

1. *Research supported in part by the Office of Naval Research under Contract No. N00014-67-A-0321-0002. Visiting from University of New Mexico.*

2. *Research supported in part by the National Science Foundation under Grant No. GU-2059. Visiting from Kyushu University (Japan).*

---

## ASYMPTOTIC PROPERTIES OF GAUSSIAN PROCESSES

*by*

Clifford Qualls<sup>1</sup> *and* Hisao Watanabe<sup>2</sup>

*Department of Statistics  
University of North Carolina at Chapel Hill*

Institute of Statistics Mimeo Series No. 736

*February, 1971*

## ASYMPTOTIC PROPERTIES OF GAUSSIAN PROCESSES

By Clifford Qualls<sup>1</sup>, *University of North Carolina, Chapel Hill  
and University of New Mexico*

and Hisao Watanabe<sup>2</sup>, *University of North Carolina, Chapel Hill  
and Kyushu University (Japan)*

0. Introduction. Let  $\{X(t), -\infty < t < \infty\}$  be a real separable Gaussian process defined on a probability space  $(\Omega, A, P)$ . We assume  $EX(t) \equiv 0$ ,  $v^2(t) = EX^2(t) > 0$ , and the covariance function  $r(t,s) = E(X(t)X(s))$  is continuous respect to  $t$  and  $s$ . And we set  $\rho(t,s) = r(t,s)/(v(t)v(s))$ . In this paper, we treat two problems concerned with Gaussian processes whose correlation functions satisfy

$$(0.1) \quad \rho(t,s) = 1 - |t-s|^\alpha H(|t-s|) + o(|t-s|^\alpha H(|t-s|)) \text{ as } |t-s| \rightarrow 0,$$

where  $0 \leq \alpha \leq 2$  and  $H$  varies slowly at zero.

In §1, we relate the magnitude of the tails of the spectral function to the order of continuity of the correlation function. As a consequence, we can show directly the equivalence (in very useful cases) of the Kahane-Nisio condition and the Fernique-Marcus and Shepp condition which are necessary and sufficient conditions for  $X(t)$  to have continuous sample functions in terms of the spectral function and the correlation function, respectively.

The second problem (§2) is to extend a result of Pickands [11] which gives the asymptotic distribution of the maximum  $Z(t) = \max_{0 \leq s \leq t} X(s)$  to condition (0.1) with  $0 < \alpha \leq 2$ . Pickands treated stationary Gaussian processes whose covariance function  $\rho(|t-s|) \equiv \rho(t,s)$  satisfy condition (0.1) with  $0 < \alpha \leq 2$  for  $H(|t-s|) \equiv$  a constant. Such studies have been done for Hölder continuity of sample functions by Marcus [8], Kôno [6], and Sirao and Watanabe [13]. Our

efforts using Pickands' methods to give the asymptotic distribution of  $Z(t)$  for the case  $\alpha = 0$  were not successful. (This is not too surprising.)

In §3, we use the result of §2 to obtain the extension of the 0-1 behavior treated in Watanabe [15] and Qualls and Watanabe [12] to our present case. The method in [12] turns out to be powerful, while the method in [15] doesn't seem to be successful in our case. Section 4 treats the non-stationary case; in §2 and §3, we assume stationarity.

1. Existence of stationary Gaussian processes. We list some definitions and properties of regularly varying functions that will be required in this and following sections. Some general references on regular variation are Karamata [5], Adamonic [1], and Feller [2].

Definition 1.1. A positive function  $H(x)$  defined for  $x > 0$  varies slowly at infinity (at zero), if for all  $t > 0$ ,

$$(1.1) \quad \lim_{\substack{x \rightarrow \infty \\ (x \rightarrow 0)}} \frac{H(tx)}{H(x)} = 1.$$

Definition 1.2. A positive function  $Q(x)$  defined for  $x > 0$  varies regularly at infinity (at zero) with exponent  $\alpha \geq 0$ , if for all  $t > 0$ ,

$$(1.2) \quad \lim_{\substack{x \rightarrow \infty \\ (x \rightarrow 0)}} \frac{Q(tx)}{Q(x)} = t^\alpha.$$

A function  $Q(x)$  satisfies (1.2) if and only if  $Q(x) = x^\alpha H(x)$ , where  $H(x)$  varies slowly. Also,  $H(x)$  varies slowly at infinity if and only if  $H(1/x)$  varies slowly at zero.

Let  $Q(x)$  vary regularly with exponent  $\alpha \geq 0$  and  $H(x)$  vary slowly at infinity. Then, the following properties hold.

(1.3) The limits (1.1) and (1.2) converge uniformly in  $t$  on any compact subset of the half line  $(0, \infty)$ .

(1.4) For any  $\varepsilon > 0$ , we have that

$$\lim_{x \rightarrow \infty} x^{-\varepsilon} H(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} x^{\varepsilon} H(x) = \infty.$$

(1.5) The function  $H(x)$  varies slowly if and only if

$$H(x) = a(x) \exp \left\{ \int_1^x \varepsilon(t)/t dt \right\},$$

where  $\varepsilon(x) \rightarrow 0$  and  $a(x) \rightarrow A$  as  $x \rightarrow \infty$  ( $0 < A < \infty$ ).

Definition 1.3. The slowly varying function  $H(x)$  is said to be "normalized" if  $a(x) \equiv A$  in property (1.5) above.

(1.6) If  $H(x)$  is a "normalized" slowly varying function at infinity, then for any  $\varepsilon > 0$ , there exists  $K \geq 1$  such that

$$t^{\varepsilon} \leq \frac{H(tx)}{H(x)} \leq t^{-\varepsilon}$$

for all  $x > 0$  and all positive  $t < 1$  such that  $tx \geq K$ .

(1.7) If  $H(x)$  is a "normalized" slowly varying function at zero, then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$t^{-\varepsilon} \leq \frac{H(tx)}{H(x)} \leq t^{\varepsilon}$$

for all  $x > 0$  and all  $t > 1$  such that  $tx < \delta$

(1.8) If  $H(x)$  is a "normalized" slowly varying function at infinity then for any  $\alpha > 0$  the function  $\bar{x}^{-\alpha} H(x)$  is eventually monotone decreasing.

(1.9) If  $H(x)$  varies slowly at infinity, then

$$\frac{x^{p+1} H(x)}{\int_0^x y^p H(y) dy} \rightarrow p+1 \quad \text{for} \quad p+1 \geq 0,$$

and

$$\frac{x^{p+1} H(x)}{\int_x^\infty y^p H(y) dy} \rightarrow |p+1| \quad \text{for } p+1 < 0.$$

Moreover, if  $\int_x^\infty \frac{H(y)}{y} dy$  exists then it varies slowly at infinity and

$$H(x) / \int_x^\infty \frac{H(y)}{y} dy \rightarrow 0.$$

Theorem 1.1. Let  $H(x)$  vary slowly at infinity. Let  $f$  be a positive even function with  $\int_{-\infty}^\infty f(x) dx = 1$  satisfying the following condition. Assume that there exist positive constants  $0 < \alpha < 2$ ,  $C_1$ ,  $C_2$ , and  $K$  such that

$$C_1 \frac{H(x)}{x^{\alpha+1}} \leq f(x) \leq C_2 \frac{H(x)}{x^{\alpha+1}} \quad \text{for } x \geq K.$$

Then, if we set  $\rho(t) = \int_{-\infty}^\infty e^{itx} f(x) dx$  and  $\sigma^2(h) = 2(1-\rho(h))$ , we have

$$(1.10) \quad C_1 \int_0^\infty \frac{8 \sin^2(x/2)}{x^{\alpha+1}} dx \leq \lim_{h \rightarrow 0} \frac{\sigma^2(h)}{|h|^\alpha H(1/h)} \leq \lim_{h \rightarrow 0} \frac{\sigma^2(h)}{|h|^\alpha H(1/h)} \\ \leq C_2 \int_0^\infty \frac{8 \sin^2(x/2)}{x^{\alpha+1}} dx.$$

Remark 1.1. The  $\rho(t)$  in Theorem 1.1 is a covariance function of a stationary Gaussian process.

Proof. Without loss of generality, we take  $H(x)$  to be "normalized". First, we note that

$$\begin{aligned} \sigma^2(h) &= 2(1-\rho(h)) = 8 \int_0^\infty \sin^2\left(\frac{hx}{2}\right) f(x) dx \\ &= 8 \int_0^{K_1} \sin^2\left(\frac{hx}{2}\right) f(x) dx + 8 \int_{K_1}^\infty \sin^2\left(\frac{hx}{2}\right) f(x) dx \equiv I_1(h) + I_2(h) \end{aligned}$$

for arbitrary  $K_1 \geq K$ . We may ignore  $I_1(h)$  in Theorem 1.1, since  
 $\lim_{h \rightarrow 0} I_1(h)/h^2 = 2 \int_0^{K_1} x^2 f(x) dx < \infty$ .

Now we estimate  $I_2(h)$  by  $C_1 J(h) \leq I_2(h) \leq C_2 J(h)$  ( $K_1 \geq K$ ) where  
 $J(h) = 8 \int_{K_1}^{\infty} \sin^2\left(\frac{hx}{2}\right) x^{-(\alpha+1)} H(x) dx$ . By a change of variables,

$$(1.11) \quad J(h) = h^\alpha \int_{hK_1}^{\infty} 8 \sin^2\left(\frac{x}{2}\right) \frac{H(x/h)}{x^{\alpha+1}} dx$$

$$= h^\alpha \left( \int_{hK_1}^{\delta} + \int_{\delta}^{\Delta} + \int_{\Delta}^{\infty} \right) 8 \sin^2\left(\frac{x}{2}\right) \frac{H(x/h)}{x^{\alpha+1}} dx \quad (\text{say}),$$

where  $hK_1 < \delta < \Delta < \infty$ .

Since  $\lim_{h \rightarrow 0} H(x/h)/H(1/h) = 1$  uniformly with respect to  $x$  in each finite interval  $(\delta, \Delta)$

$$\left| \int_{\delta}^{\Delta} (H(x/h) - H(1/h)) \frac{8 \sin^2(x/2)}{x^{\alpha+1}} dx \right| \leq o(1) \cdot H(1/h) \int_0^{\infty} \frac{8 \sin^2(x/2)}{x^{\alpha+1}} dx.$$

For the first integral of (1.11), we use property (1.6) to obtain for  $\delta < 1$

$$\int_{hK_1}^{\delta} \frac{H(x/h)}{x^{\alpha+1}} 8 \sin^2(x/2) dx \leq H(1/h) \int_0^{\delta} \frac{8 \sin^2(x/2)}{x^{\alpha+1+\epsilon_1}} dx,$$

where we choose  $\epsilon_1 > 0$  with  $\alpha + \epsilon_1 < 2$  (which is possible if  $\alpha < 2$ ).

For the third term of the right hand side in (1.11), we use property (1.8) to obtain, for  $\Delta$  sufficiently large,

$$\left| \int_{\Delta}^{\infty} \frac{8 \sin^2(x/2) H(x/h)}{x^{\alpha+1}} dx \right| = \left| \int_{\Delta}^{\infty} \frac{8 \sin^2(x/2)}{x^{\alpha+1-\epsilon_2}} \frac{H(x/h)}{h^{\epsilon_2} (x/h)^{\epsilon_2}} dx \right|$$

$$\leq \frac{H(\Delta/h)}{\Delta^{\epsilon_2}} \int_{\Delta}^{\infty} \frac{8 \sin^2(x/2)}{x^{\alpha-1-\epsilon_2}} dx,$$

where we choose  $\epsilon_2 > 0$  with  $\alpha - \epsilon_2 > 0$ .

Since  $H(\Delta/h)/H(1/h) \rightarrow 1$  as  $h \rightarrow 0$ , we have, as  $h \rightarrow 0$ ,

$$\left| \frac{1}{H(1/h)} \int_{hK_1}^{\infty} \frac{8 \sin^2(x/2)}{x^{\alpha+1}} H(x/h) dx - \frac{H(1/h)}{H(1/h)} \int_{\delta}^{\Delta} \frac{8 \sin^2(x/2)}{x^{\alpha+1}} dx \right|$$

$$\leq \int_0^\delta \frac{8\delta \sin^2(x/2)}{x^{\alpha+1+\varepsilon_1}} dx + o(1) \int_0^\infty \frac{8\delta \sin^2(x/2)}{x^{\alpha+1}} dx + \frac{O(1)}{\Delta^{\varepsilon_2}} \int_\Delta^\infty \frac{8\delta \sin^2(x/2)}{x^{\alpha+1-\varepsilon_2}} dx.$$

Since

$$\lim_{\delta \rightarrow 0} \int_0^\delta \frac{8\delta \sin^2(x/2)}{x^{\alpha+1+\varepsilon_1}} dx = 0 \quad \text{and} \quad \lim_{\Delta \rightarrow \infty} \int_\Delta^\infty \frac{8\delta \sin^2(x/2)}{x^{\alpha+1-\varepsilon_2}} dx = 0,$$

we have the desired result.

**Corollary 1.1.** Under the conditions of Theorem 1.1, we have

$$\begin{aligned} 0 < \frac{\alpha C_1}{C_2} \int_0^\infty \frac{8\delta \sin^2(x/2)}{x^{\alpha+1}} dx &\leq \lim_{h \rightarrow 0^+} \frac{\sigma^2(h)}{1-F(1/h)} \leq \overline{\lim}_{h \rightarrow 0^+} \frac{\sigma^2(h)}{1-F(1/h)} \\ &\leq \frac{\alpha C_2}{C_1} \int_0^\infty \frac{8\delta \sin^2(x/2)}{x^{\alpha+1}} dx < \infty, \end{aligned}$$

where  $F(x) = \int_{-\infty}^x f(u) du$ .

**Proof.** First, for  $h$  sufficiently small,

$$C_1 \int_{1/h}^\infty \frac{H(u)}{u^{\alpha+1}} du \leq 1 - F(1/h) \leq C_2 \int_{1/h}^\infty \frac{H(u)}{u^{\alpha+1}} du.$$

From this and using property (1.9) and Theorem 1.1, we obtain the corollary.  $\square$

**Remark 1.2.** We can find discussions similar to Theorem 1.1 and Corollary 1.1 in Garsia and Lamperti [3] and Ibragimov and Linik [4].

Next we consider the case  $\alpha = 0$ .

**Theorem 1.2.\*** Let  $L(x)$  vary slowly at infinity and assume  $L(x)$  is monotone for large  $x$ . Let  $f(x)$  be a symmetric probability density function such that

$$C_1 \frac{L(x)}{x} \leq f(x) \leq C_2 \frac{L(x)}{x} \quad \text{for large } x.$$

\* The proof of Theorem 2.1 was communicated to us by Professor Walter L. Smith.

Then, if we define  $\rho(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$ ,  $\sigma^2(h) = 2(1-\rho(h))$ , and  $H(x) = \int_x^{\infty} L(y)/y dy$ , we have

$$(1.12) \quad 4C_1 \leq \lim_{h \rightarrow 0} \frac{\sigma^2(h)}{H(1/h)} \leq \overline{\lim}_{h \rightarrow 0} \frac{\sigma^2(h)}{H(1/h)} \leq 4C_2,$$

and  $H(1/h)$  varies slowly at zero.

**Proof.** Without loss of generality, we take  $L(x)$  to be "normalized". Let the hypotheses of Theorem 1.2 hold for all  $x \geq K$ , where  $K > 1$  say. Without loss of generality we redefine  $L(x)$  on  $x < K$  so that  $L(x)$  is monotone (decreasing) for all  $x > 0$  and  $L(x) \rightarrow L(0) < \infty$  as  $x \rightarrow \infty$ . (Of course, now the bounds on  $f(x)$  given above can only hold for  $x$  bounded away from the origin.) Now as in the proof of Theorem 1.1, we may ignore the integral  $I_1(h)$  in

$$\begin{aligned} \sigma^2(h) &= 4 \int_0^K (1-\cos hx) f(x) dx + 4 \int_K^{\infty} (1-\cos hx) f(x) dx \\ &\equiv I_1(h) + I_2(h), \end{aligned}$$

as  $h \rightarrow 0$ .

By assumption, we have  $4C_1 J(h) \leq I_2(h) \leq 4C_2 J(h)$  where  $J(h) = \int_K^{\infty} (1-\cos hx) \frac{L(x)}{x} dx$ . We replace the study of  $J(h)$  by the study of  $J_0(h) = \int_0^{\infty} (1-\cos hx) \frac{L(x)}{x} dx$ , since the integral  $\int_0^K (1-\cos hx) \frac{L(x)}{x} dx$  again may be ignored as  $h \rightarrow 0$ .

Now  $J_0(h) = \int_0^{\infty} (1-\cos x) \frac{L(x/h)}{x} dx$  is an increasing function of  $h$  since  $L(x/h)$  is increasing. Of course

$$J_0(h) \leq 2^{-1} h^2 \int_0^{Kh} xL(x/h) dx + \int_K^{\infty} \frac{L(x)}{x} dx < \infty.$$

So we may study the Laplace transform

$$J_0^*(s) \equiv \int_0^{\infty} e^{-s\lambda} dJ_0(\lambda) = \int_0^{\infty} \frac{x^2}{s^2+x^2} \frac{L(x)}{x} dx.$$

We write



$$J_0^*(s) = \left( \int_0^{s\Delta} + \int_{s\Delta}^{\infty} \right) \frac{x^2}{s^2+x^2} \frac{L(x)}{x} dx \equiv J_1^*(s) + J_2^*(s),$$

where  $\Delta$  is large.

Since  $J_1^*(s) \leq s^{-2} \int_0^{s\Delta} xL(x)dx \sim \frac{\Delta^2 L(\Delta s)}{2}$  as  $s \rightarrow \infty$  by property (1.9), we have

$$(1.13) \quad \overline{\lim}_{s \rightarrow \infty} \frac{J_1^*(s)}{L(s)} \leq \frac{\Delta^2}{2}.$$

Also

$$\frac{\Delta^2}{1+\Delta^2} \int_{s\Delta}^{\infty} \frac{L(x)}{x} dx \leq J_2^*(s) \leq \int_{s\Delta}^{\infty} \frac{L(x)}{x} dx \equiv H(s\Delta).$$

Now, property (1.9) states that  $H(1/h)$  varies slowly at zero and that  $L(s)/H(s) \rightarrow 0$  as  $s \rightarrow \infty$  (which we use here). This together with (1.13) yields

$$\frac{\Delta^2}{1+\Delta^2} \leq \overline{\lim}_{s \rightarrow \infty} \frac{J_0^*(s)}{H(s)} \leq \overline{\lim}_{s \rightarrow \infty} \frac{J_0^*(s)}{H(s)} \leq 1;$$

and consequently

$$(1.14) \quad \lim_{s \rightarrow \infty} \frac{J_0^*(s)}{H(s)} = 1.$$

We apply a fundamental Tauberian theorem to  $J_0^*(s)$  to obtain

$$(1.15) \quad \lim_{h \rightarrow 0} \frac{J_0(h)}{H(1/h)} = 1, \text{ as desired.}$$

Remark 1.3. Given a slowly varying function  $H$  at infinity, we can find  $L(\cdot)$  by solving the functional equation  $H(x) = \int_x^{\infty} L(u)/u du$ . If  $H$  is differentiable, then we need only verify that  $L(x) = -xH'(x)$  varies slowly. For example, if  $f(x) \sim \beta/x(\log x)^{\beta+1}$  as  $x \rightarrow \infty$ ,  $\beta > 0$ , then  $\sigma^2(h) \sim 4/|\log h|^\beta$  as  $h \rightarrow 0$ .

Corollary 1.2. Under the same conditions as in Theorem 1.2, we have

$$\frac{4C_1}{C_2} \leq \overline{\lim}_{h \rightarrow 0} \frac{\sigma^2(h)}{1-F(1/h)} \leq \overline{\lim}_{h \rightarrow 0} \frac{\sigma^2(h)}{1-F(1/h)} \leq \frac{4C_2}{C_1}.$$

where  $F(x) = \int_{-\infty}^x f(y)dy$ .

Several authors give necessary and sufficient conditions for the sample functions of stationary Gaussian processes to be continuous. Marcus and Shepp [9] give a condition in terms of  $\sigma^2(h)$ , i.e., if  $\sigma$  is monotonic increasing, then

$$\int_0^\varepsilon \frac{\sigma(h)}{h(\log \frac{1}{h})^{1/2}} dh < \infty$$

is a necessary and sufficient condition for continuous sample functions. Also Nisio [10] has obtained a condition in terms of the spectral distribution function  $F$ . Based on her result, Landau and Shepp [7] have given that if  $s_n = F(2^{n+1}) - F(2^n)$  is eventually decreasing, then  $\sum_{n=1}^\infty s_n^{1/2} < \infty$  is a necessary and sufficient condition for continuity of sample functions. By using the results of this section, we can make clear the relation between the two criterion.

Now, suppose that the spectral function satisfies the conditions of either Corollary 1.1 or Corollary 1.2. Then we may suppose without harm that  $\sigma^2(h) \sim 1-F(1/h)$  as  $h \rightarrow 0$ , so that  $\sigma^2(2^{-n}) \sim \sum_{k=n}^\infty s_k$  as  $n \rightarrow \infty$ . If  $\sigma(\cdot)$  is monotone increasing in some interval  $(0, \varepsilon)$ , then

$$\int_0^\infty \frac{\sigma(h)}{h(\log \frac{1}{h})^{1/2}} dh < \infty \quad \text{if and only if} \quad \sum_{n=1}^\infty \frac{\sigma(2^{-n})}{n^{1/2}} < \infty$$

Now, there exist positive constants  $C_3$  and  $C_4$  such that

$$(1.16) \quad C_3 \sum_{n=1}^\infty \left( \frac{1}{n} \sum_{k=n}^\infty s_k \right)^{1/2} \leq \sum_{n=1}^\infty \frac{\sigma(2^{-n})}{n^{1/2}} \leq C_4 \sum_{n=1}^\infty \left( \frac{1}{n} \sum_{k=n}^\infty s_k \right)^{1/2}.$$

Here, we use the following inequality of Boas (see Marcus and Shepp [9]).

Lemma 1.1. Let  $g_n = \sum_{j=n}^\infty a_j$ , where  $a_j$  are positive and decreasing. Then

$$\frac{1}{2} \sum_{n=1}^\infty a_n^{1/2} \leq \sum_{n=1}^\infty \left( \frac{g_n}{n} \right)^{1/2} \leq 2 \sum_{n=1}^\infty a_n^{1/2}.$$

So

$$\frac{C_3}{2} \sum_{n=1}^\infty s_n^{1/2} \leq \sum_{n=1}^\infty \frac{\sigma(2^{-n})}{n^{1/2}} \leq 2C_4 \sum_{n=1}^\infty s_n^{1/2}.$$

Theorem 1.3. If we suppose that the spectral function  $F$  satisfies the conditions of either Corollary 1.1 or Corollary 1.2 and that  $s_n$  and  $\sigma(h)$  are eventually decreasing, then

$$\sum_{n=1}^{\infty} s_n^{1/2} < \infty \quad \text{if and only if} \quad \int_0^\varepsilon \frac{\sigma(h)}{h(\log \frac{1}{h})^{1/2}} dh < \infty.$$

Remark 1.4. Of course if we take the "normalized" version of the regularly varying  $\sigma(h)$  in the case  $0 < \alpha < 2$ , then it is eventually decreasing by property (1.8), as  $h \downarrow 0$ .

2. The asymptotic distribution of the maximum. For the study of the asymptotic distribution of  $Z(t) = \sup_{0 \leq s \leq t} X(s)$  in this section, we assume that the process  $X(t)$  is stationary in addition to its covariance function  $\rho(s) \equiv \rho(t, t+s)$  satisfying condition (0.1) with  $0 < \alpha \leq 2$ . We are assuming each  $X(t)$  has mean 0 and variance 1. The non-stationary case is discussed in §4. Without loss of generality, we also assume the slowly varying function  $H(s)$  in condition (0.1) is "normalized". See §1 for definitions. Let  $\sigma^2(s) \equiv E\{X(t+s) - X(t)\} = 2(1 - \rho(s))$ , define  $\tilde{\sigma}^2(s) = 2|s|^\alpha H(s)$ ,  $A_1(t) = \inf_{0 < s \leq t} \sigma(s)/\tilde{\sigma}(s)$  and  $A_2(t) = \sup_{0 < s \leq t} \sigma(s)/\tilde{\sigma}(s)$ .

The theorem of this section is an extension of Pickands' result [11]. Since Pickands' methods apply in our case, we will only sketch the proof emphasizing the points of difference.

Theorem 2.1. If condition (0.1) with  $0 < \alpha \leq 2$  holds,  $\sigma^2(s) > 0$  for  $s \neq 0$ , and  $\tilde{\sigma}(\cdot)$  is defined as above, then

$$(2.1) \quad \lim_{x \rightarrow \infty} \frac{P[Z(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} = H_\alpha \equiv \lim_{T \rightarrow \infty} T^{-1} \int_0^\infty e^s P[\sup_{0 < t < T} Y(t) > s] ds,$$

and  $0 < H_\alpha < \infty$ , where  $\{Y(t), 0 \leq t < \infty\}$  is a non-stationary Gaussian process

with  $Y(0) = 0$  a.s.,  $E\{Y(t)\} = -|t|^\alpha/2$ ,  $\text{cov}\{Y(t_1), Y(t_2)\} = (|t_1|^\alpha + |t_2|^\alpha - |t_1 - t_2|^\alpha)/2$ , and  $\psi(x) = (2\pi)^{-\frac{1}{2}} x^{-1} \exp(-x^2/2)$ .

**Remark 2.1.** By property (1.8) in §1, the  $\tilde{\sigma}(\cdot)$  defined above (or any "normalized" regularly varying function of positive exponent) is monotone on some small interval  $(0, \tilde{\delta})$ . This useful fact does not seem to be well known in this context. Of course, any function  $\tilde{\sigma}(\cdot)$  with  $\tilde{\sigma}(s) \sim \sigma(s)$  as  $s \rightarrow 0$  (or  $\sigma(\cdot)$  itself) that is monotone near the origin can be used in Theorem 2.1. In fact, any  $g(x) \sim \tilde{\sigma}^{-1}(1/x)$  as  $x \rightarrow \infty$  could be used whether  $g$  is monotone or not.

**Remark 2.2.** The condition that  $\sigma^2(s) > 0$  for  $s \neq 0$  excludes the periodic case. However, if  $\rho(s)$  is periodic with period  $s_0$ , then Theorem 2.1 holds with  $t$  in the denominator of (2.1) replaced by  $\tau = \min(t, s_0)$ . See the remarks in [12].

The proof of Theorem 2.1 is accomplished by a series of lemmas; and in particular Lemma 2.3 is a useful discrete version of Theorem 2.1. The connection between the constants of Theorem 2.1 and Lemma 2.3 is that  $H_\alpha = \lim_{a \rightarrow 0} H_\alpha(a)/a$ . The proof of the first lemma below illustrates the central idea behind the discrete version of Theorem 2.1. For this discrete version, we need a partition of the time interval  $(0, t)$  with mesh size  $\Delta(x)$  approaching 0 at the proper rate as  $x \rightarrow \infty$ . Let  $\Delta(x) = \tilde{\sigma}^{-1}(1/x)$  for all  $x \geq 1/\tilde{\sigma}(\tilde{\delta})$ .

**Lemma 2.1.** If the conditions of Theorem 2.1 hold, then for  $a > 0$ ,

$$\lim_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \geq a^{-1} \left(1 - 2 \sum_{k=1}^{\infty} (1 - \Phi(\frac{1}{2}(ka)^{\alpha/2}))\right),$$

where  $Z_x(t) = \max_{0 \leq k \leq m} X(ka \cdot \Delta(x))$ ,  $m = [t/(a\Delta(x))]$ ,  $[\cdot]$  denotes the greatest integer function, and  $\Phi(\cdot)$  is the standard Gaussian distribution function.

**Proof.** This lemma corresponds to Pickands' Lemma 2.4 [11]. Defining the events

$B_k = [X(ka \cdot \Delta(x)) > x]$  and using stationarity, we have

$$\begin{aligned} P[Z_x(t) > x] &\geq \sum_{k=0}^m PB_k - \sum_{0 \leq j < k \leq m} \sum P(B_j \cap B_k) \\ &\geq (m+1)(PB_0 - \sum_{k=1}^m P(B_0 \cap B_k)). \end{aligned}$$

Now from [11, Lemma 2.3], we record that

$$(2.2) \quad P(B_0 \cap B_k) \leq 2\psi(x) \{1 - \phi(x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}})\},$$

where  $\rho = \rho(ka \cdot \Delta(x))$ , and obtain

$$(2.3) \quad \lim_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \geq a^{-1} (1 - \overline{\lim}_{x \rightarrow \infty} 2 \sum_{k=1}^m \{1 - \phi(x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}})\}).$$

To study  $2 \sum_{k=1}^m \{1 - \phi(x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}})\}$  partition the sum into three parts according to i)  $ka \leq 1$ , ii)  $ka > 1$ ,  $ka \cdot \Delta(x) < \text{some } \delta$ , and iii)  $ka > 1$ ,  $ka \cdot \Delta(x) \geq \delta$ . First,  $\lim_{x \rightarrow \infty} \sum^{(i)} 2(1-\phi) = \sum^{(i)} \lim_{x \rightarrow \infty} 2(1-\phi)$ .

We may ignore the third sum  $\sum^{(iii)}$ . For  $ka \cdot \Delta(x) \geq \delta$ , there exists a positive  $\kappa$  such that  $1-\rho \geq \kappa$ , and

$$\begin{aligned} \sum^{(iii)} \{1 - \phi(x \frac{1-\rho}{1+\rho})^{\frac{1}{2}}\} &\leq \sum^{(iii)} \{1 - \phi(x(\kappa/2)^{\frac{1}{2}})\} \\ &\leq m\psi(x(\kappa/2)^{\frac{1}{2}}) \leq \frac{t}{a\Delta(x)} \exp(-\kappa x^2/4) \rightarrow 0 \end{aligned}$$

as  $x \rightarrow \infty$ .

For positive  $\delta_1$  sufficiently small,  $A_1(\delta_1) > 0$ . For  $\sum^{(ii)}$  when  $ka \cdot \Delta(x) < \delta_1$  estimate

$$\begin{aligned} x \frac{1-\rho}{1+\rho}^{\frac{1}{2}} &\geq \frac{x}{2} \sigma(ka \cdot \Delta(x)) = \frac{1}{2} \frac{\sigma(ka \cdot \Delta(x))}{\tilde{\sigma}(\Delta(x))} \\ &\geq \frac{A_1(\delta_1)}{2} \frac{\tilde{\sigma}(ka \cdot \Delta(x))}{\tilde{\sigma}(\Delta(x))} = \frac{A_1(\delta_1)}{2} (ka)^{\alpha/2} \left( \frac{H(ka \cdot \Delta(x))}{H(\Delta(x))} \right)^{\frac{1}{2}}. \end{aligned}$$

By property (1.7) in §1 and for  $ka > 1$ , there is a positive  $\delta_\alpha$  such that  $H(ka \cdot \Delta(x))/H(\Delta(x)) \geq (ka)^{-\alpha/2}$  provided  $ka \cdot \Delta(x) < \delta_\alpha$ . Take  $\delta = \min(\delta_\alpha, \delta_1, t)$ .

Consequently

$$\inf_{T \leq x < \infty} x \left( \frac{1-\rho}{1+\rho} \right)^{\frac{x}{2}} \geq \frac{A_1(\delta)}{2} (ka)^{\alpha/4}$$

for  $ka > 1$ ,  $ka \cdot \Delta(x) < \delta$  and  $T$  large.

Finally, defining  $a_k(x) = 2\{1 - \Phi(x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}})\}$  for  $ka \cdot \Delta(x) < \delta$  and  $a_k(x) \equiv 2\{1 - \Phi(2^{-1}(ka)^{\alpha/2})\}$  for  $ka \cdot \Delta(x) \geq \delta$ , we have

$$\sum_{k>a}^{-1} \sup_{T \leq x < \infty} a_k(x) \leq \sum_{k>a}^{-1} 2\{1 - \Phi\left(\frac{A_1(\delta)}{2} (ka)^{\alpha/4}\right)\} < \infty$$

It follows that

$$(2.4) \quad \overline{\lim}_{x \rightarrow \infty} \sum_{k=1}^{\infty} a_k(x) \leq \sum_{k=1}^{\infty} \overline{\lim}_{x \rightarrow \infty} a_k(x) = \sum_{k=1}^{\infty} 2(1 - \Phi(2^{-1}(ka)^{\alpha/2})) < \infty,$$

since

$$x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}} \sim \frac{x}{2} \sigma(ka \cdot \Delta(x)) \sim \frac{1}{2} \frac{\tilde{\sigma}(ka \cdot \Delta(x))}{\tilde{\sigma}(\Delta(x))} \rightarrow \frac{1}{2} (ka)^{\alpha/2}$$

as  $x \rightarrow \infty$ . Applying (2.4) in (2.3) completes the proof.  $\square$

The partition corresponding to  $Z_x(t)$  above was made to depend on  $a$ , and the lower estimate of the distribution of  $Z_x(t)$  (as well as the upper estimate) depends on  $a$ . Since we wish to take  $a \downarrow 0$ , it is seen that Lemma 2.1 is not sharp enough to obtain the desired result. Hence Lemma 2.2 will be needed.

Lemma 2.2. If the conditions of Theorem 2.1 hold, then for  $a > 0$

$$\lim_{x \rightarrow \infty} \frac{P[Z_x(na \cdot \Delta(x)) > x]}{\psi(x)} = H_\alpha(n, a) \equiv 1 + \int_0^\infty e^s P[\max_{1 \leq k \leq n} Y(ka) > s] ds < \infty.$$

Proof. Simplifying Pickands' proof [11, Lemma 2.2], we have

$$P[Z_x(na \cdot \Delta(x)) > x] = P[X(o) > x] + P[X(o) \leq x, \max_{1 \leq k \leq n} X(ka \cdot \Delta(x)) > x].$$

The second term equals

$$\int_{-\infty}^x P[\max_{1 \leq k \leq n} X(ka \cdot \Delta(x)) > x / X(o) = u] \phi(u) du,$$

where  $\phi(u)$  is the standard Gaussian density function. Substituting  $u = x-s/x$ , and defining  $Y_1(t) = x(X(t \cdot \Delta(x)) - x) + s$ , we obtain

$$\begin{aligned} \psi(x) & \int_0^{\infty} e^s P[\max_{1 \leq k \leq n} X(ka \cdot \Delta(x)) > x / X(o) = x-s/x] \exp(-s^2/(2x^2)) ds \\ & = \psi(x) \int_0^{\infty} e^s P[\max_{1 \leq k \leq n} Y_1(ka) > s / X(o) = x-s/x] \exp(-s^2/(2x^2)) ds. \end{aligned}$$

Note that

$$\begin{aligned} E(Y_1(t) / X(o) = x-s/x) & = x(\rho(t \cdot \Delta(x))(x-s/x) - x) + s \\ & = -x^2(1-\rho(t \cdot \Delta(x))) + s(1-\rho(t \cdot \Delta(x))) \\ & = -x^2 \tilde{\sigma}^2(\Delta(x)) \cdot |t|^\alpha / 2 + o(1) \\ & = -|t|^\alpha / 2 + o(1) \quad \text{as } x \rightarrow \infty; \end{aligned}$$

and that

$$\begin{aligned} \text{Cov}(Y_1(t_1), Y_1(t_2) / X(o) = x-s/x) & = x^2 [\rho((t_2-t_1) \cdot \Delta(x)) - \rho(t_1 \cdot \Delta(x)) \rho(t_2 \cdot \Delta(x))] \\ & = \frac{x^2}{2} [-\tilde{\sigma}^2(\Delta(x)) |t_2-t_1|^\alpha + \tilde{\sigma}^2(\Delta(x)) |t_1|^\alpha + \tilde{\sigma}^2(\Delta(x)) |t_2|^\alpha - \tilde{\sigma}^4(\Delta(x)) |t_1 t_2|^\alpha / 2] + o(1) \\ & = \frac{1}{2} [-|t_2-t_1|^\alpha + |t_1|^\alpha + |t_2|^\alpha] + o(1) \quad \text{as } x \rightarrow \infty \end{aligned}$$

Consequently  $P[\max_{1 \leq k \leq n} Y_1(ka) > s / X(o) = x-s/x] \rightarrow P[\max_{1 \leq k \leq n} Y(ka) > s]$  as  $x \rightarrow \infty$ , and an application of Boole's inequality and the Lebesgue dominated convergence theorem completes the proof.  $\square$

Lemma 2.3. If the conditions of Theorem 2.1 hold, then for  $a > 0$

$$(2.5) \quad \lim_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} = \frac{H_\alpha(a)}{a},$$

where  $0 < H_\alpha(a) \equiv \lim_{n \rightarrow \infty} H_\alpha(n, a)/n < \infty$ ,  $Z_x(t) = \max_{0 \leq k \leq m} X(ka \cdot \Delta(x))$ , and  $m = [t/(a\Delta(x))]$ .

Proof. This lemma corresponds to [11, Lemma 2.5]. For each non-negative integer

$k$ , let  $B_k = [X(ka \cdot \Delta(x)) > x]$ , and for an arbitrary positive integer  $n$ , let  $A_k = \bigcup_{j=(k-1)n}^{kn-1} B_j$ . Then

$$(2.6) \quad P\left(\bigcup_{k=1}^{m'} A_k\right) \leq P[Z_x(t) > x] \leq P\left(\bigcup_{k=1}^{m'+1} A_k\right),$$

where  $m' = [(m+1)/n]$ . By stationarity,  $P(A_k) = P(A_1)$  for all  $k \geq 1$ .

Consequently

$$P[Z_x(t) > x] \leq \sum_{k=1}^{m'+1} P(A_k) = (m'+1) P(A_1).$$

Now using Lemma 2.2, we obtain

$$(2.7) \quad \lim_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \leq \lim_{x \rightarrow \infty} \frac{(m'+1)PA_1}{(t/\Delta(x))\psi(x)} = H_\alpha(n-1, a)/na$$

On the other hand, (2.6) and stationarity imply

$$(2.8) \quad \begin{aligned} P[Z_x(t) > x] &\geq \sum_{k=1}^{m'} P(A_k) - \sum_{1 \leq k < j \leq m'} P(A_k \cap A_j) \\ &\geq m' P(A_1) - m' \sum_{j=2}^{m'} P(A_1 \cap A_j) \\ &\geq m' \left\{ PA_1 - \sum_{k=0}^{n-1} \sum_{\ell=n}^m P(B_k \cap B_\ell) \right\}. \end{aligned}$$

As in the proof of Lemma 2.1, inequality (2.2) applied to  $P(B_k \cap B_\ell) = P(B_0 \cap B_{\ell-k})$  and inequality (2.8) yield

$$(2.9) \quad \lim_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \geq \left\{ H_\alpha(n-1, a) - \lim_{x \rightarrow \infty} \sum_{k=0}^{n-1} \sum_{\ell=n}^m P(B_k \cap B_\ell) / \psi(x) \right\} / na$$



$$\geq \left\{ H_\alpha(n-1, a) - \sum_{k=1}^{n-1} \sum_{\ell=n}^{\infty} d_{\ell-k} \right\} / na$$

where  $d_j = 2\{1 - \Phi(\frac{1}{2}(ja)^{\alpha/2})\}$ . By (2.4) the  $\sum_{j=0}^{\infty} d_j < \infty$ , and therefore

$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \sum_{\ell=n}^{\infty} d_{\ell-k} / na = 0$ , by Kronecker's lemma.

Combining (2.7) and (2.9), we have

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \frac{H_\alpha(n-1, a)}{na} &\leq \overline{\lim}_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \\ &\leq \overline{\lim}_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \\ &\leq \overline{\lim}_{n \rightarrow \infty} \frac{H_\alpha(n-1, a)}{na}, \end{aligned}$$

and the conclusion of Lemma 2.2.

Now (2.7) implies  $H_\alpha(a) < \infty$ . By Lemma 2.1,  $H_\alpha(a) > 0$  for a sufficiently large, say for all  $a > \text{some } a_0$ . For any  $a > 0$ , there exists an integer  $m$  such that  $ma > a_0$ . Now  $H_\alpha(n, am) \leq H_\alpha(nm, a)$  implies  $H_\alpha(am) \leq mH_\alpha(a)$  and  $H_\alpha(a) > 0$ .  $\square$

Lemma 2.4. Under the same conditions as Theorem 2.1, it follows for  $a > 0$  and  $2^{-\alpha/4} < b < 1$  that

$$\overline{\lim}_{x \rightarrow \infty} \frac{P[X(0) \leq x, Z(a \cdot \Delta(x)) > x]}{\psi(x)} \leq M(a), \quad \text{and} \quad \lim_{a \rightarrow 0} \frac{M(a)}{a} = 0,$$

where

$$M(a) = (a/2)^{\alpha/2} \sum_{k=0}^{\infty} 2^{k(1-\alpha/2)} R((2/a)^{\alpha/2} (2^{\alpha/2} b)^k - 2^{-1} (a/2)^{\alpha/2} 2^{-\alpha k/2}),$$

and

$$R(x) = \int_x^{\infty} (1 - \Phi(s)) ds.$$

Proof. We combine Pickands' Lemmas 2.7 and 2.8 [11]. Note that

$$[X(0) \leq x, Z(a \cdot \Delta(x)) > x] \subseteq \bigcup_{k=0}^{\infty} D_k \quad \text{and} \quad D_k \subseteq \bigcup_{j=0}^{2^k-1} E_{j,k},$$

where

$$D_k = \left[ \max_{0 \leq j \leq 2^{k-1}} X(ja \cdot \Delta(x)/2^k) \leq x, \max_{0 \leq j \leq 2^{k+1}-1} X(ja \cdot \Delta(x)/2^{k+1}) > x + b^k/x \right],$$

and

$$E_{j,k} = [X(ja \cdot \Delta(x)/2^k) \leq x, X((2j+1)a \cdot \Delta(x)/2^{k+1}) > x + b^k/x].$$

By using [11, Lemma 2.6], we obtain  $P(E_{j,k}) \leq \psi(x) x \rho^{-1} (1-\rho^2)^{\frac{1}{2}} R(y)$ , where  $\rho = \rho(a \cdot \Delta(x)/2^{k+1})$ , and  $y = y(x) = b^k x^{-1} \rho (1-\rho^2)^{-\frac{1}{2}} - x(1+\rho)^{-1} (1-\rho^2)^{\frac{1}{2}}$ . Consequently

$$(2.10) \quad \overline{\lim}_{x \rightarrow \infty} \frac{P[X(0) \leq x, Z(a \cdot \Delta(x)) > x]}{\psi(x)} \leq \overline{\lim}_{x \rightarrow \infty} \sum_{k=0}^{\infty} 2^{k-1} \sum_{j=0}^{\infty} P(E_{j,k}) / \psi(x) \\ \leq \overline{\lim}_{x \rightarrow \infty} \sum_{k=0}^{\infty} 2^k x \rho^{-1} (1-\rho^2)^{\frac{1}{2}} R(y).$$

In order to apply the technique used in Lemma 2.1, we need to show

$$\sum_{k=0}^{\infty} 2^k \sup_{T \leq x < \infty} x \rho^{-1} (1-\rho^2)^{\frac{1}{2}} R(y) < \infty \quad \text{for some } T > 0.$$

That this sum is finite follows from the estimates:

$$x \rho^{-1} (1-\rho^2)^{\frac{1}{2}} \leq x \rho^{-1} \sigma(a \cdot \Delta(x)/2^{k+1}) \leq x S^{-1} A_2 \tilde{\sigma}(a \cdot \Delta(x)/2^{k+1}) \\ \leq S^{-1} A_2 (a 2^{-k-1})^{\alpha/2} (H(a \cdot \Delta(x)/2^{k+1})/H(\Delta(x)))^{\frac{1}{2}} \\ \leq S^{-1} A_2 (2^{-k} a/2)^{\alpha/2} (a/2^{k+1})^{-\alpha/4} \quad \text{by (1.6) in §1} \\ = S^{-1} A_2 (a/2)^{\alpha/4} 2^{-\alpha k/4},$$

and

$$y(x) = b^k x^{-1} \rho (1-\rho^2)^{-\frac{1}{2}} - x(1+\rho)^{-1} (1-\rho^2)^{\frac{1}{2}} \geq b^k S x^{-1/\sigma} - 2^{-1} x \sigma \\ \geq b^k S A_2^{-1} \tilde{\sigma}(\Delta(x)) / \tilde{\sigma}(a \Delta(x)/2^{k+1}) - 2^{-1} A_2 \tilde{\sigma}(a \Delta(x)/2^{k+1}) / \tilde{\sigma}(\Delta(x)) \\ \geq b^k S A_2^{-1} (2^k 2/a)^{\alpha/4} - 2^{-1} A_2 (2^k \cdot 2/a)^{-\alpha/4} \quad \text{by (1.6)}.$$

Here  $S \equiv \inf_{0 \leq s \leq a \cdot \Delta(x)} \rho(s) \geq 1 - \frac{1}{2}$ , and  $A_2 = A_2(a \cdot \Delta(x)) \leq 1 + \frac{1}{2}$  for  $x >$  some  $T$ .

Therefore, (2.10) yields

$$\overline{\lim}_{x \rightarrow \infty} \frac{P[X(0) \leq x, Z(a \cdot \Delta(x)) > x]}{\psi(x)} \leq \sum_{k=0}^{\infty} \overline{\lim}_{x \rightarrow \infty} 2^k x \rho^{-1} (1-\rho^2)^{\frac{1}{2}} R(y) = M(a),$$

since  $x(1-\rho^2)^{\frac{1}{2}} \rightarrow (a/2)^{\alpha/2} 2^{-\alpha k/2}$  as  $x \rightarrow \infty$ .

In order to see that  $M(a)/a \rightarrow 0$  as  $a \rightarrow 0$ , use the estimates  $R(x) \leq \psi(x)/x \leq \exp(-x^2/2)$  for  $x^2 \geq (2\pi)^{-1/2}$ ; note that one may disregard any finite number of the leading terms of  $M(a)$ ; and the factor  $e^{-ca^\alpha}$  out of the infinite sum  $M(a)$ . Here  $c > 0$  is a properly chosen constant.  $\square$

Proof of Theorem 2.1. See [11, Lemma 2.9]. There it is shown that  $H_\alpha = \lim_{a \rightarrow 0} H_\alpha(a)/a$ , and then

$$\lim_{a \rightarrow 0} H_\alpha(a)/a = \lim_{T \rightarrow \infty} T^{-1} \left( 1 + \int_0^\infty e^s P[\sup_{0 < t < T} Y(t) > s] ds \right). \quad \square$$

Remark 2.3. Pickands' Theorem 2.1 [11] concerning the expected number of  $\varepsilon$ -up-crossings is hereby generalized also.

3. An asymptotic 0-1 behavior. In this section, we use the results of §2 to obtain an extension of the results in Qualls and Watanabe [12]. We again postpone discussion of the non-stationary case to §4. Using the notation of §2, we have

Theorem 3.1. If  $\rho(t)$  satisfies (0.1) with  $0 < \alpha \leq 2$ ,  $\tilde{\sigma}(\cdot)$  is defined as in §2, and

$$(3.1) \quad \rho(t) = O(t^{-\gamma}) \text{ as } t \rightarrow \infty, \text{ for some } \gamma > 0;$$

then, for any positive non-decreasing function  $\phi(t)$  on some interval  $[a, \infty)$ ,

$$PE_\phi \equiv P\{\exists t_0(\omega) > a: X(t) \leq \phi(t) \text{ for all } t \geq t_0\} = 1 \text{ or } 0$$

as the integral

$$I(\phi) \equiv \int_a^\infty (\phi(t) \tilde{\sigma}^{-1}(1/\phi(t)))^{-1} \exp(-\phi^2(t)/2) dt$$

converges or diverges.

Remark 3.1. Monotone  $\tilde{\sigma}(\cdot)$  other than the one defined in §2 can be used; see Remark 2.1. Note that condition (3.1) implies  $\rho(t)$  is not periodic; consequently Theorem 2.1 is applicable.

Proof. For every  $\varepsilon > 0$ , assumption (0.1) implies that  $s^{\frac{\alpha+\varepsilon}{2}} \leq \tilde{\sigma}(s) \leq s^{\frac{\alpha-\varepsilon}{2}}$  and that  $s^{\frac{2}{\alpha-\varepsilon}} \leq \tilde{\sigma}^{-1}(s) \leq s^{\frac{2}{\alpha+\varepsilon}}$  for all positive  $s \leq$  some  $\delta$ . In particular, the integrand of  $I(\phi)$  is eventually a decreasing function of  $\phi$ .

(1) *The case when  $I(\phi) < \infty$ .*

Let  $t_n = n\Delta$ , where  $\Delta > 0$  and  $n = 0, 1, 2, \dots$ . By Theorem 2.1 and for fixed  $\Delta > 0$ , we have

$$\begin{aligned} & \sum_{n=n_0}^{\infty} P\left\{ \sup_{t_n \leq t \leq t_{n+1}} X(t) \geq \phi(t_n) \right\} \\ & \leq C_1 \sum_{n=n_0}^{\infty} (t_{n+1} - t_n) (\phi(t_n) \tilde{\sigma}^{-1}(1/\phi(t_n)))^{-1} \exp(-\phi^2(t_n)/2) \\ & = C_1 \sum_{n=n_0}^{\infty} (t_n - t_{n-1}) (\phi(t_n) \tilde{\sigma}^{-1}(1/\phi(t_n)))^{-1} \exp(-\phi^2(t_n)/2) \\ & \leq C_1 \int_{n_0\Delta}^{\infty} (\phi(t) \tilde{\sigma}^{-1}(1/\phi(t)))^{-1} \exp(-\phi^2(t)/2) dt < \infty, \end{aligned}$$

for  $n_0$  sufficiently large. Here  $C_1 > 0$  is a certain constant. So, the Borel-Cantelli lemma yields

$$P\left\{ \exists n_\phi(\omega): \sup_{t_n \leq t \leq t_{n+1}} X(t) \leq \phi(t_n) \text{ for all } n \geq n_\phi \right\} = 1;$$

and consequently  $PE_\phi = 1$ .  $\square$

(2) *The case when  $I(\phi) = \infty$ .*

For this part of the proof, we need the following lemma.

Lemma 3.1. If Theorem 3.1 when  $I(\phi) = \infty$  holds under the additional assumption that

$2\log t \leq \phi^2(t) \leq 3\log t$ , for all large  $t$ ,

then it holds without this additional assumption.

Proof. From the bounds on  $\tilde{\sigma}^{-1}(\cdot)$  given above, there are positive constants  $C_2$  and  $C_3$  such that

$$(3.2) \quad C_2 \int_a^\infty \phi(t)^{\frac{2}{\alpha+\epsilon}-1} \exp(-\phi^2(t)/2) dt \leq I(\phi) \leq C_3 \int_a^\infty \phi(t)^{\frac{2}{\alpha-\epsilon}-1} \exp(-\phi^2(t)/2) dt.$$

When  $0 < \alpha < 2$ , choose  $\epsilon > 0$  such that  $\alpha+\epsilon < 2$  and  $\alpha-\epsilon > 0$ . When  $\alpha = 2$ , it is well known that the  $H(s)$  in  $\sigma^2(s)$  cannot tend to zero; consequently, we may choose  $\epsilon = 0$  in the left hand side of (3.2) when  $\alpha = 2$ . We obtain for  $0 < \alpha \leq 2$  that

$$(3.3) \quad I(\phi) \geq C_2' \int_a^\infty \exp(-\phi^2(t)/2) dt \equiv C_2' J(\phi).$$

Let  $\phi(t)$  be an arbitrary positive non-decreasing function such that  $I(\phi) = \infty$ . Let  $\hat{\phi}(t) = \min(\max(\phi(t), (2\log t)^{\frac{1}{2}}), (3\log t)^{\frac{1}{2}})$ . To show  $I(\hat{\phi}) = \infty$ , we may assume  $\phi(t)$  crosses  $u(t) = (2\log t)^{\frac{1}{2}}$  infinitely often as  $t \rightarrow \infty$ . Otherwise, either  $\phi \leq u$  and  $I(\hat{\phi}) = I(u) = \infty$ , or  $\phi > u$  and  $I(\hat{\phi}) \geq I(\phi) = \infty$ , for some large  $a$ .

The proof of Lemma 1.4 in [12] now shows that  $J(\hat{\phi}) = \infty$ ; and by (3.3) that  $I(\hat{\phi}) = \infty$ .

That  $P[X(t) > \hat{\phi}(t) \text{ i.o.}] = 1$  implies  $P[X(t) > \phi(t) \text{ i.o.}] = 1$  follows from "Theorem 3.1 when  $I(v) < \infty$ " with  $v(t) = (3\log t)^{\frac{1}{2}}$ ; details are given in Lemma 4.1 in [15].  $\square$

The proof of the second part of Theorem 3.1 now proceeds in the same way as in Qualls and Watanabe [12]. We will use the same notation as in [12].

Define a sequence of intervals by  $I_n = [n\Delta, n\Delta + \beta]$  for  $\Delta > 0$  and  $0 < \beta < \Delta$ . Let  $G_k = \{t_{k,v} = k\Delta + v/n_k : v=0,1,\dots, [8n_k]\}$  be points in  $I_k$  where  $n_k = [(\tilde{\sigma}^{-1}(1/\phi(k\Delta + \beta)))^{-1}]$ . Let  $E_k = [\max_{s \in G_k} X(s) \leq \phi(k\Delta + \beta)]$ . Now using Lemma 2.3 in the same way as in [12], we see that  $I(\phi) = \infty$  implies  $\Sigma P(E_k^c) = \infty$ .

So, we only need to prove the asymptotic independence of the  $E_k$ 's, that is,

$$(3.4) \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \left| P(\cap_{k=1}^m E_k) - \prod_{k=1}^m P E_k \right| = 0.$$

Now by the use of Lemma 3.1 and the bounds on  $\tilde{\sigma}^{-1}(\cdot)$ , we have  $\phi^2(k\Delta + \beta) \geq 2 \log(k\Delta + \beta)$  and

$$n_k \leq \phi(k\Delta + \beta)^{\frac{2}{\alpha - \varepsilon}} \leq (3 \log(k\Delta + \beta))^{\frac{1}{\alpha - \varepsilon}},$$

where  $\varepsilon > 0$  and  $\alpha - \varepsilon > 0$ . Now the proof of (3.4) given in [12] applies without change.  $\square$

Remark 3.2. By the use of inequalities (3.2) and Theorem 3.1, we can easily show that for every  $\varepsilon > 0$ ,

$$P \left[ \frac{1}{2} + \frac{1}{\alpha + \varepsilon} \leq \lim_{x \rightarrow \infty} \frac{(2 \log t)^{\frac{1}{2}} (Z(t) - (2 \log t)^{\frac{1}{2}})}{\log \log t} \leq \frac{1}{2} + \frac{1}{\alpha - \varepsilon} \right] = 1;$$

and consequently

$$P \left[ \lim_{x \rightarrow \infty} \frac{(2 \log t)^{\frac{1}{2}} (Z(t) - (2 \log t)^{\frac{1}{2}})}{\log \log t} = \frac{1}{2} + \frac{1}{\alpha} \right] = 1.$$

It is interesting that this is true whatever  $H$  may be (as long as it satisfies the assumptions of Theorem 3.1).

Remark 3.3. Generally speaking, it seems to be difficult to compute  $I(\phi)$  in the criterion of Theorem 3.1 in concrete examples. Of course, inequalities (3.2) may be used except in the critical cases.

4. The non-stationary case. It is not surprising that Slepian's result [14, Theorem 1] can be used to generalize §3. See §2 of [12]. It is more interesting that Slepian's result can be used to generalize §2.

Let  $X(t)$  be a separable Gaussian process with zero mean function and correlation function  $\rho(t,s)$ . We adopt the notation of §2 with modifications to the non-stationary case. At first, we only assume that

$$(4.1) \quad 1 - C_1 h^\alpha H(h) \leq \rho(\tau, \tau+h) \leq 1 - C_2 h^\alpha H(h)$$

for  $0 < h < \delta_{12}$  and  $0 \leq \tau \leq t$ , where  $0 < \alpha \leq 2$  and  $H$  is slowly varying at zero. Without loss of generality, we take  $H$  to be "normalized". By §1 there exist separable zero mean stationary processes  $Y_1(t)$  and  $Y_2(t)$  with covariance functions satisfying  $q_1(h) \sim 1 - C_3 h^\alpha H(h)$  and  $q_2(h) \sim 1 - C_4 h^\alpha H(h)$  as  $h \rightarrow 0$ , respectively. For  $C_4 < C_2 < C_1 < C_3$ , we have

$$(4.2) \quad q_1(h) \leq \rho(\tau, \tau+h) \leq q_2(h) \quad \text{for } 0 < h < \delta_{34}$$

and  $0 \leq \tau \leq t$ . We shall use the subscripts 1 and 2 through out to correspond to the stationary processes  $Y_1(\cdot)$  and  $Y_2(\cdot)$ , respectively.

Remark 4.1. Section §1 does not cover the case  $\alpha = 2$ . For a covariance, say  $q_1(h)$ , with  $\alpha = 2$ , we know  $H(h)$  has a limit, that the zero limit must be excluded, and that a finite limit is easily handled. The interesting case is when  $H(s) \rightarrow \infty$ . So when  $\alpha = 2$  we assume (4.2) instead of (4.1).

In order to exclude any type of periodic case, we assume  $\kappa = \sup\{\rho(\tau, \tau+h) : \delta_{34} \leq h, 0 \leq \tau+h \leq t\} < 1$ .

Theorem 4.1. If  $X(\cdot)$  satisfies (4.1) and  $\kappa < 1$ , then for  $a > 0$

$$(4.3) \quad C_2^{1/\alpha} \frac{H_\alpha(a)}{a} - 2(C_1^{1/\alpha} - C_2^{1/\alpha}) \frac{H_\alpha(a)}{a} \leq \lim_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)}$$

$$\begin{aligned}
 & \leq \overline{\lim}_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} \leq C_1^{1/\alpha} \frac{H_\alpha(a)}{a}, \\
 (4.4) \quad C_2^{1/\alpha} H_\alpha - 2(C_1^{1/\alpha} - C_2^{1/\alpha}) H_\alpha & \leq \overline{\lim}_{x \rightarrow \infty} \frac{P[Z(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} \\
 & \leq \overline{\lim}_{x \rightarrow \infty} \frac{P[Z(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} \leq C_1^{1/\alpha} H_\alpha,
 \end{aligned}$$

where  $\tilde{\sigma}^2(h) \equiv 2|h|^\alpha H(h)$ .

Moreover if  $X(\cdot)$  satisfies (0.1) with  $0 < \alpha \leq 2$  and  $\kappa < 1$ , then for  $a > 0$

$$(4.5) \quad \lim_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} = \frac{H_\alpha(a)}{a},$$

and

$$(4.6) \quad \lim_{x \rightarrow \infty} \frac{P[Z(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} = H_\alpha.$$

Proof. The proof consists of showing that the results only depend on the "local condition" for  $\rho(\tau, \tau+h)$  instead of on the total time interval  $(0, t)$ . Let the integer  $M$  be large enough that  $\delta = t/M$  is less than  $\delta_{34}/2$ . Define

$$A_j = [ \max_{(j-1)\delta \leq ka \cdot \Delta(x) < j\delta} X(ka \cdot \Delta(x)) > x ].$$

That  $PA_j \leq P_1 A_j \equiv P_1 A_1$  is Slepian's result in the non-stationary case together with the fact that  $P_1$  is a stationary measure. Since  $\Delta(x) \sim C_3^{1/\alpha} \Delta_1(x)$  as  $x \rightarrow \infty$ , Theorem 2.1 (or rather Lemma 2.3) yields

$$(4.7) \quad \overline{\lim}_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t(x)/\Delta(x)} \leq \overline{\lim}_{x \rightarrow \infty} \frac{P_1[Z_x(\delta) > x]}{\delta\psi(x)/\Delta(x)} = C_3^{1/\alpha} \frac{H_\alpha(a)}{a}$$

Similarly

$$(4.8) \quad \overline{\lim}_{x \rightarrow \infty} \frac{P[Z(t) > x]}{t\psi(x)/\Delta(x)} \leq \overline{\lim}_{x \rightarrow \infty} \frac{P_1[Z(\delta) > x]}{\delta\psi(x)/\Delta(x)} = C_3^{1/\alpha} H_\alpha.$$



For lower bounds, we consider

$$(4.9) \quad P[Z_x(t) > x] \geq \sum_{j=1}^M PA_j - \sum_{1 \leq i < j \leq M} P(A_i \cap A_j).$$

For  $j-i \geq 2$  in the double sum, we use a well-known device (see, e.g., Lemma 1.5 in [12]) to obtain

$$\left| P(A_i \cap A_j) - PA_i PA_j \right| \leq K \sum_{k=1}^m \sum_{k=1}^m |\rho| \frac{\exp(-x^2/(1+\rho))}{(1-\rho^2)^{\frac{1}{2}}} \leq K^1 m^2 \exp(-x^2/(1+\kappa)),$$

where  $m = [t/(a\Delta(x))]$ .

Dividing by  $t\psi(x)/\Delta(x)$ , we see that the error term and  $PA_i PA_j$  approach zero as  $x \rightarrow \infty$ ; and therefore we may ignore this part of the double sum. For  $j-i = 1$ ,

$$(4.10) \quad \sum_{j=1}^{M-1} P(A_j \cap A_{j+1}) < M(2P_1(A_1) - P_2(A_1 \cup A_2)).$$

Using (4.10) in (4.9), we have

$$(4.11) \quad \begin{aligned} \lim_{x \rightarrow \infty} \frac{P[Z(t) > x]}{t\psi(x)/\Delta(x)} &\geq \lim_{x \rightarrow \infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \\ &\geq \lim_{x \rightarrow \infty} \frac{P_2[Z_x(\delta) > x]}{\delta\psi(x)/\Delta(x)} - 2 \lim_{x \rightarrow \infty} \left\{ \frac{P_1[Z_x(\delta) > x]}{\delta\psi(x)/\Delta(x)} - \frac{P_2[Z_x(2\delta) > x]}{2\delta\psi(x)/\Delta(x)} \right\} \\ &= C_4^{1/\alpha} \frac{H_\alpha(a)}{a} - 2(C_3^{1/\alpha} - C_4^{1/\alpha}) \frac{H_\alpha(a)}{a}. \end{aligned}$$

Choosing  $C_3 = C_1$ ,  $C_4 = C_2$ , and letting  $a \rightarrow 0$  in (4.7), (4.8) and (4.11), we obtain (4.3) and (4.4). Choosing  $C_1 = C_2 = 1$  in (4.3) and (4.4), we obtain (4.5) and (4.6).  $\square$

Of course, Theorem 3.1 of §3 can be generalized easily to a result analogous to Theorems 2.1 and 2.3 of [12]. We write the following theorem without proof.

Theorem 4.2. If  $X(\cdot)$  satisfies (4.1) with  $0 < \alpha \leq 2$  for  $0 < h < \delta$  and all  $\tau > T$ , and

$$(4.12) \quad \rho(\tau, \tau+s) = O(s^{-\gamma}) \text{ uniformly in } \tau \text{ as } s \rightarrow \infty \text{ for some } \gamma > 0,$$

then, for any positive non-decreasing function  $\phi(t)$  on some  $[a, \infty)$ ,

$$P[X(t) > v(t)\phi(t) \text{ i.o. in } t] = 0 \text{ or } 1$$

as the integral  $I(\phi) < \infty$  or  $= \infty$ .

## References

- (1) Adamović, D. Sur quelques propriétés des fonctions à croissance lente de Karamata. I, II. *Mat. Vesnik* 3(18) 1966, 123-136, 161-172.
- (2) Feller, W. *An Introduction to Probability Theory and Its Applications, Vol. II*, Wiley, New York, 1966.
- (3) Garsia, A. and Lamperti, J. A discrete renewal theorem with infinite mean. *Comment. Math. Helv.* 37 (1963), 221-234.
- (4) Ibragimov, I.A. and Linik, Yu.V. *Independent and Stationary connected variables*, Nauka Moscow 1965. (In Russian).
- (5) Karamata, J. Sur un mode de croissance régulière des fonctions. *Mathematica (Cluj)* 4 (1930), 38-53.
- (6) Kôno, N. On the modulus of continuity of sample functions of Gaussian processes. *Jour. Math. Kyoto Univ.* 10 (1970), 493-536.
- (7) Landau, H.J. and Shepp, L.A. On the supremum of a Gaussian process. (To appear.)
- (8) Marcus, M.B. Hölder condition for Gaussian processes with stationary increments. *Trans. AMS* 134 (1968), 29-52.
- (9) Marcus, M.B. and Shepp, L.A. Continuity of Gaussian processes. *Trans. AMS* 151 (1970), 377-392.
- (10) Nisio, M. On the continuity of stationary Gaussian processes. *Nagoya. Math. Jour.* 34 (1969), 89-104.
- (11) Pickands, J., III. Upcrossing probabilities for stationary Gaussian processes. *Trans. AMS* 145 (1969), 51-73.
- (12) Qualls, C. and Watanabe, H. An asymptotic 0-1 behavior of Gaussian processes. *Univ. of N. Car., Chapel Hill, Inst. of Stat. Mimeo No. 725* (1970).
- (13) Sirao, T. and Watanabe, H. On the upper and lower class for stationary Gaussian processes. *Trans. AMS* 147 (1970), 301-331.
- (14) Slepian, D. The one-sided barrier problem for Gaussian noise. *Bell Sys. Tech. J.* 41 (1962), 463-501.
- (15) Watanabe, H. An asymptotic property of Gaussian process. *Trans. AMS* 148 (1970), 233-248.

## Footnotes

AMS Subject Classifications: Primary 60G15, 42A68, 60G10; Secondary 60F20, 60G17.

*Key Phrases:* Stationary processes (existence), 2. Fourier transforms, 3. Tauberian theorem for Fourier transforms, 4. Regular variation, 5. Slow variation, 6. Supremum of a Gaussian process, 7. Extreme values of normal processes, 8. Upcrossing probabilities, 9. Law of the iterated logarithm, 10. Zero-one laws, 11. Continuity of sample functions.

1. Research supported in part by the Office of Naval Research under Contract N00014-67-A-0321-0002.

2. Research supported in part by the National Science Foundation under Grant GU-2059.