$\phi(u(i\Delta+\beta),\ u(j\Delta+\beta);\ \lambda\rho) \le (2\pi)^{-1}(1-\delta^2)^{-\frac{1}{2}} \left[\frac{1}{i\Delta+\beta}\right]^{2(1+\lambda\rho)^{-1}} .$$

But $2(1+\lambda\rho)^{-1} \ge 2(1\underline{+}\delta)^{-1} > 1$ implies the remaining part of the series dominating $A_{m,\infty}$ is also convergent. Q.E.D.

Remark. If conditions 3) and 3') hold simultaneously, then $\alpha \le \alpha'$. The nonstationary case theorem exactly analogous to Theorem 1.1 holds if conditions 3) and 3') with $\alpha = \alpha'$ and condition 4) hold.

Of course Theorem 6 of [3], which is used by Watanabe as a proof of the asymptotic 0-1 behavior of Brownian motion and its indefinite integral, can be improved by using a hypothesis anologous to condition 4) of Theorem 2.3.

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