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WAVE-LENGTH AND AMPLITUDE FOR A STATIONARY PROCESS AFTER A HIGH MAXIMUM *

by

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1. Introduction.

Let $\{\xi(t), t\in R\}$ be a stationary, zero-mean Gaussian process with covariance function r and assume $r(0) = 1$, $-r''(0) = \lambda_2$. The object of this paper is to study the distribution of the two wave-characteristics wave-length and amplitude, i.e. the horizontal and vertical distances between "a randomly chosen" local maximum and the following minimum, especially when the maximum is very high. The main tool is a random process

$$\xi_u(t) = ur(t) - \eta_u(\lambda_2 r(t)+r''(t)) + \Delta(t)$$

(where η_u is a certain random variable and Δ is a certain non-stationary Gaussian process). The sample paths of the process $\{\xi_u(t), t\in R\}$ have (almost surely) a local maximum of height u at $t = 0$ and they can be used to describe the behaviour of the original process ξ in the neighborhood of "a randomly chosen" local maximum of height u. (For "horizontal window", h.w., conditional processes, see Kac and Slepian [4], Geman [1] and, for this special topic, Lindgren [6].)

Let the wave-length $\tau_u > 0$ be the time for the first local minimum of ξ_u, and let $\delta_u = u-\xi_u(\tau_u)$ be the corresponding amplitude.

The asymptotic behaviour of (τ_u, δ_u) as $u \to -\infty$ has been treated by Lindgren [7] who also gives moment approximations to the distribution for moderate u-values, [8].

Here we will concentrate upon the case $u \to +\infty$ for the cases i) and ii) defined below. Then the dominant term in the definition of $\xi_u(t)$ will be

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$ur(t)$ and it is seen that the behaviour of the process after a very high maximum is well determined by the behaviour of its covariance function; the process "follows its covariance function". We have then mainly the following three cases where we say that r has a stationary point at t_o if $r'(t_o) = 0$.

i) r has a first local minimum at $t_o > 0$ and has no stationary points in $(0, t_o)$.

ii) r has a first local minimum at $t_o > 0$ and has at least one stationary point in $(0, t_o)$.

iii) r has no stationary points in $(0, \infty)$.

In case i) we will prove that, after suitable normalizations, both $\tau_u - t_o$ and $\delta_u - u(1-r(t_o))$ have asymptotic normal distributions while in case ii) there will be a positive probability that τ_u falls near some of the stationary points less than t_o. Then the asymptotic normal distributions have to be modified. This is done in Section 3 and 4 respectively. In case iii), then for every $t > 0$ the probability will tend to one that $\xi_u(\cdot)$ is strictly decreasing in $(0,t)$ and hence $\tau_u \to \infty$ in probability as $u \to \infty$.

The results and methods of proofs in case iii) are quite different from those in case i) and ii) and therefore case iii) will be dealt with in a separate paper.

2. SOME DEFINITIONS AND GENERAL RESULTS.

Suppose the covariance function r is four times continuously differentiable with

$$r(0) = 1, \quad -r''(0) = \lambda_2, \quad r^{IV}(0) = \lambda_4$$

$$r^{IV}(t) = \lambda_4 + O(\left|\log|t|\right|^{-a}) \quad \text{as} \quad t \to 0 \quad \text{for some} \quad a > 1$$

$$r(t) \to 0 \quad \text{as} \quad t \to \infty$$

and put $\beta = (\lambda_4-\lambda_2^2)/\lambda_2$. Adopt the following functions from [6]:

$$C(s,t) = r(s-t) - [\lambda_2(\lambda_4-\lambda_2^2)]^{-1} \{\lambda_2\lambda_4 r(s)r(t) + \lambda_2^2 r(s)r''(t) + (\lambda_4-\lambda_2^2)r'(s)r'(t) + \lambda_2^2 r''(s)r(t) + \lambda_2 r''(s)r''(t)\}$$

$$c(s,t) = \frac{\partial^2 C(s,t)}{\partial s\partial t} = -r''(s-t) - [\lambda_2(\lambda_4-\lambda_2^2)]^{-1} \{\lambda_2\lambda_4 r'(s)r'(t) + \lambda_2^2 r'(s)r'''(t) + (\lambda_4-\lambda_2^2)r''(s)r''(t) + \lambda_2^2 r'''(s)r'(t) + \lambda_2 r'''(s)r'''(t)\}$$

$$\Psi(x) = \phi(x) + x\Phi(x)$$

where ϕ and Φ are the standardized normal density and cumulative distribution functions. Also let η_u be a random variable (r.v.) with the density

$$(2.1)\qquad q_u^*(y) = \begin{cases} 0 & y < -u/\beta \\ \dfrac{\lambda_2\beta(u/\beta+y)\exp(-\lambda_2\beta y^2/2)}{\sqrt{2\pi}\ \Psi(u\sqrt{\lambda_2/\beta})} & y \geq -u/\beta \end{cases}$$

and let $\{\Delta(t), t\in R\}$ be a non-stationary, zero-mean, Gaussian process, independent of η_u and with the covariance function $C(s,t)$. (That C is a non-negative definite function is proved in Lemma 2 of [6].) The process Δ can be chosen to have, with probability one, twice continuously differentiable sample functions, the derivatives of which

$$\delta(t) = \Delta'(t)$$

constitute a non-stationary, zero-mean, Gaussian process with the covariance function $c(s,t)$. This can be proved in a similar way as Lemma 1.1 of [7] if one makes use of a weaker condition for the existence of sample derivatives given by Leadbetter and Weissner [5]. Note that the distributions of $\Delta(t)$ and $\delta(t)$ do not depend on u.

Define the processes

$$(2.2)\qquad \xi_u(t) = ur(t) - \eta_u(\lambda_2 r(t)+r''(t)) + \Delta(t)$$

$$(2.3)\qquad \xi_u'(t) = ur'(t) - \eta_u(\lambda_2 r'(t)+r'''(t)) + \delta(t)$$

and the wave-length and amplitude

(2.4) τ_u = first local minimum of $\xi_u(t)$

= first upcrossing zero of $\xi_u'(t)$

$\delta_u = u - \xi_u(\tau_u)$.

We sum up some properties of ξ_u and ξ_u'.

PROPOSITION 2.1.

a) ξ_u has a local maximum of height u at 0 (a.s.).

b) $\xi_u'(t) < 0$ for all sufficiently small positive t (a.s.).

c) Given that ξ has a local maximum with height u at t_o (in h.w. sense) $\xi(t_o+t)$ and $\xi'(t_o+t)$ have the same distribution as $\xi_u(t)$ and $\xi_u'(t)$.

d) The wave-length and amplitude of ξ after a local maximum with height u (in h.w. sense) have the same distribution as τ_u and δ_u.

PROOF. Rewriting $\xi_u(t)$ as

$$\xi_u(t) = u \cdot \frac{\lambda_4 r(t)+\lambda_2 r''(t)}{\lambda_4 - \lambda_2^2} + \Delta(t) - \zeta \cdot \frac{\lambda_2 r(t)+r''(t)}{\lambda_4 - \lambda_2^2}$$

where the r.v. $\zeta = \lambda_2\beta(u/\beta + \eta_u)$ has the density $q_u(z) = q_u^*((z-\lambda_2 u)/\lambda_2\beta)/\lambda_2\beta$ $(z > 0)$, we recognize (2.2) and (2.3) as the processes given by (1.1) and (1.2) in [7]. Then part a) and part b) follow from the proof of Lemma 1.1 in [8]. Part c) and part d) are essentially the ergodic theorems 1.1 and 1.2 in [8]. We only have to assertain that $\{\xi(t), t\epsilon R\}$ is an ergodic process. But since $r(t) \to 0$ as $t \to \infty$ the spectral distribution of $\xi(t)$ can have no discrete part. From a theorem of Maruyama [9] and Grenander [2], it follows that $\xi(t)$ is ergodic.

LEMMA 2.1. The r.v. $\eta_u\sqrt{\lambda_2\beta}$ has an asymptotic standard normal distribution as $u \to \infty$ and its density tends to ϕ with dominated convergence.

PROOF. The function $x/\Psi(x)$ increases to one as $x \to \infty$. Therefore the density

$$(2\pi)^{-\frac{1}{2}}\, exp(-\tfrac{1}{2}x^2)(u\sqrt{\lambda_2/\beta} + x\sqrt{\lambda_2\beta})/\Psi(u\sqrt{\lambda_2/\beta})$$

of $\eta_u\sqrt{\lambda_2\beta}$ tends to $\phi(x)$ with dominated convergence.

The lemma implies that the term $\eta_u(\lambda_2 r'(t)+r'''(t))$ of $\xi_u'(t)$ is of moderate order as $u \to \infty$, and so is the $\delta(t)$-term in any bounded interval (since it has continuous sample functions). Hence we expect that, for large u, $\xi_u'(t)$ can be zero only if $r'(t)$ is very close to zero. To express it more precisely, we have the following lemma.

LEMMA 2.2. If I is a bounded, measurable set of non-negative times and if

$$I_\varepsilon = I \cap \{t;\ t \ge \varepsilon\}$$

then $\inf_{t\in I_\varepsilon} |r'(t)| > 0$ for all $\varepsilon > 0$ implies

$$P(\tau_u \in I) \to 0 \quad \text{as} \quad u \to \infty.$$

PROOF. There is an $\varepsilon > 0$ such that $r'(t) < 0$ for $0 < t \le \varepsilon$. Since

$$P(\tau_u \in I) \le P(\tau_u \le \varepsilon) + P(\tau_u \in I_\varepsilon)$$

it is sufficient to prove that a) $P(\tau_u \le \varepsilon) \to 0$, b) $P(\tau_u \in I_\varepsilon) \to 0$.

a) r' and r''' are continuously differentiable with $r'(0) = r'''(0) = 0$, $r''(0) = -\lambda_2 < 0$, so that $\inf_{0\le t\le\varepsilon} t^{-1}|r'(t)| > 0$, $\sup_{0\le t\le\varepsilon} t^{-1}|\lambda_2 r'(t) + r'''(t)| < \infty$. Since, furthermore, δ has continuously differentiable sample functions with $\delta(0) = 0$ (a.s.), we have that $\sup_{0\le t\le\varepsilon} t^{-1}|\delta(t)|$ is a well-defined, finite r.v.. Thus, for $0 < t \le \varepsilon$,

$$\xi_u'(t) \le -ut \cdot \inf\left|\frac{r'(t)}{t}\right| + |\eta_u| t \cdot \sup\left|\frac{\lambda_2 r'(t)+r'''(t)}{t}\right| + t \cdot \sup\left|\frac{\delta(t)}{t}\right|$$

which is strictly negative if

$$|\eta_u| < \frac{u\cdot \inf|r'(t)/t| - \sup|\delta(t)/t|}{\sup|(\lambda_2 r'(t)+r'''(t))/t|} .$$

This occurs with high probability if u is large and therefore $P(\tau_u \le \varepsilon) = 1 - P(\xi_u'(t)<0 \text{ for } 0<t\le\varepsilon)$ is arbitrarily close to zero for large u.

b) It remains to prove that $P(\tau_u \in I_\varepsilon) \to 0$ as $u \to \infty$. Take T so that $I_\varepsilon \in [\varepsilon, T]$. If we write $m = \inf_{I_\varepsilon} |r'(t)| > 0$, $M = \sup_{[\varepsilon,T]}|\lambda_2 r'(t) + r'''(t)| < \infty$ we can estimate $|\xi_u'(t)|$ in terms of $|\eta_u|$ and $\sup_{[\varepsilon,T]}|\delta(t)|$. Thus for $|\eta_u| \le um/2M$ we get $\inf_{I_\varepsilon}|\xi_u'(t)| \ge um - |\eta_u|M - \sup_{I_\varepsilon}|\delta(t)|$ and conclude that

$$P(\sup_{[\varepsilon,T]} |\delta(t)| < um/2) \le P(\inf_{I_\varepsilon}|\xi_u'(t)| > 0) + P(|\eta_u| > um/2M)$$

$$P(\tau_u \in I_\varepsilon) = 1 - P(\inf_{I_\varepsilon}|\xi_u'(t)|>0) \le 1 - P(\sup_{[\varepsilon,T]}|\delta(t)|<um/2)$$
$$+ P(|\eta_u|>um/2M).$$

The last probability in the right hand side tends to zero as $u \to \infty$. Since the sample functions of δ are continuous they are bounded over the compact interval $[\varepsilon,T]$ and thus the first probability tends to one. This proves that $\lim_{u\to\infty} P(\tau_u \in I_\varepsilon) = 0$.

3. ASYMPTOTIC NORMALITY IN CASE i).

We specify the conditions for case i).

C1 There is a time $t_o > 0$ and a positive integer k_o such that the covariance function r is $2k_o$ times continuously differentiable near t_o and

$$r'(t) < 0 \quad \text{for} \quad 0 < t < t_o$$
$$r^{(j)}(t_o) = 0 \quad \text{for} \quad j = 1, 2, \ldots, 2k_o-1$$
$$r^{(2k_o)}(t_o) > 0.$$

The asymptotic distribution of (τ_u, δ_u) will now be stated in terms of two independent normal r.v. χ_o and ψ_o, defined as follows. Let

η be $N(0, 1/\sqrt{\lambda_2\beta})$ and independent of $(\delta(t_o), \Delta(t_o))$, and let

$$(3.1) \quad \begin{aligned} \chi_o &= \eta r'''(t_o) - \delta(t_o) \\ \psi_o &= \eta(\lambda_2 r(t_o) + r''(t_o)) - \Delta(t_o). \end{aligned}$$

Then (χ_0, ψ_0) has a bivariate normal distribution with mean zero and the covariances

$$\begin{aligned} V(\chi_o) &= r'''(t_o)^2 V(\eta) + V(\delta(t_o)) \\ &= r'''(t_o)^2/\lambda_2\beta + c(t_o,t_o) = \lambda_2 - r''(t_o)^2/\lambda_2 \\ Cov(\chi_o,\psi_o) &= r'''(t_o)(\lambda_2 r(t_o)+r''(t_o))V(\eta) - Cov(\delta(t_o),\Delta(t_o)) \\ &= r'''(t_o)(\lambda_2 r(t_o)+r''(t_o))/\lambda_2\beta - \left.\frac{\partial C(s,t)}{\partial s}\right|_{s=t=t_o} = 0 \\ V(\psi_o) &= (\lambda_2 r(t_o)+r''(t_o))^2 V(\eta) + V(\Delta(t_o)) \\ &= (\lambda_2 r(t_o)+r''(t_o))^2/\lambda_2\beta + C(t_o,t_o) = 1 - r(t_o)^2 . \end{aligned}$$

Here we have made repeated use of the fact that $r'(t_o) = 0$.

Note that since both r and $-r''$ are covariance functions we have $|r(t_o)| \le r(0) = 1$, and $|-r''(t_o)| \le |-r''(0)| = \lambda_2$ so that $1-r(t_o)^2 \ge 0$, and $\lambda_2 - r''(t_o)^2/\lambda_2 \ge 0$.

THEOREM 3.1. If condition C1 is fulfilled with t_o and k_o then, as $u \to \infty$

a) $\tau_u \xrightarrow{P} t_o$

b) $(u(\tau_u - t_o)^{2k_o-1}, \ \delta_u - u(1-r(t_o))) \xrightarrow{L} \left(\frac{(2k_o-1)!}{r^{(2k_o)}(t_o)} \chi_o, \psi_o\right)$.

Here $\xrightarrow{P}$ and $\xrightarrow{L}$ means convergence in probability and law respectively. The theorem can be restated as follows.

THEOREM 3.2. If condition C1 is fulfilled with t_o and k_o then

a') $\tau_u \xrightarrow{P} t_o$ as $u \to \infty$

b') $u(\tau_u - t_o)^{2k_o-1}$ is $AsN(0, \frac{(2k_o-1)!}{r^{(2k_o)}(t_o)} \sqrt{\lambda_2 - r''(t_o)^2/\lambda_2}\ )$

c') $\delta_u - u(1-r(t_o))$ is $AsN(0, \sqrt{1 - r(t_o)^2})$

d') τ_u and δ_u are asymptotically independent.

Proof of Theorem 3.1.

a) First we notice that, for all sufficiently small $\varepsilon > 0$, the closed interval $[0, t_o-\varepsilon]$ fulfills the requirements of Lemma 2.2. Therefore, $P(\tau_u \le t_o-\varepsilon) \to 0$ as $u \to \infty$. Furthermore, the covariance derivative $r'(t_o+\varepsilon)$ is strictly positive for small positive ε, and since the sample derivative

$$\xi_u'(t_o+\varepsilon) = ur'(t_o+\varepsilon) - \eta_u(\lambda_2 r'(t_o+\varepsilon) + r'''(t_o+\varepsilon)) + \delta(t_o+\varepsilon)$$

is positive if

$$\eta_u(\lambda_2 r'(t_o+\varepsilon) + r'''(t_o+\varepsilon)) - \delta(t_o+\varepsilon) \le ur'(t_o+\varepsilon)$$

we conclude that $P(\xi_u'(t_o+\varepsilon)>0) \to 1$ as $u \to \infty$. Since ξ_u' has continuous sample functions (a.s.) $\xi_u'(t_o+\varepsilon) > 0$ implies $\tau_u < t_o + \varepsilon$ except for a set of probability zero. Thus

$$P(t_o - \varepsilon \le \tau_u \le t_o + \varepsilon) \ge P(\xi_u'(t_o+\varepsilon) > 0) - P(\tau_u \le t_o - \varepsilon)$$

and, as asserted for every $\varepsilon > 0$,

$$P(t_o - \varepsilon \le \tau_u \le t_o + \varepsilon) \to 1 \quad \text{as} \quad u \to \infty.$$

b) From part a), we know that $\nabla = \tau_u - t_o$ tends to zero in probability as $u \to \infty$. Therefore, we can expand the functions r, r', r'', and r''' as well as the random functions δ and Δ in Taylor series in the neighborhood of t_o. If we write k instead of k_o and employ the symbol $o_p(1)$ for any r.v. $\xrightarrow{P} 0$ as $u \to \infty$, then $\nabla = o_p(1)$ and we have

$$r(\tau_u) = r(t_o) + \frac{\nabla^{2k}}{(2k)!}(r^{(2k)}(t_o)+o_p(1)) = r(t_o) + o_p(1)$$

$$r'(\tau_u) = \frac{\nabla^{2k-1}}{(2k-1)!}(r^{(2k)}(t_o)+o_p(1)) = o_p(1)$$

$$r''(\tau_u) = r''(t_o) + o_p(1) \quad \text{and} \quad r'''(\tau_u) = r'''(t_o)+ o_p(1)$$

$$\delta(\tau_u) = \delta(t_o) + o_p(1) \quad \text{and} \quad \Delta(\tau_u) = \Delta(t_o) + o_p(1).$$

It will soon be evident that $u\nabla^{2k} = o_p(1)$ so that

$$ur(\tau_u) = ur(t_o) + o_p(1)$$

$$ur'(\tau_u) = \frac{u\nabla^{2k-1}}{(2k-1)!}(r^{(2k)}(t_o) + o_p(1))$$

$$\eta_u(\lambda_2 r(\tau_u)+r''(\tau_u)) = \eta_u(\lambda_2 r(t_o)+r''(t_o)) + o_p(1)$$

$$\eta_u(\lambda_2 r'(\tau_u)+r'''(\tau_u)) = \eta_u r'''(t_o)+ o_p(1).$$

Here we used that $\eta_u \cdot o_p(1) = o_p(1)$. Combining the expansions and writing

$$\chi_o^{\ u} = \eta_u r'''(t_o) - \delta(t_o)$$

$$\psi_o^{\ u} = \eta_u(\lambda_2 r(t_o)+r''(t_o)) - \Delta(t_o)$$

we obtain

$$\xi_u(\tau_u) = ur(t_o) - \psi_o^{\ u} + o_p(1)$$

$$\xi_u'(\tau_u) = \frac{u\nabla^{2k-1}}{(2k-1)!}(r^{(2k)}(t_o)+o_p(1)) - \chi_o^{\ u} + o_p(1).$$

Up to now we have not used that $\xi_u'(\tau_u) = 0$. Doing so now and noticing that $\delta_u - u(1-r(t_o)) = ur(t_o)-\xi_u(\tau_u)$ we get

$$u\nabla^{2k-1} = \frac{(2k-1)!}{r^{(2k)}(t_o) + o_p(1)}\{\chi_o^{\ u}+o_p(1)\}$$

(3.2)

$$\delta_u - u(1-r(t_o)) = \psi_o^{\ u} + o_p(1).$$

Lemma 2.1 directly gives that $(\chi_o^{\ u},\psi_o^{\ u}) \stackrel{L}{\to} (\chi_o,\psi_o)$ and this now solves our problems. Firstly it justifies that $u\nabla^{2k} = o_p(1)$. Secondly we can use the bivariate version of the Cramér-Slutsky theorem:

$$\left.\begin{array}{l}(\xi_n,\eta_n) \overset{L}{\to} (\xi,\eta) \\ a_n^{\,i} \overset{P}{\to} a^i \quad \text{(constant)} \\ \quad i = 1,2,3,4\end{array}\right\} \Longrightarrow \begin{array}{l}(a_n^{\,1}\xi_n+a_n^{\,2},\ a_n^{\,3}\eta_n+a_n^{\,4}) \overset{L}{\to} \\ (a^1\xi+a^2,\ a^3\eta+a^4)\end{array}$$

which will give us part b) of the theorem if applied to the variables in (3.2).

4. MODIFIED ASYMPTOTIC NORMALITY IN CASE II).

In this case, the covariance function has a (series of) "terrace" point(s) before its first local minimum and at these terrace points the covariance derivative r' has a tangency of zero. Even if the sample derivative $\xi_u'(t)$ given by (2.2) closely follows the function $ur'(t)$, the question whether it will cross the zero level or not near the terrace point depends on the sign of $-\eta_u r'''(t) + \delta(t)$. The probabilities of the respective outcomes are nontrivial.

Example: $r(t) = \frac{\sin t}{t} \cdot \cos^2\frac{t}{2}$ has a terrace point at $t = \pi$ and its first minimum at a point $t > \pi$.

We specify the conditions for case ii).

C2 There is a finite number of times $0 < t_1 < t_2 < \dots < t_n < t_o$ and positive integers $k_1, k_2, \dots, k_n, k_o$ such that r has continuous derivatives up to order $2k_i + 1$ near t_i, $i = 1,2,\dots,n$ and up to order $2k_o$ near t_o. Furthermore

$$r'(t) < 0 \qquad \text{for} \quad 0 < t < t_o, \quad t \neq t_1,\dots,t_n$$

$$r^{(j)}(t_o) = 0 \qquad \text{for} \quad j = 1,\dots,2k_o-1$$

$$r^{(2k_o)}(t_o) > 0$$

$$\left.\begin{array}{ll} r^{(j)}(t_i) = 0 & \text{for } j = 1,\dots,2k_i \\ r^{(2k_i+1)}(t_i) < 0 & \end{array}\right\} \text{ for } i = 1,\dots,n .$$

With (3.1) we introduced two independent normal r.v. χ_o and ψ_o equal to $\eta r'''(t_o) - \delta(t_o)$ and $\eta(\lambda_2 r(t_o)+r''(t_o)) - \Delta(t_o)$ where $\eta_u \xrightarrow{L} \eta$ and η is $N(0, 1/\sqrt{\lambda_2\beta})$ and independent of all $\delta(t_j)$-s and $\Delta(t_j)$-s. In this section, we put

$$
(4.1)\qquad
\begin{aligned}
\chi_i &= \eta r'''(t_i) - \delta(t_i) && i = 0, 1, \ldots, n \\
\psi_i &= \eta(\lambda_2 r(t_i)+r''(t_i)) - \Delta(t_i) && i = 0, 1, \ldots, n.
\end{aligned}
$$

Thus $(\chi,\psi) = (\chi_o,\chi_1,\ldots,\chi_n,\psi_o,\psi_1,\ldots,\psi_n)$ is $(2n+2)$-variate normal with mean zero and with the covariances

$$
\begin{aligned}
Cov(\chi_i,\chi_j) &= r'''(t_i)r'''(t_j)/\lambda_2\beta + c(t_i,t_j) \\
&= - r''(t_i-t_j) - r''(t_i)r''(t_j)/\lambda_2 \\
Cov(\chi_i,\psi_j) &= r'''(t_i)(\lambda_2 r(t_j)+r''(t_j))/\lambda_2\beta + \left.\frac{\partial C(s,t)}{\partial s}\right|_{\substack{s = t_i \\ t = t_j}} \\
&= r'(t_i-t_j) \\
Cov(\psi_i,\psi_j) &= (\lambda_2 r(t_i)+r''(t_i))(\lambda_2 r(t_j)+r''(t_j))/\lambda_2\beta + C(t_i,t_j) \\
&= r(t_i-t_j) - r(t_i)\, r(t_j).
\end{aligned}
$$

It should be observed that χ_i and ψ_i are independent and have the variances $\lambda_2 - r''(t_i)^2/\lambda_2$ and $1 - r(t_i)^2$ as before.

If we recall the proof of Theorem 3.1 and try to use χ_i, ψ_i in a limit theorem, we have to modify the procedure. Since $\xi_u'(t)$ is zero near one of the stationary points t_j only if $\eta_u r'''(t_j)-\delta(t_j)$ is negative, we have actually not normal but conditional normal r.v.

We devise the following method to pick up the right time and the right r.v. (χ_j,ψ_j). Cover the times t_o, t_1, ..., t_n by disjoint ε-intervals

$$I_j^{\varepsilon} = [t_j-\varepsilon,\ t_j+\varepsilon] \qquad j = 0, ., \ldots, n.$$

Usually ε is held fixed and then we suppress it. Let the indicator variable κ^* be defined by

$$\kappa^* = \begin{cases} j & \text{if } \tau_u \in I_j \qquad j = 0, ., \ldots, n \\ 0 & \text{if } \tau_u \notin \bigcup_0^n I_j . \end{cases}$$

A corresponding indicator variable κ for the contemplated limit distribution is defined by

$$\kappa = \begin{cases} j & \text{if } \chi_j < 0, \ \chi_i \geq 0 \quad \text{for } i = 1, 2, \ldots, j-1 \\ 0 & \text{if } \chi_i \geq 0 \qquad\qquad \text{for } i = 1, 2, \ldots, n . \end{cases}$$

The r.v. κ has the distribution

$$P(\kappa=j) = p_j = P(\chi_j<0, \ \chi_i \geq 0 \ \text{ for } \ i = 1,2,\ldots,j-1)$$

$$P(\kappa=0) = p_0 = 1 - \sum_1^n p_j$$

where p_1, p_2, and p_3 can be expressed in terms of elementary functions. For a reference to normal probability integrals see Gupta [3].

If we write

$$\varepsilon_j = \begin{cases} 0 & \text{if } \ j = 0 \\ 1 & \text{if } \ j = 1,2,\ldots,n \end{cases}$$

then $2k_{\kappa^*} + \varepsilon_{\kappa^*}$ and $2k_\kappa + \varepsilon_\kappa$ are the order of the first non-vanishing derivatives of r at the randomly selected times t_{κ^*} and t_κ respectively. It is now clear how to observe the r.v.

$$u(\tau_u - t_{\kappa^*})^{2k_{\kappa^*}+\varepsilon_{\kappa^*}-1}$$

and

$$\delta_u - u(1-r(t_{\kappa^*})) = ur(t_{\kappa^*}) - \xi_u(\tau_u)$$

since once we have the value of τ_u we can pick up the $t_{\kappa*}$-value and rise the difference $\tau_u - t_{\kappa*}$ to the appropriate power. Similarly we can observe the r.v.

$$\frac{(2k_\kappa+\varepsilon_\kappa-1)!}{r^{(2k_\kappa+\varepsilon_\kappa)}(t_\kappa)}\chi_\kappa \quad \text{and} \quad \psi_\kappa$$

by taking the first negative χ_κ in the sequence $\chi_1,\ldots,\chi_n$, or, if they are all positive, taking χ_o, and the corresponding ψ_κ.

THEOREM 4.1. If condition C2 is fulfilled with $t_o,t_1,\ldots,t_n$ and $k_o,k_1,\ldots,k_n$ then, as $u \to \infty$

a) $\tau_u \overset{L}{\to} t_\kappa$

b) $(u(\tau_u - t_{\kappa*})^{2k_{\kappa*}+\varepsilon_{\kappa*}-1},\ \delta_u - u(1-r(t_{\kappa*}))) \overset{L}{\to} \left(\frac{(2k_\kappa+\varepsilon_\kappa-1)!}{r^{(2k_\kappa+\varepsilon_\kappa)}(t_\kappa)}\chi_\kappa,\ \psi_\kappa\right)$.

REMARK. Part a) of the theorem gives the probabilities with which τ_u is near the different t_j-s, while part b) says something about the distance between τ_u and the t_j it happens to be near.

PROOF. From Lemma 2.2, it follows that for every $\varepsilon > 0$

$$P(\tau_u \notin \bigcup_0^n I_j^\varepsilon) \to 0 \quad \text{as} \quad u \to \infty.$$

To get a comprehensible and short notation write (c.f. (4.1))

$$\begin{aligned} \chi_j^u &= \eta_u r'''(t_j) - \delta(t_j) \\ \psi_j^u &= \eta_u(\lambda_2 r(t_j) + r''(t_j)) - \Delta(t_j) \end{aligned} \qquad j = 0,1,\ldots,n$$

so that

$$(4.2) \qquad (\chi^u, \psi^u) = (\chi_o^u,\ldots,\chi_n^u, \psi_o^u,\ldots,\psi_n^u) \overset{L}{\to} (\chi, \psi).$$

Also, for $j = 0,1,\ldots,n$, define the events

$$A_o: \{\chi_i \geq 0, \quad i = 1,2,\ldots,n\}$$

$$A_j: \{\chi_j < 0, \quad \chi_i \geq 0, \quad i = 1,2,\ldots,j-1\}$$

$$A_j(x,y): \left\{\frac{(2k_j+\varepsilon_j-1)!}{r^{(2k_j+\varepsilon_j)}(t_j)} \chi_j < x, \quad \psi_j < y\right\}$$

$$B_j: \{\tau_u \in I_j\}$$

$$B_j(x,y): \{u(\tau_u-t_j)^{2k_j+\varepsilon_j-1} < x, \quad \delta_u - u(1-r(t_j)) < y\}.$$

The point in the proof is that we can express the conditions for the events B_j and $B_j(x,y)$ in terms of certain relations for the variables χ_i^u, ψ_i^u $(i = 1,\ldots,j)$ which are very similar to the relations which define the events A_j and $A_j(x,y)$.

In what follows, we concentrate upon the case $j = 1,\ldots,n$. The case $j = 0$ is quite analogous. To start with, we derive the following bounds for the random functions

$$ur(t_j) - \xi_u(t_j+h) = ur(t_j) - ur(t_j+h) + \eta_u(\lambda_2 r(t_j+h) + r''(t_j+h)) - \Delta(t_j+h)$$

$$\xi_u'(t_j+h) = ur'(t_j+h) - \eta_u(\lambda_2 r'(t_j+h)+r'''(t_j+h)) + \delta(t_j+h).$$

Starting with the non-random terms, we notice that, for any $\theta > 0$ there is an $\varepsilon > 0$ such that for $|h| \leq \varepsilon$, $j = 1,2,\ldots,n$

$$(4.3) \qquad \begin{aligned} M_j^-(h) &\leq ur(t_j) - ur(t_j+h) \leq M_j^+(h) \\ -m_j^+(h) &\leq ur'(t_j+h) \leq -m_j^-(h) \end{aligned}$$

where

$$M_j^+(h) = -(1+\theta\cdot \mathit{sign}h) \frac{uh^{2k_j+1}}{(2k_j+1)!} r^{(2k_j+1)}(t_j)$$

$$M_j^-(h) = -(1-\theta\cdot \mathit{sign}h) \frac{uh^{2k_j+1}}{(2k_j+1)!} r^{(2k_j+1)}(t_j)$$

$$m_j^+(h) = -(1+\theta)\frac{uh^{2k_j}}{(2k_j)!} r^{(2k_j+1)}(t_j)$$

$$m_j^-(h) = -(1-\theta)\frac{uh^{2k_j}}{(2k_j)!} r^{(2k_j+1)}(t_j).$$

In order to obtain bounds for the random terms fix a $T \geq t_o+\varepsilon$ and let $\theta' > 0$ be arbitrary. Then there is an M such that the event

$$N: \{|\eta_u| \leq M, \ \sup_{0\leq t\leq T}|\delta(t)| \leq M, \ \sup_{0\leq t\leq T}|\delta'(t)| \leq M\}$$

has a probability $P(N) \geq 1-\theta'$. Considering only outcomes in N we get

$$\text{(4.4)}\quad \begin{aligned} \eta_u(\lambda_2 r(t_j+h)+r''(t_j+h)) - \Delta(t_j+h) &= \psi_j^u + h\cdot G_j(h) \\ \eta_u(\lambda_2 r'(t_j+h)+r'''(t_j+h)) - \delta(t_j+h) &= \chi_j^u + h\cdot H_j(h) \end{aligned}$$

where

$$\begin{aligned} |G_j(h)| &= |\eta_u(\lambda_2 r'(t_j+h')+r'''(t_j+h')) - \delta(t_j+h')| \\ &\leq M(\sup|\lambda_2 r'|+\sup|r'''|+1) \leq M(\lambda_2^{3/2}+(\lambda_2\lambda_4)^{\frac{1}{2}}+1) \leq K \\ |H_j(h)| &\leq M(\sup|\lambda_2 r''|+\sup|r^{IV}|+1) \leq K \end{aligned}$$

with some K depending on θ'.

Adding (4.3) and (4.4) we obtain, for all outcomes in N, the following estimates, valid for $|h| \leq \varepsilon$, $j = 1,\ldots,n$.

$$\text{(4.5a)}\quad \psi_j^u + M_j^-(h) - |h|K \leq ur(t_j) - \xi_u(t_j+h) \leq \psi_j^u + M_j^+(h) + |h|K$$

$$\text{(4.5b)}\quad \chi_j^u + m_j^-(h) - |h|K \leq \xi_u'(t_j+h) \leq \chi_j^u + m_j^+(h) + |h|K.$$

As is easily proved by differentiation, the lower bound functions in (4.5) are uniformly bounded from below (remember that $r^{(2k_j+1)}(t_j) < 0$) so that there is a constant $K' > 0$ such that for all h, $j = 1,\ldots,n$

$$(4.6)\qquad \begin{aligned} M_j^-(h) - |h|K &\geq -K'/u \\ m_j^-(h) - |h|K &\geq -K'/u. \end{aligned}$$

Now we can proceed to the announced equivalences. If $j > 0$, the only interesting case is $x \geq 0$. For all outcomes in N, the event B_j implies that $\xi_u'(t) < 0$ for all $t \in \bigcup_{i=1}^{j-1} I_j$, and $\xi_u'(t) > 0$ for some $t \in I_j$, which in turn, together with (4.5b) and (4.6) gives

$$(4.7)\qquad \chi_i^u > 0 \text{ for } i = 1,\dots,j-1 \quad\text{and}\quad \chi_j^u \leq K'/u.$$

If $x > 0$ and the event $B_j(x,y)$ occurs and especially $u(\tau_u - t_j)^{2k_j} < x$, then $\xi_u'(t_j+h)$ is zero for some $|h| < h_x = (x/u)^{1/2k_j}$. But since $m_j^+(h) + |h|K$ decreases as h tends to zero, we have that $\chi_j^u + m_j^+(h_x) + h_xK < 0$ implies that $\chi_j^u + m_j^+(h) + |h|K < 0$ for all $|h| < h_x$. The upper bound in (4.5b) then gives that $\xi_u'(t_j+h) > 0$ for such h-values. Thus the event $B_j(x,y)$ implies that $\chi_j^u + m_j^+(h_x) + h_xK \geq 0$ or equivalently

$$(4.8)\qquad \frac{(2k_j)!}{r^{(2k_j+1)}(t_j)}\,\chi_j^u < (1+\theta)x - (\tfrac{x}{u})^{1/2k_j} K''.$$

The event $B_j \wedge B_j(x,y)$ also implies that $ur(t_j) - \xi_u(\tau_u) < y$ and this, together with (4.6) and the lower bound in (4.5a) gives

$$(4.9)\qquad \psi_j^u \leq y + K'/u.$$

We sum up the inequalities (4.7)-(4.9) and obtain

$$\begin{aligned} P(B_j \wedge B_j(x,y)) \leq\ & P\{\chi_i^u > 0,\ i=1,\dots,j-1 \wedge \chi_j^u \leq K'/u \\ & \wedge \frac{(2k_j)!}{r^{(2k_j+1)}(t_j)}\chi_j^u < (1+\theta)x - (\tfrac{x}{u})^{1/2k_j} K'' \\ & \wedge \psi_j^u \leq y + K'/u\} + P(N^*) \end{aligned}$$

Letting $u \to \infty$ we get from (4.2)

$$(4.10) \qquad \limsup_{u \to \infty} P(B_j \wedge B_j(x,y)) \leq P(A_j \wedge A_j((1+\theta)x,y)) + \theta'.$$

A reverse inequality can be derived in a similar way, again considering only outcomes in N. The relations (4.5b) and (4.6) give that if

$$\chi_i^u > K'/u \quad \text{for} \quad i = 1,\dots,j-1 \quad \text{and} \quad \chi_j^u < 0$$

then

$$\tau_u \in I_j \quad \text{or} \quad \tau_u \notin \bigcup_0^n I_i.$$

If, furthermore, the lower bound in (4.5b) is positive for $h = -h_x = -(x/u)^{1/2k_j}$, i.e. if

$$\frac{(2k_j)!}{r^{(2k_j+1)}(t_j)} \chi_j^u < (1-\theta)x - (\tfrac{x}{u})^{1/2k_j} K'''$$

then the derivative $\xi_u'(t_j+h)$ is negative at $h \leq -h_x$ and positive at $h = 0$ so its first zero must fall in the interval $(-h_x, 0)$ i.e.

$$u(\tau_u - t_j)^{2k_j} < x.$$

A bound similar to (4.6) can be obtained for M_j^+ to the effect that $M_j^+(h) + |h|K \leq h_x K'$ if $|h| \leq h_x$. Therefore the upper bound in (4.5a) gives that if

$$\psi_j^u < y - (x/u)^{1/2k_j} K'$$

then

$$\delta_u - u(1-r(t_j)) < y.$$

Summing the implications, we obtain

$$P(B_j \wedge B_j(x,y)) \geq P\{\chi_i^{\,u} > K'/u,\ i=1,\ldots,j-1 \wedge \chi_j^{\,u} < 0$$

$$\wedge \frac{(2k_j)!}{r^{(2k_j+1)}(t_j)} \chi_j^{\,u} < (1-\theta)x - (\tfrac{x}{u})^{1/2k_j} K'''$$

$$\wedge\ \psi_j^{\,u} < y - (\tfrac{x}{u})^{1/2k_j} K^4\} - P(N^*) - P(\tau_u \notin \bigcup_0^n I_i)$$

and, if $u \to \infty$

(4.11) $$\liminf_{u \to \infty} P(B_j \wedge B_j(x,y)) \geq P(A_j \wedge A_j((1-\theta)x,y)) - \theta'.$$

Now a little reflection shows that the left hand limits in (4.10) and (4.11) do not depend on ε. The right hand bounds can be made arbitrarily close to each other by first taking small θ and θ', then a sufficiently large M and a small ε for the arguments to go through. Thus, for $j = 1,\ldots,n$

$$\lim_{u \to \infty} P(B_j \wedge B_j(x,y)) = P(A_j \wedge A_j(x,y)).$$

Since the same relation can be shown to hold for $j = 0$, we have proved part b) of the theorem. Part a) follows in an obvious way from the proof of part b).

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