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** *and the University of Lund, Sweden.*

WAVE-LENGTH AND AMPLITUDE FOR A STATIONARY PROCESS AFTER A HIGH MAXIMUM; DECREASING COVARIANCE FUNCTION*

by

Georg Lindgren**
Department of Statistics
University of North Carolina at Chapel Hill

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1. Introduction and results.

In a sequence of previous papers, the author has treated the distribution of the wave-length τ_u, and the amplitude δ_u of a stationary Gaussian process $\{\xi(t), t\epsilon R\}$ after a local maximum with the height u. This paper is a direct continuation of [4] and it deals with the asymptotic distribution of τ_u and δ_u as $u \to \infty$ in the case where the covariance function r of ξ is strictly decreasing for $t > 0$, here called case iii). We adopt without further comment all notations and results from [4], to which the reader is referred, and in which further references are found.

In this paper, we assume that $r'(t)$ is strictly negative for $t > 0$ and that $r(t)$ and its first four derivatives tend to zero as $t \to \infty$. Then, for every fixed but large t, the dominant term in $\xi_u'(t)$ (which describes ξ' near a local maximum with height u) is $ur'(t)$ while $\eta_u(\lambda_2 r'(t) + r'''(t))$ is negligible. Furthermore, for large s and t, the covariance function $c(s,t)$ of the non-stationary Gaussian process $\delta(t)$ which appears in the definition of $\xi_u'(t)$ is approximately $-r''(s-t)$ which we recognize as the covariance function of the original process derivative $\xi'(t)$, and therefore we expect $\delta(t)$ to behave almost like $\xi'(t)$ for large t-values. Thus it seems plausible that the distribution of the time for the first upcrossing zero of the process $\xi_u'(t)$ can be expressed in terms of that of the process $ur'(t) + \xi'(t)$. In addition to the general conditions on r given in [4, Section 2], we now require the following

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condition to be fulfilled.

C3 a) $r'(t) < 0$ for $t > 0$

b) $r'(t) = 0(t^{-\gamma})$ as $t \to \infty$ for some $\gamma > 1$

c) $t^{-2}(\lambda_4 - r^{IV}(t))$ increases as t decreases to zero and $\int_0^{t_o} t^{-1}(\lambda_4 - r^{IV}(t))^{\frac{1}{2}}\, dt < \infty$ for some $t_o > 0$

d) For $k = 1, 2, 3, 4$ it holds that $(-1)^k r^{(k)}(t)$ is convex, positive and decreasing for large t, $t^k r^{(k)}(t) \to 0$ as $t \to \infty$, there is a constant $C \geq 0, \leq \infty$ such that $-r^{(k)}(t)/r^{(k-1)}(t)$ tends monotonically to C as $t \to \infty$.

If $C = 0$ then $-r^{(k)}(t)/r^{(k-1)}(t) = 0(t^{-1})$ as $t \to \infty$.

It should be noted that if the condition C3.d is fulfilled for $k = 4$, then it is fully fulfilled.

Before we can formulate the results we need some notations. First define the function

$$m_{uy}(t) = ur'(t) - y(\lambda_2 r'(t) + r'''(t))$$

and the r.v.

ν_t = the number of upcrossing zeros of $\xi_u'(s)$ in $(0,t]$

$\nu_t(y)$ = the number of upcrossing zeros of $m_{uy}(s) + \delta(s)$ in $(0,t]$

μ_t = the number of upcrossing zeros of $ur'(s) + \xi'(s)$ in $(0,t]$.

Then our interest centers on

$$P(\tau_u \leq t) = 1 - \int_y P(\nu_t(y)=0) q_u^*(y)\, dy$$

$$= 1 - \int_y P(\sup_{[0,t]} m_{uy}(s) + \delta(s) < 0) q_u^*(y)\, dy$$

$$1 - P(\mu_t = 0) = 1 - P(\sup_{[0,t]} ur'(s) + \xi'(s) < 0)$$

while we use

$$E(\nu_t) = \int_y E(\nu_t(y))\, q_u^*(y)\, dy$$

$$= \int_y \int_{s=0}^{t} \omega(s)\phi(m_{uy}(s)/\sigma(s))\Psi(\eta_{uy}(s))\, q_u^*(y)\, ds\, dy$$

$$E(\mu_t) = \int_0^t \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi(ur'(s)/\sqrt{\lambda_2})\, \Psi(ur''(s)/\sqrt{\lambda_4})\, ds$$

where (see e.g. [3])

$$\sigma^2 = \sigma^2(s) = V(\delta(s)) = c(s,s)$$

$$\gamma^2 = \gamma^2(s) = V(\delta'(s)) = \left.\frac{\partial^2 c(s,t)}{\partial s \partial t}\right|_{t=s}$$

$$\mu = \mu(s) = Cov(\delta(s),\delta'(s))/\sigma(s)\,\gamma(s) = \left.\frac{\partial c(s,t)}{\partial s}\right|_{t=s} \Big/ \sigma(s)\,\gamma(s)$$

$$\omega = \omega(s) = \gamma(s)\,(1-\mu^2(s))^{\frac{1}{2}}/\sigma(s)$$

$$\eta_{uy} = \eta_{uy}(s) = \{m_{uy}'(s) - \gamma(s)\mu(s)m_{uy}(s)/\sigma(s)\}/\gamma(s)\sqrt{1-\mu^2(s)}$$

$$= \{ur''-y(\lambda_2 r''+r^{IV})/\gamma\sqrt{1-\mu^2} - \mu\{ur'-y(\lambda_2 r'+r''')\}/\sigma\sqrt{1-\mu^2}\ .$$

We now split up the discussion into three parts according to the value of $\lim_{t\to\infty} -r''(t)/r'(t) = C$; $C = 0$, $0 < C < \infty$, $C = \infty$. Common for the three cases is that we define a function, $T = T(u) \to \infty$ ($= T^0$, T^C, T^∞ respectively) so that the expected values of ν_T and μ_T tend to θ, say, and then we obtain a non-trivial limit of the probability $P(\tau_u - T \le x)$ as $u \to \infty$.

If $C = 0$, then the function $-ur'(t)$ behaves almost like a large constant for large u and t, and we might expect a behaviour similar to the asymptotic Poisson-character of the stream of crossings of a very high level derived by Cramér and others, cf. [3, Ch.12], although we here have a very high function. In fact, Theorem 1.1b contains the first term in the asymptotic Poisson distribution. It is possible to establish the full Poisson distribution by means of the standard proof, generalised as in Section 4 of this paper, under e.g. the following conditions on the function g_u:

$$\inf_{0 \le t \le T(u)} g_u(t) \to \infty, \qquad \sup_{0 \le t \le T(u)} |g_u'(t)| \to 0$$

$$\sup_{0 \le s,\, t \le T(u)} |g_u^2(s) - g_u^2(t)| \text{ is bounded.}$$

The results by Qualls [8] on multiple level crossings are instructive in this respect.

THEOREM 1.1. If condition C3 is fulfilled, if $-r''(t)/r'(t) \to 0$ as $t \to \infty$, and if $T_\theta^{\,o} = T_\theta^{\,o}(u) \to \infty$ as $u \to \infty$ so that

$$E(\mu_{T_\theta^o}) \to \theta$$

then, as $u \to \infty$

a) $E(\nu_{T_\theta^o + x}) \to \theta$ for all x

b) $P(\tau_u - T_\theta^{\,o} \le x) \to 1 - e^{-\theta}$ for all x.

We prefer to formulate the results in a slightly different way in the case $0 < C \le \infty$. With the choice of T^C and T^∞ proposed in Theorem 1.2 in can be shown that

$$-\,ur'(T^C + t) \to e^{-Ct}$$

$$-\,ur'(T^\infty + t) \to \begin{cases} \infty & \text{for } t < 0 \\ 0 & \text{for } t \ge 0 \end{cases}$$

respectively, and the theorem reflects this fact.

THEOREM 1.2. If condition C3 is fulfilled, if $-r''(t)/r'(t) \to C$ as $t \to \infty$, with $0 < C \le \infty$, and if $T^C = T^C(u)$, $T^\infty = T^\infty(u) \to \infty$ so that

$$-\,ur'(T^C) = 1 \qquad (0 < C < \infty)$$

$$-\,ur'(T^\infty) = \sqrt{-r'(T^\infty)/r''(T^\infty)} \qquad (C = \infty)$$

then, as $u \to \infty$

a) $E(\nu_{T^C+x}) \to \int_{-\infty}^{x} \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi(e^{-Ct}/\sqrt{\lambda_2})\, \Psi(Ce^{-Ct}/\sqrt{\lambda_4})\, dt \quad (0 < C < \infty)$

$$E(\nu_{T^\infty+x}) \to \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} + \frac{x}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}} & \text{for } x \ge 0 \end{cases} \quad (C = \infty)$$

b) $P(\tau_u - T^C \le x) \to 1 - P(\sup_{t \le x} \xi'(t) - e^{-Ct} < 0) \quad (0 < C < \infty)$

$$P(\tau_u - T^\infty \le x) \to \begin{cases} 0 & \text{for } x < 0 \\ 1 - P(\sup_{0 \le t \le x} \xi'(t) < 0) & \text{for } x \ge 0 \end{cases} \quad (C = \infty)$$

Part b of Theorem 1.2 is included in the following theorem concerning the joint limit distribution if τ_u and δ_u.

THEOREM 1.3. If the conditions of Theorem 1.2 are fulfilled, and if the r.v. τ is defined by

$$\tau = \begin{cases} \inf\{t;\ \xi'(t) \ge e^{-Ct}\} & (0 < C < \infty) \\ \inf\{t \ge 0;\ \xi'(t) \ge 0\} & (C = \infty) \end{cases}$$

then, as $u \to \infty$

$$(\tau_u - T^C,\ \delta_u - u) \overset{L}{\to} (\tau,\ -C^{-1} e^{-C\tau} - \xi(\tau)) \quad (0 < C < \infty)$$

$$(\tau_u - T\ ,\ \delta_u - u) \overset{L}{\to} (\tau,\ -\xi(\tau)) \quad (C = \infty).$$

The following theorem holds regardless of the value of C.

THEOREM 1.4. If condition C3 is fulfilled then, as $u \to \infty$

$$\delta_u/u \overset{P}{\to} 0.$$

The proofs of Theorem 1.1a and 1.2a are found in Section 3, that of Theorem 1.1b in Section 4, and that of Theorem 1.2b in Section 5. Section 6 contains the rest of the proof of Theorem 1.3 as well as a general proof of Theorem 1.4.

2. SOME AUXILIARY RESULTS.

In this section, we show that for increasing u-values we can disregard what happens in an increasing region $(0, T_{-})$.

LEMMA 2.1. For any $t_o > 0$ as $u \to \infty$

a) $E(\mu_{t_o}) \to 0$

b) $P(\tau_u < t_o) \leq E(\nu_{t_o}) \to 0.$

PROOF. We first prove that the results hold for one particular $t_o{*}$ and then we extend it to all t_o.

a) Take $t_o{*}$ so that $r''(t) \leq 0$ for $0 \leq t \leq t_o{*}$. Then, as $u \to \infty$

$$E(\mu_{t_o{*}}) = \int_0^{t_o{*}} \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right) \Psi\left(\frac{ur''(t)}{\sqrt{\lambda_4}}\right) dt \leq K\int_0^{t_o{*}} \phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right) dt \to 0$$

b) Let

$$a(t) = (\lambda_4 r'(t) + \lambda_2 r'''(t))/\lambda_2\beta$$

$$b(t) = (\lambda_2 r'(t) + r'''(t))/\lambda_2\beta.$$

Then it is easily proved that there is a $t_o{*} > 0$ such that $-a(t)$ and $b(t)$ are non-negative and strictly increasing in $[0, t_o{*}]$. Since $m_{uy}(t) = ua(t) - \lambda_2\beta(y+u/\beta)b(t)$, we conclude that $m_{uy}'(t)$ is strictly negative for $0 < t \leq t_o{*}$, and for all $y > -u/\beta$. Since the function Ψ is increasing and fulfills $\Psi(x) \leq 1+|x|$ we get

$$\Psi(\eta_{uy}(t)) \leq \Psi\left(-\frac{\mu}{\sqrt{1-\mu^2}} \cdot \frac{m_{uy}(t)}{\sigma(t)}\right) \leq 1 + \frac{|\mu|}{\sqrt{1-\mu^2}} \cdot \frac{|m_{uy}(t)|}{\sigma(t)}.$$

Thus

$$E(\nu_{t_o^*}) = \int_{y=-u/\beta}^{\infty} \int_{t=0}^{t_o^*} \omega(t)\ \phi\left(\frac{m_{uy}(t)}{\sigma(t)}\right)\Psi(\eta_{uy}(t))q_u^*(y)\ dt\ dy$$

(2.1)
$$\leq \int_y \int_t \omega\ \phi(m_{uy}/\sigma)q_u^*(y)\ dt\ dy$$

$$+ \int_y \int_t \frac{\gamma|\mu|}{\sigma^2}\ |m_{uy}|\ \phi(m_{uy}/\sigma)q_u^*(y)\ dt\ dy.$$

We will now show that the second integral in this sum tends to zero. The first integral is treated similarly. We first note that $\gamma|\mu|$ is bounded. Furthermore, simple calculations show that

$$\frac{d}{dt}\sigma^2(t) = -2\beta a'(t)b(t), \qquad \sigma^2(t) \leq -2\beta a(t)b(t).$$

Now we can estimate $|m_{uy}|/\sigma$ and $|m_{uy}|/\sigma^2$.

Write, for short, $R(t) = \lambda_4 - r^{IV}(t)$ and recall that $R(t)$ decreases, but $R(t)/t^2$ increases as t decreases to zero. Thus

$$-\ tR(t)/a(t) = -tR(t)/\int_0^t a'(s)\ ds \propto tR(t)/\int_0^t R(s)\ ds \geq 1$$

but

$$\leq tR(t)/\int_0^t s^2 \cdot R(t)/t^2\ ds = 3$$

so that $tR(t)/|a(t)|$ is bounded away from zero and infinity for small t. Since $b(t) \sim K \cdot t$, we get

$$R(t) \cdot \frac{b^2(t)}{\sigma^2(t)} \geq \frac{R(t)b(t)}{|a(t)|} \cdot \frac{|a(t)|b(t)}{-2\beta a(t)b(t)} \geq K_b'$$

$$R(t)^{-1} \cdot \frac{a^2(t)}{\sigma^2(t)} = \frac{a^2(t)}{R^2(t)b^2(t)} \cdot \frac{R(t)b^2(t)}{\sigma^2(t)} \geq K_a'.$$

Since, furthermore,

$$-\frac{dta(t)}{dt}\Big/\frac{d\sigma^2(t)}{dt} = \frac{-a(t)-ta'(t)}{2\beta a'(t)b(t)} \leq K_1 \cdot \frac{-a(t)}{tR(t)} + K_2 \leq K_a''$$

we get

$$- ta(t)/\sigma^2(t) \quad \le K_a''$$

$$tR(t)b(t)/\sigma^2(t) \quad \le \quad - \frac{ta(t)}{\sigma^2(t)} \cdot \frac{R(t)b^2(t)}{-a(t)} \quad \le \quad K_b''.$$

Thus it is always possible to find positive constants K_i such that for $0 < t \le t_o^*$, $y > -u/\beta$

$$\frac{|m_{uy}(t)|}{\sigma(t)} \ge K_1(y + u/\beta)\ R(t)^{-\frac{1}{2}} + K_2 u\ R(t)^{\frac{1}{2}} \ge K_1(y + u/\beta)\ R(t)^{-\frac{1}{2}}$$

$$\frac{|m_{uy}(t)|}{\sigma^2(t)} \le t^{-1}\{K_3(y + u/\beta)\ R(t)^{-1} + K_4 u\}.$$

Introducing the explicit expression [4, (2.1)] for $q_u^*(y)$, we get the following upper bound for the second integral in (2.1):

$$K \int_{y=-u/\beta}^{\infty} \int_{t=0}^{t_o^*} t^{-1} \{K_3(y+u/\beta)R(t)^{-1} + K_4 u\}\ exp(-K_1{}^2(y+u/\beta)^2/2R(t)) \cdot$$

$$\cdot\ (y+u/\beta)exp(-\lambda_2\beta y^2/2)\ \Psi(u\sqrt{\lambda_2/\beta})^{-1}\ dt\ dy$$

$$\le K_5 \Psi(u\sqrt{\lambda_2/\beta})^{-1} \int_y \int_t \frac{(y+u/\beta)^2}{tR(t)}\ exp(-K_1{}^2(y+u/\beta)^2/2R(t))\ dt\ dy$$

$$+ K_6 \int_y \int_t \frac{(y+u/\beta)}{t}\ exp(-K_1{}^2(y+u/\beta)^2/2R(t)) \cdot exp(-\lambda_2\beta y^2/2)\ dt\ dy$$

$$\le K_7 \Psi(u\sqrt{\lambda_2/\beta})^{-1} \int_{t=0}^{t_o^*} \int_{x=0}^{\infty} \frac{x^2}{tR(t)}\ exp(-K_1{}^2x^2/2R(t))\ dx\ dt$$

$$+ K_8 \int_t \int_x \frac{x}{t}\ exp(-K_1{}^2x^2/2R(t)) \cdot exp(-\lambda_2\beta(x-u/\beta)^2/2)\ dx\ dt.$$

The double integrals in these expressions are bounded by

$$\int_{t=0}^{t_o^*} \frac{\sqrt{R(t)}}{t}\, dt \qquad \text{and} \qquad \int_{t=0}^{t_o^*} \frac{R(t)}{t}\, dt \qquad \text{respectively}$$

which both are finite. Since $\Psi(u\sqrt{\lambda_2/\beta}) \to \infty$ dominated convergence gives that the total bound tends to zero.

We now extend the results to general t_o.

a) Since $\inf_{(t_o^*, t_o)} |r'(t)| > 0$, $\sup_{(t_o^*, t_o)} |r''(t)| < \infty$, we get, for some $K_1, K_2 > 0$

$$\int_{t_o^*}^{t_o} \phi(ur'(t)/\sqrt{\lambda_2})\ \Psi(ur''(t)/\sqrt{\lambda_4})\ dt \leq \int_{t_o^*}^{t_o} \phi(K_1 u)(1+K_2 u)\ dt \to 0.$$

b) Taking $\inf$ and $\sup$ over $t \geq t_o^*$ we get that $\sigma_- = \inf \sigma(t) > 0$, $\sigma_+ = \sup \sigma(t) < \infty$, $\gamma_- = \inf \gamma(t) > 0$, $\gamma_+ = \sup \gamma(t) < \infty$, $\varepsilon = \sup |\mu(t)|(1-\mu^2(t))^{\frac{1}{2}} < \infty$. This follows from the ergodicity of $\xi(t)$, since that assumption implies that the distribution of $(\delta(t), \delta'(t))$ is non-singular and that, as $t \to \infty$

(2.2) $\sigma^2(t)$, $\gamma^2(t)$, $\mu^2(t) \to \lambda_2, \lambda_4, 0$ respectively.

Since $|m_{uy}'(t)| \leq K_1'u + K_2'|y|$ and $|m_{uy}(t)| \leq K_1''u + K_2''|y|$ for all u and y we get $\Psi(\eta_{uy}(t)) \leq 1 + |\eta_{uy}(t)| \leq K_1 u + K_2|y|$.

If we further notice that $\sup_{(t_o^*, t_o)} |\lambda_2 r'(t) + r'''(t)| / \inf_{(t_o^*, t_o)} |r'(t)|$ is finite, we get

$$|m_{uy}(t)| \geq u|r'(t)| - |y| \cdot |\lambda_2 r'(t) + r'''(t)| \geq K_3'u - K_4'|y| \geq K_3 u$$

for all y with $|y| \leq K_4 u$. Separating $|y| \geq K_4 u$ and $\leq K_4 u$ we get

$$\int_{y > -u/\beta} \int_{t=t_o^*}^{t_o} \omega\ \phi(m_{uy}/\sigma)\ \Psi(\eta_{uy})\ q_u^*(y)\ dt\ dy$$

$$\leq K_5 \int_{|y| \leq K_4 u} \int_{t=t_o^*}^{t_o} \phi(K_6 u)\ (u+|y|)\ q_u^*(y)\ dt\ dy$$

$$+ K_7 \int_{|y| \geq K_4 u} \int_{t=t_o^*}^{t_o} (u+|y|)\ q_u^*(y)\ dt\ dy \to 0 \quad \text{as} \quad u \to \infty$$

and the lemma is proved. □

The following lemma plays an important role in the future development.

LEMMA 2.2. There exist a constant K, independent of u and y, such that, for all interval I sufficiently remote from the origin

$$\int_I \phi(m_{uy}(t)/\sigma(t))\,\Psi(\eta_{uy}(t))\,dt \;\le\; K(|I|+1)$$

where $|I|$ is the length of the interval.

PROOF. Since $m_{uy}'(t) = ur''(t)\,\{1 - \frac{y}{u}(\lambda_2 + r^{IV}(t)/r''(t))\}$ and $r^{IV}(t)/r''(t)$ is monotonic for large t, $m_{uy}'(t)$ can change sign at most once in I. Let I_+ and I_- denote those parts of I in which $m_{uy}'(t) \ge 0$ and < 0 respectively. With the same notation σ_+, σ_- etc. as in the proof of Lemma 2.1, we get

$$\int_{I_-} \le \int_{I_-} \phi(m_{uy}(t)/\sigma_+)\,\Psi(\varepsilon|m_{uy}(t)|/\sigma_-)\,dt$$

$$\le \int_{I_-} \phi(m_{uy}(t)/\sigma_+)\,\{1 + \varepsilon|m_{uy}(t)|/\sigma_-\}\,dt \;\le\; K_-|I_-|$$

since $\phi(x)$ and $x\phi(x)$ are bounded.

$$\int_{I_+} \le \int_{I_+} \phi(m_{uy}(t)/\sigma_+)\,\Psi(m_{uy}'(t)/\gamma_-\sqrt{1-\varepsilon^2} + \varepsilon|m_{uy}(t)|/\sigma_-)\,dt$$

$$\le \int_{I_+} \phi(m_{uy}(t)/\sigma_+)\,\{m_{uy}'(t)/\gamma_-\sqrt{1-\varepsilon^2} + 1 + \varepsilon|m_{uy}(t)|/\sigma_-\}\,dt$$

$$\le K_-|I_+| + K\int x\phi(x/\sigma_+)\,dx \;\le\; K_+(|I_+|+1).$$

A combination of the results yields the lemma. □

Now take t_o large enough to assure that $-r'(t)$ decreases for $t \ge t_o$ and define a function $T_- = T_-(u) \to \infty$ such that (for $\delta > 0$)

$$(2.3)\qquad \begin{cases} -ur'(T_-) = \sqrt{(2+\delta)\lambda_2 \log T_-} & (\text{if } C = 0) \\ -ur'(T_-) \to \infty & (\text{if } C > 0). \end{cases}$$

REMARK. It is only in the case $C = -\lim_{t\to\infty} r''(t)/r'(t) = 0$ that we require $-ur'(T_-)$ to tend to infinity in a specified way.

LEMMA 2.3. If T_- satisfies (2.3) then, as $u \to \infty$

a) $E(\mu_{T_-}) \to 0$

b) $P(\tau_u \le T_-) \le E(\nu_{T_-}) \to 0.$

PROOF. By Lemma 2.1, it suffices to show that the corresponding integrals over (t_o, T_-) tend to zero.

a) If $-r''(t)/r'(t) \to 0$ then

$$\int_{t_o}^{T_-} \phi(ur'(t)/\sqrt{\lambda_2})\, \Psi(ur''(t)/\sqrt{\lambda_4})\, dt$$

$$\le \int_{t_o}^{T_-} \phi(ur'(t)/\sqrt{\lambda_2})\, \{1 + ur''(t)/\sqrt{\lambda_4}\}\, dt$$

$$\le T_-\phi(ur'(T_-)/\sqrt{\lambda_2}) + \lambda_4^{-\frac{1}{2}} \int_{ur'(t_o)}^{ur'(T_-)} \phi(x/\sqrt{\lambda_2})\, dx$$

$$\le \frac{T_-}{\sqrt{2\pi}} \exp(-(2+\delta)\lambda_2 \log T_-/2\lambda_2) + (\lambda_2/\lambda_4)^{-\frac{1}{2}}\, \Phi(ur'(T_-)/\sqrt{\lambda_2}) \to 0$$

by the definition of T_-.

If $-r''(t)/r'(t) \to C > 0$ then $\inf_{(t_o,T_-)} ur''(t) \to \infty$ and we can use the simple estimate $\Psi(ur''(t)/\sqrt{\lambda_4}) \le Kur''(t)$.

b) Write for a moment $y_u = T_-$ and expand $m_{uy}(t)$ and $\eta_{uy}(t)$ for $|y| \le y_u$:

$$m_{uy}(t) = ur'(t)\, \{1 - R_{uy}(t)\}$$

$$\eta_{uy}(t) = ur''(t)\lambda_4^{-\frac{1}{2}}\, \{1 - S_{uy}(t)\}$$

where the residuals

$$R_{uy}(t) = \frac{y}{u}\,(\lambda_2 + r'''(t)/r'(t))$$

$$S_{uy}(t) = 1 - \frac{\sqrt{\lambda_4}}{\gamma\sqrt{1-\mu^2}} + \frac{1}{\gamma\sqrt{1-\mu^2}} \cdot \frac{y}{u} \cdot (\lambda_2 + r^{IV}(t)/r''(t))$$

$$+ \frac{\mu}{\sigma\sqrt{1-\mu^2}} \left\{\frac{r'(t)}{r''(t)} - \frac{y}{u}\left[\lambda_2 \frac{r'(t)}{r''(t)} + \frac{r'''(t)}{r''(t)}\right]\right\}$$

will be shown to converge to zero uniformly for $t_o \le t \le T_-$, $|y| \le y_u$ as $u \to \infty$, $t_o \to \infty$.

First we note that $\mu(t) = o(r'(t))$ or $o(r^{IV}(t))$ as $t \to \infty$ according to if $C < \infty$ or $C > 0$, (sic!). Then the assertion is proved if we can show that the following quantities tend to zero as $t \to \infty$ and $u \to \infty$, respectively: $r'(t)^2/r''(t)$, $r'(t)r^{IV}(t)/r''(t)$, $r'(t)r'''(t)/r''(t)$, $r'''(t)r^{IV}(t)/r''(t)$; y_u/u, $\sup_{t_o \le t \le T_-} y_u r'''(t)/ur'(t)$, $\sup_{t_o \le t \le T_-} y_u r^{IV}(t)/ur''(t)$. That the first four quotients tend to zero is a consequence of the assumed convexity of $(-1)^k\, r^{(k)}(t)$. For the last three expressions, we note that $r'(t) = O(t^{-\gamma})$ implies $-ur'(T_-) \le MuT_-^{-\gamma}$ so that for large u we have $y_u = T_- \le u^{1/\gamma}$. Thus $y_u/u \to 0$, and the convergence is obvious in the case $C < \infty$. If $C = \infty$ we have for $t \le T_-$ that

$$\frac{y_u r'''(t)}{ur'(t)} \le \frac{T_- r'''(T_-)}{ur'(T_-)}, \qquad \frac{y_u r^{IV}(t)}{ur''(t)} \le \frac{T_- r^{IV}(T_-)}{ur''(T_-)}$$

where $|T_- r'''(T_-)| \le T_- r^{IV}(T_-) \to 0$ and $ur''(T_-) \ge |ur'(T_-)| \to \infty$. Since $\sigma^2(t) \to \lambda_2$ we have, as $u \to \infty$, $t_o \to \infty$

$$\frac{m_{uy}(t)}{\sigma(t)} = \frac{ur'(t)}{\sqrt{\lambda_2}}(1 + o(1)) \tag{2.4}$$

$$\eta_{uy}(t) = \frac{ur''(t)}{\sqrt{\lambda_4}}(1 + o(1)) \tag{2.5}$$

uniformly in $t_o \le t \le T_-$, $|y| \le y_u$.

Now let $\delta > 0$ be as in (2.3) and take $\varepsilon > 0$ so that $(1-\varepsilon)(1+\frac{1}{2}\delta) > 1$. If $t_o \le t \le T_-$, $|y| \le y_u$, we can then assume that $|m_{uy}(t)|/\sigma(t) \ge (1-\varepsilon)|ur'(t)|/\sqrt{\lambda_2}$, $|\eta_{uy}(t)| \le (1+\varepsilon)\, ur''(t)/\sqrt{\lambda_4}$. Using

Lemma 2.2 if $|y| > y_u$ we get

$$\int_{y>-u/\beta} \int_{t=t_o}^{T_-} \phi(m_{uy}(t)/\sigma(t))\, \Psi(\eta_{uy}(t))\, q_u{*}(y)\, dt\, dy$$

$$\leq \int_{|y|\leq y_u} \int_{t=t_o}^{T_-} \phi\left(\frac{(1-\varepsilon)ur'(t)}{\sqrt{\lambda_2}}\right) \Psi\left(\frac{(1+\varepsilon)ur''(t)}{\sqrt{\lambda_4}}\right) q_u{*}(y)\, dt\, dy$$

$$+ \int_{|y|>y_u} K\, T_-\, q_u{*}(y)\, dy.$$

For the first integral, we use the same technique as in part a), while for the second we use that $T_-\int_{|y|>y_u} q_u{*}(y)dy \sim 2T_-(1-\Phi(y_u)) \sim 2T_-\phi(T_-)/T_- \to 0$ as $T_- \to \infty$. □

For later use, we extend part of the proof to a new lemma.

LEMMA 2.4. If $T = T^o, T^C, T^\infty \to \infty$ as in Theorem 1.1 and 1.2 respectively, T_- fulfills (2.3), and $y_u = T$ then

$$\frac{m_{uy}(t)}{\sigma(t)} = \frac{ur'(t)}{\sqrt{\lambda_2}}(1 + o(1))$$

$$\eta_{uy}(t) = \frac{ur''(t)}{\sqrt{\lambda_4}}(1 + o(1))$$

where $o(1) \to 0$ uniformly in $|y| \leq y_u$, $T_- \leq t \leq T+x$ if $C < \infty$ and in $|y| \leq y_u$, $T_- \leq t \leq T$ if $C = \infty$.

PROOF. If $C < \infty$, then $-ur'(T) \leq MuT^{-\gamma}$ implies that $y_u = T \leq M'u^{1/\gamma}$, and we can proceed as in the proof of (2.4) and (2.5). If $C = \infty$, then $ur''(T) \leq MuT^{-2}$ by condition C3.d which implies that $y_u/u \to 0$. It remains to show that $Tr'''(T)/ur'(T) \to 0$. If we square the quotient, we obtain $T^2r'''(T)^2/(-r'(T)/r''(T))$ which tends to zero since $r'''(T)r''(T)/r'(T)$ does. □

3. PROOF OF THEOREM 1.1A AND 1.2A.

Write $y_u = T = T(u)$ $(= T^o, T^C, T^\infty)$.

LEMMA 3.1. As $u \to \infty$

$$\int_{|y|>y_u} \int_{t=0}^{T+x} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})\, q_u^*(t)\, dt\, dy \to 0.$$

PROOF. The lemma is a direct consequence of Lemma 2.3, Lemma 2.2 and the fact that $T\int_{|y|>T} q_u^*(y)dy \to 0$ as $T \to \infty$. □

PROOF OF THEOREM 1.1A. In the light of Lemma 2.3 and Lemma 3.1 it only remains to prove that

$$\int_{|y|\le y_u} \int_{T_-}^{T+x} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})\, q_u^*(y)\, dt\, dy \to \theta.$$

We do this by proving $\int_{|y|\le y_u} \int_{T_-}^{T} \to \theta$ and $\int_{|y|\le y_u} \int_{T}^{T+x} \to 0$. First we note that for $t > T_-$ the derivative $ur''(t) \le ur''(T_-) = -ur'(T_-)\cdot$ $\cdot r''(T_-)/(-r'(T_-) = K\sqrt{\ell og T_-}\; r''(T_-)/(-r'(T_-) \to 0$ by condition C3.d. By Lemma 2.4, it then holds that $\eta_{uy}(t) \to 0$ uniformly in $T_- \le t \le T+x$, $|y| \le y_u$, and thus $\Psi(\eta_{uy}) \to \Psi(0)$ uniformly.

$\int_{|y|\le y_u} \int_{T}^{T+x}$: Here the inner integral is taken over an interval of fixed length and the integrand is bounded by $K_1\phi(K_2ur'(t))\, q_u^*(y)$. Since $ur''(t) \to 0$ uniformly, a little reflexion will show that $-ur'(t) \ge -ur'(T+x)$ $\sim -ur'(T)$ must tend to infinity if the integral $\int_0^T \phi(ur'(t)/\sqrt{\lambda_2})\Psi(ur''(t)/\sqrt{\lambda_4})dt$ is to be bounded. Therefore the said bound of the integrand is $q_u^*(y) \cdot o(1)$ uniformly in $(T, T+x)$ and the total integral tends to zero.

$\int_{|y|\le y_u} \int_{T_-}^{T}$: We have seen that $\omega(t) \to \sqrt{\lambda_4/\lambda_2}$ and $\Psi(\eta_{uy}) \sim$ $\Psi(ur''/\sqrt{\lambda_2}) \to \Psi(0)$ uniformly as $u \to \infty$. It remains to prove that $\phi(m_{uy}(t)/\sigma(t))/\phi(ur'(t)/\sqrt{\lambda_2}) \to 1$ uniformly, i.e. to prove that

$$\frac{m_{uy}^2(t)}{\sigma^2(t)} - \frac{u^2 r'(t)^2}{\lambda_2} \to 0 \quad \text{uniformly in} \quad T_- \le t \le T, \ |y| \le y_u.$$

The difference is

$$\sigma^{-2}(t)\Big\{u^2 r'(t)^2(\lambda_2 - \sigma^2(t))/\lambda_2 + y^2(\lambda_2 r'(t) + r'''(t))^2$$

$$- 2uyr'(t)(\lambda_2 r'(t) + r'''(t))\Big\} = A + B - C, \quad \text{say.}$$

Since $\lambda_2 - \sigma^2(t) = O(r'(t)^2)$, $r'(t) = O(t^{-\gamma})$, $r'''(t) = O(r'(t))$, and $T_- \le y_u = T \le u^{1/\gamma}$, we have

$$A \le K_1 u^2 r'(T_-)^4 \le K_2 T_-^{-2} \log T_- \to 0$$

$$B \le K_3 y_u^2 r'(T_-)^2 \le K_4 (y_u/u)^2 \log T_- \le K_5 u^{-2(1-1/\gamma)} \log u \to 0$$

$$|C| \le K_6 u y_u r'(T_-)^2 \le K_7 (y_u/u) \log T_- \to 0.$$

Thus all the three factors in the considered integral are uniformly asymptotically close to the corresponding factors in the integral $\int \lambda_4/\lambda_2 \ \phi(ur'/\sqrt{\lambda_2}) \ \Psi(0) \ dt$ which in turn is approximately θ. By this, the theorem is proved. □

For future use, we formulate the obtained results as a lemma.

LEMMA 3.2. If $-r''(t)/r'(t) \to 0$ as $t \to \infty$ and if T_- and T are defined as in (2.3) and Theorem 1.1 respectively then

$$\gamma(t)\sqrt{1-\mu^2(t)}/\sigma(t) \to \sqrt{\lambda_4/\lambda_2}$$

$$\phi(m_{uy}(t)/\sigma(t))/\phi(ur'(t)/\sqrt{\lambda_2}) \to 1$$

$$\Psi(\eta_{uy}(t)) \sim \Psi(ur''(t)/\sqrt{\lambda_4}) \to \Psi(0) = 1/\sqrt{2\pi}$$

uniformly in $T_- \le t \le T$, $|y| \le T$ as $u \to \infty$

For the proof of Theorem 1.2a, we need another lemma.

LEMMA 3.3. As $u \to \infty$, for every $t_o > 0$, $t_o' > 0$

$$-ur'(T^C+t)/exp(-Ct) \to 1 \quad \text{uniformly in } |t| \le t_o$$

$$-ur'(T^\infty+t) \to \begin{cases} \infty & \text{uniformly in } -t_o \le t \le -t_o' \\ 0 & \text{uniformly in } t \ge 0 \end{cases}$$

PROOF. If $-r''(t)/r'(t) \to C < \infty, > 0$ then for every $\varepsilon > 0$ there is a t^* such that $-(C-\varepsilon)r'(t) \le r''(t) \le -(C+\varepsilon)r'(t)$ for $t > t^*$. By continuity arguments, this can be shown to imply that $-r'(s)exp(-(C+\varepsilon)(t-s)) \le -r'(t) \le -r'(s)exp(-(C-\varepsilon)(t-s))$ for $t \ge s \ge t^*$, which gives the lemma in this case.

If $-r''(t)/r'(t) \to \infty$ the result follows if one considers the definition of T^∞. □

PROOF OF THEOREM 1.2A. Let $T_- \to \infty$ so that $T_- - T \to -\infty$, $-ur'(T_-) \to \infty$, and so that the convergences of the foregoing lemma hold uniformly in $T_- - T^C \le t \le t_o$ and $T_- - T^\infty \le t \le -t_o'$, $t \ge 0$ respectively. Using this function in Lemma 2.3, we only need to prove that

$$\int_{|y|\le y_u} \int_{T_-}^{T+x} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})\, q_u^*(y)\, dt\, dy$$

tends to the respective expressions as $u \to \infty$.

$-r''(t)/r'(t) \to C < \infty$: By Lemma 2.4 we have $|m_{uy}(t)|/\sigma(t) \ge K_1 u|r'(t)|$, $K_2 ur''(t) \ge \eta_{uy}(t) \ge K_3 ur''(t) \ge K_4 > 0$ for $T_- \le t \le T+x$. Thus

$$\int_{T_-}^{T+x} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})ds \le K_5 \int_{T_-}^{T+x} \phi(K_1 ur'(t))ur''(t)dt$$

$$= K_6 \int_{ur'(T_-)}^{ur'(T+x)} \phi(k_1 s)ds \le K_7 \quad \text{independent of } u, y.$$

A little reflexion also shows that $m_{uy}^2(t)/\sigma^2(t) - exp(-2C(t-T))/\lambda_2$ and

$\eta_{uy}(t) - C\exp(-t+T)$ can be made uniformly small in $T_- \le t \le T+x$. Dominated convergence then gives the assertion.

$-r''(t)/r'(t) \to \infty$: If $x < 0$ Lemma 2.3 and Lemma 3.3 give that $E(\nu_{T+x}) \to 0$.

If $x = 0$, then

$$\begin{aligned} E(\nu_T(y)) &\ge P(\nu_T(y) \ge 1) \\ &\ge P(ur'(T) - y(\lambda_2 r'(T) + r'''(T)) + \delta(T) > 0) \\ &= \Phi((ur'(T) - y(\lambda_2 r'(T) + r'''(T))/\sigma(T)). \end{aligned}$$

If $|y| \le y_u = T$, we have $yr'''(T) \to 0$, $yr'(T) \to 0$ so that all terms in the argument of the Φ-function tend to zero. Thus

$$\liminf_{u \to \infty} E(\nu_T) \ge \liminf_{u \to \infty} \int_{|y| \le y_u} E(\nu_T(y))\, q_u^*(y)\, dy \ge \frac{1}{2}.$$

A reverse inequality follows again from Lemma 2.4. For $\varepsilon > 0$ and u large

$$\begin{aligned} &\int_{T_-}^{T} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})\, dt \\ &\quad \le (1+\varepsilon) \sqrt{\frac{\lambda_4}{\lambda_2}} \int_{T_-}^{T} \phi\left((1-\varepsilon) \frac{ur'(t)}{\sqrt{\lambda_2}}\right) \Psi\left((1+\varepsilon) \frac{ur''(t)}{\sqrt{\lambda_4}}\right) dt \\ &\quad \le (1+\varepsilon)^2 \frac{1}{\sqrt{\lambda_2}} \int_{T_-}^{T} \phi\left((1-\varepsilon) \frac{ur'(t)}{\sqrt{\lambda_2}}\right) ur''(t)\, dt \\ &\quad = \frac{(1+\varepsilon)^2}{1-\varepsilon} \left\{ \Phi\left((1-\varepsilon) \frac{ur'(T)}{\sqrt{\lambda_2}}\right) - \Phi\left((1-\varepsilon) \frac{ur'(T_-)}{\sqrt{\lambda_2}}\right) \right\} \to \frac{1}{2} \frac{(1+\varepsilon)^2}{1-\varepsilon}. \end{aligned}$$

Thus $\limsup E(\nu_T) \le \frac{1}{2}$ and this part of the theorem is proved.

It remains to prove that, if $x > 0$, then

$$\int_{|y| \le y_u} \left\{ \int_{T}^{T+x} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})\, dt \right\} q_u^*(y)\, dy \to \frac{x}{2\pi} \sqrt{\frac{\lambda_4}{\lambda_2}}.$$

Lemma 2.2 shows that the inner integral is bounded by a constant K, independent of u and y. The dominated convergence theorem applied to the outer integral gives that result if, for fixed y

$$\int_T^{T+x} \phi(m_{uy}/\sigma)\ \Psi(\eta_{uy})\ dt \ \to\ x/2\pi.$$

To see this convergence, let s be fixed, $0 < s < x$, and $t = T+s$. Then $m_{uy}(t)$ and $\eta_{uy}(t)$ both tend to zero. Thus the integrand tends pointwise to $\phi(0)\ \Psi(0) = 1/2\pi$. Also this convergence is dominated since

$$\eta_{uy}(t) \ \le\ K_1 ur''(t) + K_2,$$

$$\phi(m_{uy}/\sigma)\ \Psi(\eta_{uy}) \ \le\ K_3 + ur''(t),$$

$$\int_T^{T+x} ur''(t)\ dt \ =\ ur'(T+x) - ur'(T) \ \to\ 0.$$

The theorem is proved. □

4. Proof of Theorem 1.1b.

We introduce the event E_I where $I \subseteq R$:

$$E_I\colon \{m_{uy}(t) + \delta(t) < 0 \quad \text{for all} \quad t \in I\}$$

and prove that for fixed y

$$(4.1) \qquad P(E_{(0,T_\theta^o]}) \ \to\ e^{-\theta}\ .$$

Let $T = T_\theta^o$. Since the expected number of crossings in $(T, T+x]$ tends to zero, we then have , by (4.1)

$$\lim_{u\to\infty} P(\tau_u > T+x) \ =\ \lim_{u\to\infty} \int_y P(E_{(0,T+x]})\ q_u{*}(y)\ dy$$

$$=\ \lim_{u\to\infty} \int_y P(E_{(0,T]})\ q_u{*}(y)\ dy \ =\ \int_y e^{-\theta} \lim q_u{*}(y)\ dy \ =\ e^{-\theta}$$

by dominated convergence.

To prove (4.1), we start along well-known lines and divide the interval $[T_-, T]$ into n subintervals, each of length Δ:

$$T_- = t_o < t_1 = t_o + \Delta < \ldots < t_k = t_o + k\Delta < \ldots < t_n = T.$$

Let α be a small number and split the intervals into two parts

$$I_k = [t_k, t_k+(1-\alpha)\Delta], \qquad J_k = (t_k+(1-\alpha)\Delta, t_{k+1}).$$

Let $t_{k,j} = t_k+j(1-\alpha)\Delta/n_k$, $j = 0,1,\ldots,n_k$ be a subdivision of the I_k-intervals and write for short

$$E_k = E_{I_k}$$

$$F_k: \{m_{uy}(t_{k,j}) + \delta(t_{k,j}) < 0 \quad \text{for} \quad j = 0, 1, \ldots, n_k\}.$$

Then (4.1) is a consequence of the following four lemmas.

LEMMA 4.1. If $\alpha \to 0$ as $u \to \infty$ then

$$\lim_{u\to\infty} P(E_{(0,T]}) = \lim_{u\to\infty} P\left(\bigcap_{k=1}^{n} E_k\right).$$

LEMMA 4.2. If $n_k = \log t_k$ and $\Delta \to 0$ as $u \to \infty$, then

$$\lim_{u\to\infty} P\left(\bigcap_{k=1}^{n} E_k\right) - P\left(\bigcap_{k=1}^{n} F_k\right) = 0.$$

LEMMA 4.3. If $n_k = \log t_k$ and $\Delta \to 0$ sufficiently slowly as $u \to 0$, then

$$\lim_{u\to\infty} P\left(\bigcap_{k=1}^{n} F_k\right) - \prod_{k=1}^{n} P(F_k) = 0$$

LEMMA 4.4. If $n_k = \log t_k$, $\alpha \to 0$, and $\Delta \to 0$ sufficiently slowly as $u \to \infty$, then

$$\lim_{u\to\infty} \prod_{k=1}^{n} P(F_k) = e^{-\theta}.$$

Before we proceed to the proof of these lemmas, we state and prove the following auxiliary results.

LEMMA 4.5. For $\delta > 0$, y fixed and $T_- \le t \le T$ it holds

$$\sqrt{(2-\delta)\log t} \le |m_{uy}(t)|/\sigma(t) \le \sqrt{(2+\delta)\log t}$$

if u is large enough.

PROOF. By Lemma 2.4, we have that $|m_{uy}(t)|/\sigma(t) \sim -ur'(t)/\sqrt{\lambda_2}$ for $T_- \le t \le T$. Since $-ur'(t) \le -ur'(T_-) = \sqrt{(2+\delta)\lambda_2 \log T_-} \le \sqrt{(2+\delta)\lambda_2 \log t}$ we have the right hand inequality. Since $-t\cdot r''(t)/r'(t)$ is bounded, the following inequality holds in the interval $(T - T/\log T, T)$:

$$-ur'(t) \le -ur'(T) + \frac{T}{\log T} \circ \sup ur''(t) \le -ur'(T) + K/\sqrt{\log T} .$$

Therefore

$$\begin{aligned}\int_0^T \phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right) dt &\ge \int_{T-T/\log T}^T \phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right) dt \\ &\ge K \circ \frac{T}{\log T} \cdot \exp\left\{-\frac{1}{2\lambda_2}(-ur'(T) + K'/\sqrt{\log T})^2\right\} \\ &\ge K \cdot \frac{T}{\log T} \circ \exp\left\{-\frac{1}{2\lambda_2} u^2 r'(T)^2 + K''\right\}\end{aligned}$$

from which it follows that $-ur'(T) \ge \sqrt{(2-\delta)\lambda_2 \log T}$ (since the left hand integral has a finite limit). Thus

$$-ur'(t) \ge -ur'(T) \ge \sqrt{(2-\delta)\lambda_2 \log T} \ge \sqrt{(2-\delta)\lambda_2 \log t}$$

as asserted. □

LEMMA 4.6. The covariance function $\bar{c}$ of the normalized process $\bar{\delta}(t) = \delta(t)/\sigma(t)$ fulfills

a) $\bar{c}(t, t+h) = 1 - \frac{\lambda_4}{\lambda_2} \cdot \frac{h^2}{2} + h \cdot r'(t)^2 \cdot H_h(t) + O(h^3 \cdot r'(t)^2 + h^2 \cdot r'(t)^4)$

where $H_h(t)$ is bounded as $h \to 0$, $t \to \infty$.

b) There is a $\gamma > 1$ such that for every $h_o > 0$ there exists M so that

$$|\bar{c}(t, t+h)| \leq Mh^{-\gamma} \quad \text{for} \quad h \geq h_o, \quad t \geq 0.$$

PROOF. Part a) follows from the explicit representation of $c(t, t+h)$. In part b), we only have to take γ less than the γ which exists according to condition C3.b. □

PROOF OF LEMMA 4.1. Write ν_J for the number of upcrossing zeros of the process $m_{uy}(t) + \delta(t)$ for $t \in J$. Then

$$0 \leq P\left(\bigcap_{k=1}^{n} E_k\right) - P(E_{(0,T]}) \leq P(\nu_{T_-}(y) \geq 1) + P(\nu_J \geq 1)$$

where $P(\nu_{T_-}(y) \geq 1) \leq E(\nu_{T_-}(y)) \to 0$ as $u \to \infty$. Furthermore, by Lemma 3.2

$$P(\nu_J \geq 1) \leq E(\nu_J) = \int_J \omega \, \phi(m_{uy}/\sigma) \, \Psi(\eta_{uy}) \, dt$$

$$\leq K \int_J \phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right) dt \to 0 \quad \text{if} \quad \alpha \to 0. \quad \square$$

PROOF OF LEMMA 4.2. With

ν_k = the number of upcrossing zeros of $m_{uy}(t) + \delta(t)$ in I_k

ν_k' = the number of "upcrossing zeros" of the sequence

$m_{uy}(t_{k,j}) + \delta(t_{k,j})$ for $j = 0, 1, \ldots, n_k$

we have

$$0 \leq P\left(\bigcap_{k=1}^{n} F_k\right) - P\left(\bigcap_{k=1}^{n} E_k\right) \leq P\left(\bigcap_{k=1}^{n} E_k^c F_k\right) \leq \sum_{k=1}^{n} P(E_k^c F_k)$$

$$\leq \sum_{k=1}^{n} P(\nu_k \geq 1, \ \nu_k' = 0) \leq \sum_{k=1}^{n} E(\nu_k - \nu_k').$$

To make this difference small, we take a small $\varepsilon > 0$ and choose u large enough to apply Lemma 3.2. Then

$$E(\nu_k) = \int_{I_k} \omega\, \phi(m_{uy}/\sigma)\, \Psi(\eta_{uy})\, dt$$

$$\leq (1+\varepsilon) \int_{I_k} \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right) \Psi(0)\, dt$$

$$E(\nu_k') = \sum_{j=1}^{n_k} P(m_{uy}(t_{k,j-1}) + \delta(t_{k,j-1}) < 0 < m_{uy}(t_{k,j}) + \delta(t_{k,j}))$$

$$(4.2) \qquad = \sum_{j=1}^{n_k} Q_{kj} \circ \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi(m_{uy}(t_{k,j})/\sigma(t_{k,j}))\, \Psi(0) \cdot (t_{k,j}-t_{k,j-1})$$

$$\geq (1-\varepsilon) \sum_{j=1}^{n_k} Q_{kj} \circ \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi(ur'(t_{k,j})/\sqrt{\lambda_2})\, \Psi(0) \cdot (t_{k,j}-t_{k,j-1}).$$

Below we will show that

$$Q_{kj} = \frac{P(m_{uy}(t_{k,j-1}) + \delta(t_{k,j-1}) < 0 < m_{uy}(t_{k,j}) + \delta(t_{k,j}))}{\sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi(m_{uy}(t_{k,j})/\sigma(t_{k,j}))\, \Psi(0) \cdot (t_{k,j}-t_{k,j-1})}$$

is uniformly greater than $1 - \varepsilon$ in all I_k if u is large. If we use the remainder of (4.2) as an approximating Riemann-sum (remember that $ur'(t)$ is monotonic) we get

$$E(\nu_k') \geq (1-\varepsilon)^3 \int_{I_k} \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right) \Psi(0)\, dt$$

and

$$\sum_{k=1}^{n} E(\nu_k-\nu_k') \leq \{(1-\varepsilon) - (1-\varepsilon)^3\} \sum_{k=1}^{n} \int_{I_k} \sqrt{\frac{\lambda_4}{\lambda_2}}\, \phi\left(\frac{ur'(t)}{\sqrt{\lambda_2}}\right) \Psi(0)\, dt \leq 4\varepsilon \cdot \theta$$

from which the lemma follows.

It remains to show that Q_{kj} is uniformly not less than $1 - \varepsilon$. Suppress the index k and write

$$x_j = -m_{uy}(t_{k,j})/\sigma(t_{k,j}),$$

$$\Delta_k = t_{k,j} - t_{k,j-1},$$

$$\bar{\delta}_j = \delta(t_{k,j})/\sigma(t_{k,j}),$$

$$\zeta_j = (\bar{\delta}_j - \bar{\delta}_{j-1})/\Delta_k,$$

and let $f_j(\cdot,\cdot)$ be the density function of the r.v. $(\bar{\delta}_{j-1},\zeta_j)$. Then the probability in Q_{kj} can be written as

$$\Delta_k^{-1} P(x_j - \Delta_k \cdot \frac{\bar{\delta}_j - \bar{\delta}_{j-1}}{\Delta_k} < \bar{\delta}_{j-1} < x_{j-1})$$

$$\geq \Delta_k^{-1} \int_{z=0}^{\infty} \int_{x=x_j-\Delta_k z}^{x_{j-1}} f_j(x,z)\, dx\, dz \quad = \quad (y = (x_j - x)/\Delta_k z)$$

$$= \int_{z=0}^{\infty} z \int_{y=(x_j-x_{j-1})/\Delta_k z}^{1} f_j(x_j - \Delta_k zy,\ z)\, dy\, dz$$

$$\geq \int_{z=0}^{\infty} z \int_{y=0}^{1} f_j(x_j - \Delta_k zy,\ z)\, dy\, dz. \tag{4.3}$$

Here we used that $x_j - x_{j-1} < 0$ for large u.

Since $\bar{\delta}_{j-1}$ and ζ_j have a joint normal distribution with mean zero and the covariance matrix

$$D_j = \begin{pmatrix} 1 & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} 1 & \Delta_k^{-1} (\bar{c}(t_{k,j-1}, t_{k,j}) - 1) \\ \cdot\cdot & 2\Delta_k^{-2} (1 - \bar{c}(t_{k,j-1}, t_{k,j})) \end{pmatrix}$$

we have

$$f_j(x_j - \Delta_k zy,\ z) = \frac{1}{2\pi\sqrt{\det D_j}} \exp(-A/2\det D_j)$$

where

$$A = d_{22}(x_j - \Delta_k zy)^2 - 2d_{12}(x_j - \Delta_k zy)z + z^2 .$$

Here $\max_{0\le y\le 1} A = A_{y=0} = d_{22}x_j^2 - 2d_{12}x_j z + z^2$ as soon as $0 \le z \le x_j/\Delta_k$ and we get the following lower bound for the integral (4.3).

$$\int_{z=0}^{x_j/\Delta_k} z \frac{1}{2\pi\sqrt{\det D_j}} \exp(-(d_{22}x_j^2 - 2d_{12}x_j z + z^2)/2\det D_j)\, dz$$

(4.4)

$$= \frac{1}{2\pi\sqrt{\det D_j}} \exp(-\tfrac{1}{2}x_j^2) \int_{z=0}^{x_k/\Delta_k} z \exp(-(z-d_{12}x_j)^2/2\det D_j)\, dz.$$

If $\bar{c} = \bar{c}(t_{k,j-1}, t_{k,j})$ then, by Lemma 4.6a, we have $\det D_j = \Delta_k^{-2}(1-\bar{c}^2) = \lambda_4/\lambda_2 - \Delta_k^{-1} r'(t_{k,j-1})^2(1+o(1))+o(1)$ as $\Delta_k \to 0$, $t_{k,j-1} \to \infty$. If $n_k = \log t_k$ then $\Delta_k = \Delta/n_k \sim \Delta/\log t_{k-1}$ and

$$0 \le \Delta_k^{-1} r'(t_{k,j-1})^2 \le 2r'(t_{k-1})^2 \log t_{k-1}/\Delta \to 0$$

since $r'(t) = O(t^{-\gamma})$ as $t \to \infty$. Thus $\det D_j \to \lambda_4/\lambda_2$.

Remark. We observe that the results hold even if $\Delta \to 0$ sufficiently slowly.

Furthermore, $x_j/\Delta_k \ge K\sqrt{\log t_k}/\Delta_k \to \infty$ and

$$|d_{12}x_j| = \Delta_k^{-1}(1-\bar{c}) \frac{|m_{uy}(t_{k,j})|}{\sigma(t_{k,j})} \le K(\Delta_k + r'(t_{k,j})^2)\, u^2 r'(t_{k,j})^2$$

$$\le K(\Delta/\log t_k + t_k^{-2\gamma}) \log t_k \to 0$$

if $\Delta \to 0$. This implies that the integral in (4.4) tends to λ_4/λ_2.

These considerations give that the lower bound in (4.3) is at least $(1-\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}}\,\phi(x_j)\Psi(0)$ i.e. $Q_{kj} \ge 1-\varepsilon$. □

Proof of Lemma 4.3. Write, for $i = 0,1,\dots,n_k$, $j = 0,1,\dots,n_\ell$

$$\bar{c} = \bar{c}_{ij} = \bar{c}(t_{k,i}, t_{\ell,j}), \quad x_{ki} = -m_{uy}(t_{k,i})/\sigma(t_{k,i}).$$

We will estimate the difference $P(\cap F_k) - \Pi P(F_k)$ by a method originally used by Berman [1] and improved by Cramér and Leadbetter [3], et al. The present version is due to Qualls and Watanabe [6].

$$\left|P\left(\bigcap_{k=1}^{n} F_k\right) - \prod_{k=1}^{n} P(F_k)\right| \le \sum_{1\le k<\ell\le n} \sum_{i=0}^{n_k} \sum_{j=0}^{n_\ell} |\bar{c}_{ij}| \cdot \tag{4.5}$$
$$\cdot \int_0^1 \phi(x_{ki}, x_{\ell j}; \lambda\bar{c}_{ij})\, d\lambda$$

where $\phi(\circ,\circ;\rho)$ is the density of a standardized bivariate normal r.v. with the correlation coefficient ρ:

$$\phi(x, y; \lambda\bar{c}) = \frac{1}{2\pi\sqrt{1-\lambda^2\bar{c}^2}} \, exp(-\tfrac{1}{2}B(\lambda))$$

$$B(\lambda) = (x^2 - 2\lambda\bar{c}xy + y^2)/(1 - \lambda^2\bar{c}^2).$$

Since, by Lemma 4.5, $x_{k,i}, x_{\ell,j} \ge \sqrt{(2-\delta)\ell og T}$, we have that $B(\lambda) \ge (2-\delta)\ \ell og\ T \cdot (2-2\lambda\bar{c})/(1-\lambda^2\bar{c}^2) \ge (2-\delta)(1+|\bar{c}|)^{-1}\ \ell og\ T$ and we get the following upper bound for the integral in (4.5):

$$(2\pi)^{-1}\ (1-\bar{c}^2)^{-\frac{1}{2}}\ exp\left\{-\ \frac{2-\delta}{1+|\bar{c}|}\ \ell og\ T\right\}.$$

Here $(1-\bar{c}^2)^{-\frac{1}{2}} \le \varepsilon^{-\frac{1}{2}}$, and we can take δ so small that $(2-\delta)(1+|\bar{c}|)^{-1} \ge 1+\delta' > 1$ for all $\bar{c}$. By Lemma 4.6b, there is an M such that $|\bar{c}(t,t+h)| \le Mh^{-\gamma}$ for all $h \ge \alpha\Delta$. Since $|t_{k,i}-t_{\ell,j}| \ge |t_k-t_\ell| - (1-\alpha)\Delta \ge \Delta|k-\ell| - (1-\alpha)\Delta$, we therefore have

$$|\bar{c}| \le M|t_{k,i}-t_{\ell,j}|^{-\gamma} \le M_\alpha \Delta^{-\gamma}\ |k-\ell|^{-\gamma}.$$

Using $n_k \le \ell og\ T$, we get the following upper bound for the double sum:

$$\sum_{1\le k<\ell\le n}\sum n_k n_\ell\ M_\alpha{}'\Delta^{-\gamma}\ |k-\ell|^{-\gamma}\ \varepsilon^{-\frac{1}{2}}\ exp(-(1+\delta')\ell og T)$$
$$\le M_\alpha{}''\Delta^{-\gamma} \sum_{1\le k<\ell\le n}\sum (\ell og\ T)^2\ |k-\ell|^{-\gamma}\ T^{-1-\delta'}$$

$$\leq M_\alpha''\Delta^{-\gamma} \sum_{k=1}^{n} (\log T)^2 T^{-1-\delta'} \sum_{v=1}^{\infty} v^{-\gamma}$$

$$\leq M_\alpha'''\Delta^{-\gamma} T\Delta^{-1} (\log T)^2 T^{-1-\delta'} = M_\alpha'''\Delta^{-1-\gamma} (\log T)^2 T^{-\delta'} .$$

Thus the double sum in (4.5) is bounded by a function of T that tends to zero as $T \to \infty$. The bound may become large when Δ and α are small but can be kept small by regulating the rate with which they tend to zero. Therefore the lemma is proved. □

PROOF OF LEMMA 4.4. We have to prove that if

$$\overline{m}(t) = m_{uy}(t)/\sigma(t), \qquad \overline{\delta}(t) = \delta(t)/\sigma(t),$$

and the event F_k is $\{\overline{m}(t_{k,i})+\overline{\delta}(t_{k,i}) < 0 \quad \text{for } i = 0,1,\ldots,n_k\}$ then

$$-\sum_{k=1}^{n} \log P(F_k) \to \theta.$$

The idea in the proof is to approximate $\overline{\delta}(t)$ in each I_k by two different sums of stationary, separable stochastic processes:

$$\delta^+(t) = \sqrt{1-p_k^2}\, \delta_1^{\,+}(t) + p_k\delta_2(t)$$

$$\delta^-(t) = \sqrt{1-p_k^2}\, \delta_1^{\,-}(t) + p_k\delta_2(t)$$

where $\delta_1^{\,+}$ and $\delta_1^{\,-}$ are "smooth" simple Gaussian processes with mean zero and the covariance functions

$$c_1^{\,+},\ c_1^{\,-}(h) = \cos\left\{(1\pm\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}} \cdot h\right\} \qquad \text{respectively}$$

and δ_2 is a process of "Markov" type with the covariance function

$$c_2(h) = 1 - K|h| \qquad |h| < 1/K$$

for some constant K, depending only on the upper bound of $H_h(t)$ in

Lemma 4.6a. The weights p_k are taken to be $-r'(t_k)$. A little reflexion shows that, for every $\varepsilon > 0$ there is a $(1-\alpha)\Delta > 0$ such that for all large u, $t \in I_k$, and $0 < h < (1-\alpha)\Delta$ it holds

$$\bar{c}(t, t+h) \begin{cases} < (1-p_k^2)\, c_1^-(h) + p_k^2\, c_2(h) \\ > (1-p_k^2)\, c_1^+(h) + p_k^2\, c_2(h). \end{cases}$$

Write $\Delta m_k = \varepsilon/u|r'(t_k)|$ and

$$m_k^- = (1-p_k^2)^{-\frac{1}{2}} (-ur'(t_k)/\sqrt{\lambda_2} + 2\Delta m_k)$$

$$m_k^+ = (1-p_k^2)^{-\frac{1}{2}} (-ur'(t_{k+1})/\sqrt{\lambda_2} - 2\Delta m_k).$$

Then, for large u and for $t \in I_k$, $-\lambda_2^{-\frac{1}{2}} ur'(t_{k+1}) - \Delta m_k \le -\bar{m}(t_{k+1}) \le -\bar{m}(t) \le -\bar{m}(t_k) \le -\lambda_2^{-\frac{1}{2}} ur'(t_k) + \Delta m_k$.

We now use the ordering property of the probabilities of extreme crossings derived by Slepian [7, Theorem 1] which says that, of two normalized Gaussian processes, the one with the uniformly larger covariance function has the larger probability of staying below a certain positive boundary. Thus

$$\begin{aligned} P(F_k) &= P(\bar{\delta}(t_{k,i}) < -\bar{m}(t_{k,i}), \quad i = 0, 1, \ldots, n_k) \\ &\ge P(\delta^+(t_{k,i}) < -\bar{m}(t_{k,i}), \quad i = 0, 1, \ldots, n_k) \\ &\ge P(\sqrt{1-p_k^2}\, \delta_1^+(t_{k,i}) < -\bar{m}(t_{k+1}) - \Delta m_k, \quad i = 0, 1, \ldots, n_k) \cdot \\ &\quad \cdot P(p_k \delta_2(t_{k,i}) < \Delta m_k, \quad i = 0, 1, \ldots, n_k) \\ &\ge P(\delta_1^+(h) < m_k^+, \quad 0 < h < (1-\alpha)\Delta) \cdot \\ &\quad \cdot P(\delta_2(h) < \Delta m_k/p_k, \quad 0 < h < (1-\alpha)\Delta) = P_1^+ \cdot P_2, \quad \text{say}, \end{aligned}$$

and

$$P(F_k) \le P(\delta^-(t_{k,i}) < -\bar{m}(t_{k,i}), \quad i = 0, 1, \ldots, n_k)$$

$$\leq P(\sqrt{1-p_k^2}\,\delta_1^+(t_{k,i}) < -\bar{m}(t_k) + \Delta m_k, \quad i = 0, 1, \ldots, n_k)$$
$$+ 1 - P(p_k\delta_2(t_{k,i}) < \Delta m_k, \quad i = 0, 1, \ldots, n_k)$$
$$\leq P(\delta_1^-(h) < m_k^-, \quad 0 < h < (1-\alpha)\Delta)$$
$$+ P\left\{\begin{matrix}\delta_1^- \text{ has at least two } m_k^-\text{-crossings in one of the} \\ \text{intervals } (t_{k,i}, t_{k,i+1}), \ i = 0, 1, \ldots, n_k-1\end{matrix}\right\}$$
$$+ 1 - P(\delta_2(h) < \Delta m_k/p_k, \quad 0 < h < (1-\alpha)\Delta)$$
$$= P_1^- + Q_1 + (1-P_2), \quad \text{say.}$$

The probabilities P_1^+ and P_1^- are known, see Slepian [7]:

$$P_1^+ = P(\sup_{0\leq h\leq(1-\alpha)\Delta} \delta_1^+(h) \leq m_k^+)$$
$$= \Phi(m_k^+) - \frac{(1-\alpha)\Delta}{2\pi}(1+\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}}\,exp(-\tfrac{1}{2}m_k^{+2})$$
$$\geq 1 - \frac{(1-\alpha)\Delta}{2\pi}(1+\varepsilon)^2\sqrt{\frac{\lambda_4}{\lambda_2}}\,exp(-\tfrac{1}{2}m_k^{+2})$$

since $m_k^+ \to \infty$. Similarly

$$P_1^- = \Phi(m_k^-) - \frac{(1-\alpha)\Delta}{2\pi}(1-\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}}\,exp(-\tfrac{1}{2}m_k^{-2})$$
$$\leq 1 - \frac{(1-\alpha)\Delta}{2\pi}(1-\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}}\,exp(-\tfrac{1}{2}m_k^{-2}).$$

Since $u^2r'(t_{k+1})^2/(1-p_k^2) - u^2r'(t_k)^2 \sim u^2r'(t_k)^2/(1-p_k^2) - u^2r'(t_k)^2 = u^2r'(t_k)^4/(1-p_k^2) \to 0$ and

$$m_k^{+2}, m_k^{-2} = (1-p_k^2)\left\{\frac{u^2r'(t_k)^2}{\lambda_2} \mp \frac{4\varepsilon}{\sqrt{\lambda_2}} + \frac{4\varepsilon^2}{u^2r'(t_k)^2}\right\}$$

we get, for large u, the bounds

$$P_1^+ \geq 1 - \frac{(1-\alpha)\Delta}{2\pi}(1+3\varepsilon)^3\sqrt{\frac{\lambda_4}{\lambda_2}}\,exp(-u^2r'(t_k)^2/2\lambda_2)$$

$$P_1^- \leq 1 - \frac{(1-\alpha)\Delta}{2\pi}(1-3\varepsilon)\sqrt{\frac{\lambda_4}{\lambda_2}}\,exp(-u^2r'(t_k)^2/2\lambda_2).$$

Since the process δ_1^- is a simple *cos*-process direct integrations show that

$$Q_1 = P\left\{\begin{array}{l}\text{there is a sub-interval of length } (1-\alpha)\Delta/n_k \text{ in } I_k \text{ in} \\ \text{which } \delta_1^- \text{ has at least two crossings of the level } m_k^-\end{array}\right\}$$

$$\leq \mathit{const}\cdot n_k \cdot P(\text{at least two crossings in } (0, \Delta/n_k))$$

$$\leq \mathit{const}\cdot n_k\cdot(\Delta/n_k)^3 m_k^- \mathit{exp}(-\tfrac{1}{2}m_k^{-2}) = o(1)\cdot \mathit{exp}(-u^2 r'(t_k)^2/2\lambda_2).$$

In total, we get

$$P_1^- + Q_1 \geq 1 - \frac{(1-\alpha)\Delta}{2\pi}(1-3\varepsilon)^2\sqrt{\frac{\lambda_4}{\lambda_2}}\ \mathit{exp}(-u^2 r'(t_k)^2/2\lambda_2).$$

For the probability P_2, we use a result due to Pickands [5, Lemma 2.5] which implies that if $\Delta m_k/p_k \to \infty$ and $\alpha \to 0$ then

$$1 - P_2 = P(\mathit{sup}_{0\leq h\leq(1-\alpha)\Delta}\ \delta_2(h) > \Delta m_k/p_k)$$

$$\sim \mathit{const}\cdot\Delta(\Delta m_k/p_k)^2\ (\Delta m_k/p_k)^{-1}\ \mathit{exp}(-\Delta m_k^{\ 2}/2p_k^{\ 2}).$$

Here $\Delta m_k/p_k$ $(=\varepsilon/u^2 r'(t_k)^2)$ obviously tends to infinity and if we use Lemma 4.5 it is easily proved that

$$1 - P_2 = o(1)\cdot \mathit{exp}(-u^2 r'(t_k)^2/2\lambda_2).$$

We sum up the obtained estimates:

$$P(F_k) \begin{array}{c}\geq\\ \leq\end{array} 1 - \frac{(1-\alpha)\Delta}{2\pi}(1\pm 3\varepsilon)^3\sqrt{\frac{\lambda_4}{\lambda_2}}\,\mathit{exp}(-u^2 r'(t_k)^2/2\lambda_2)$$

which give

$$-\sum_{k=1}^{n}\mathit{log}P(F_k) \leq (1-\alpha)(1+3\varepsilon)^3\sum_{k=1}^{n}\frac{\Delta}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}}\,\mathit{exp}(-u^2 r'(t_k)^2/2\lambda_2) + R_n$$

$$\leq (1-\alpha)(1+3\varepsilon)^3\sum_{k=1}^{n}\Delta\sqrt{\frac{\lambda_4}{\lambda_2}}\,\phi(ur'(t_k)/\sqrt{\lambda_2})\ \Psi(0) + R_n$$

$$\leq \quad (1-\alpha)(1+3\varepsilon)^3\theta + R_n$$

where $R_n \leq const\cdot \sum_{k=1}^{n} \Delta^2 exp(-u^2 r'(t_k)^2/\lambda_2) \to 0$ as $\Delta \to 0$ and $u \to \infty$. Similarly,

$$-\sum_{k=1}^{n} log P(F_k) \quad \geq \quad (1-\alpha)(1-3\varepsilon)^3 \sum_{k=1}^{n} \frac{\Delta}{2\pi} \sqrt{\frac{\lambda_4}{\lambda_2}}\, exp(-u^2 r'(t_k)^2/2\lambda_2)$$

$$\geq \quad (1-\alpha)(1-3\varepsilon)^3\theta + o(1).$$

Since ε is arbitrary and α is permitted to tend slowly to zero, we have the asserted result. □

5. Proof of Theorem 1.2B.

Let T be T^C or T^∞ according to if $C < \infty$ or $C = \infty$. The crucial point in the proof is the observation that the translated process

$${}^T\delta(t) = \delta(T+t)$$

converges weakly to $\xi'(t)$. More precisely, let P_T and P be probability measures for the processes $\{{}^T\delta(t), |t| \leq t_o\}$ and $\{\xi'(t), |t| \leq t_o\}$ respectively on the space $\{C, \mathcal{C}\}$ of continuous functions with the topology for uniform convergence.

Lemma 5.1. $P_T \Rightarrow P$ as $T \to \infty$, i.e. P_T converges weakly to P.

Proof. The process ${}^T\delta$ has the covariance function $c(T+s, T+t)$ which tends to $-r''(s-t)$ as $T \to \infty$, so that all finite dimensional distributions of P_T tend to those of P. To establish that $P_T \Rightarrow P$, we have to prove that the sequence $\{{}^T\delta\}$ is "tight", see e.g. Billingsley [2, Ch. 2]. Sufficient conditions for this are

a) $sup_T \; V({}^T\delta(0)) < \infty$

b) there are T_o, h_o, K such that, for $T \geq T_o$, $|h| \leq h_o$
$V({}^T\delta(t+h) - {}^T\delta(t)) < Kh^2$.

Here a) is obvious since $V({}^T\delta(0)) = V(\delta(T)) = c(T,T)$ which tends to $\lambda_2 < \infty$ as $T \to \infty$. For b), we have

$$
\begin{aligned}
V({}^T\delta(t+h) - {}^T\delta(t)) &= c(T+t+h,\, T+t+h) \\
&\quad + c(T+t,\, T+t) - 2c(T+t+h,\, T+t) \\
&= K_1(r'(t+h)-r'(t))^2 + K_2(r'(t+h) \\
&\quad -r'(t))(r'''(t+h)-r'''(t)) \\
&\quad + K_3(r'''(t+h)-r'''(t))^2 + K_4(r''(t+h)-r''(t))^2 \\
&\leq Kh^2
\end{aligned}
$$

for small h and all T.

PROOF OF THEOREM 1.2B. Let $\varepsilon > 0$ be arbitrarily small and define

$$
m_\varepsilon^+(t) = \begin{cases} (1+\varepsilon)\exp(-Ct) & (C < \infty) \\ \begin{cases} \infty & t < 0 \\ \varepsilon & t \geq 0 \end{cases} & (C = \infty) \end{cases}
$$

$$
m_\varepsilon^-(t) = \begin{cases} (1-\varepsilon)\exp(-Ct) & (C < \infty) \\ \begin{cases} t/\varepsilon & t < 0 \\ 0 & t \geq 0 \end{cases} & (C = \infty). \end{cases}
$$

Also write, for fixed y

$$
{}^T m_u(t) = m_{uy}(T+t).
$$

Lemma 3.3 then gives that for sufficiently large u

$$
m_\varepsilon^-(t) \leq -{}^T m_u(t) \leq m_\varepsilon^+(t) \qquad |t| \leq t_o.
$$

It is also always possible to find a t_o such that

$$1 - P(\xi'(t) < m_\varepsilon^+(t) \quad \text{for} \quad t < -t_o)$$
$$< \; 1 - P(\xi'(t) < m_\varepsilon^-(t) \quad \text{for} \quad t < -t_o) \; < \; \varepsilon$$

and for large T

$$1 - P({}^T\delta(t) < -{}^Tm_u(t) \quad \text{for} \quad -T < t < -t_o) \; < \; \varepsilon.$$

Then

$$P(m_{uy}(t)+\delta(t) < 0 \quad \text{for} \quad 0 < t < T+x)$$
$$= \; P({}^Tm_u(t)+{}^T\delta(t) < 0 \quad \text{for} \quad -T < t < x)$$
$$\le \; P({}^T\delta(t) < m_\varepsilon^+(t) \quad \text{for} \quad -t_o < t < x)$$

which, by Lemma 5.1, tends to

$$P(\xi'(t) < m_\varepsilon^+(t) \quad \text{for} \quad -t_o < t < x)$$
$$\le \; P(\xi'(t) < m_\varepsilon^+(t) \quad \text{for} \quad t < x) + \varepsilon.$$

Therefore

$$\limsup_{u \to \infty} P(\tau_u > T+x) \; \le \; P(\xi'(t) < m_\varepsilon^+(t) \quad \text{for} \quad t < x). \tag{5.1}$$

Similarly, we get the lower bound

$$P({}^T\delta(t) < -\,{}^Tm_u(t) \quad \text{for} \quad -t_o < t < x)$$
$$- \; (1-P({}^T\delta(t) < -{}^Tm_u(t) \quad \text{for} \quad -T < t < t_o))$$
$$\ge \; P({}^T\delta(t) < m_\varepsilon^-(t) \quad \text{for} \quad -t_o < t < x) - \varepsilon$$

which has the limit

$$P(\xi'(t) < m_\varepsilon^-(t) \quad \text{for} \quad -t_o < t < x) - \varepsilon$$

$$\geq P(\xi'(t) < m_\varepsilon^-(t) \quad \text{for} \quad t < x) - 2\varepsilon$$

and

(5.2) $$\liminf_{u \to \infty} P(\tau_u > T+x) \geq P(\xi'(t) < m_\varepsilon^-(t) \quad \text{for} \quad t < x).$$

Since the right hand sides in (5.1) and (5.2) can be made arbitrarily close to the required probability by taking ε small we have proved the asserted convergence. □

6. Proofs of Theorem 1.3 and 1.4.

Proof of Theorem 1.3. We already know that $\bar{\tau} = \tau_u - T \xrightarrow{L} \tau$. Since $\delta_u - u = -\xi_u(\tau_u)$, we have

$$(\tau_u - T,\ \delta_u - u) = (\bar{\tau},\ -ur(T+\bar{\tau}) + \eta_u(\lambda_2 r(T+\bar{\tau}) + r''(T+\bar{\tau}))$$
$$- \Delta(T+\bar{\tau})) = (\bar{\tau},\ -ur(T+\bar{\tau}) + o_p(1) - {}^T\Delta(\bar{\tau})),$$

where $o_p(1) \xrightarrow{P} 0$ and $ur(T+\bar{\tau}) \sim C^{-1} \exp(-C\bar{\tau})$ or 0 according to if $0 < C < \infty$ or $C = \infty$. Since ${}^T\Delta(t) = \Delta(T+t) = {}^T\Delta(0) + \int_0^t {}^T\Delta(s)\,ds$ we can extend Lemma 5.1 and use the tightness criterion on each of the conditional processes $({}^T\delta(\cdot) | {}^T\Delta(0) = x)$. We conclude that ${}^T\Delta(\bar{\tau})$ behaves like $\xi(0) + \int_0^{\bar{\tau}} \xi'(s)\,ds = \xi(\bar{\tau})$, and get the theorem. □

Proof of Theorem 1.4.

$$\delta_u/u = (u-\xi_u(\tau_u))/u = 1 - r(\tau_u) + u^{-1}\eta_u(\lambda_2 r(\tau_u) + r''(\tau_u))$$
$$- u^{-1}\,\Delta(\tau_u).$$

Since τ_u tends to infinity, all the variables $r(\tau_u)$, $r''(\tau_u)$ and $u^{-1}\eta_u$ tend to zero in probability. We estimate $u^{-1}\Delta(\tau_u)$ by

$$|u^{-1}\,\Delta(\tau_u)| \;=\; u^{-1}\,\left|\int_0^{\tau_u} \delta(t)dt\right| \;\le\; u^{-1}\,\tau_u\, \mathit{sup}_{0<t<\tau_u}\, |\delta(t)|.$$

Then, for all t_u, $x_u > 0$ we have

$$P(|u^{-1}\,\Delta(\tau_u)| \;\le\; u^{-1}\, t_u x_u) \;\ge\; P(\tau_u \le t_u,\, \mathit{sup}_{0<t<t_u}\, |\delta(t)| \le x_u)$$

$$\ge\; 1 - P(\tau_u > t_u) - P(\mathit{sup}_{0<t<t_u} |\delta(t)| > x_u).$$

By assumption, there is a $\gamma > 1$ and a constant $K > 0$ such that $|r'(t)| \le Kt^{-\gamma}$ for large t. Take $\delta > 0$ and $1 < \gamma' < \gamma$ and define

$$t_u \;=\; u^{1/\gamma'}, \quad x_u \;=\; \sqrt{(2+\delta)\lambda_2\, \mathit{log}\; u}.$$

Then $|ur'(t_u)| \le Kut_u^{-\gamma} \to 0$ and we conclude that if $-r''(t)/r'(t) \to 0$, then $P(\tau_u > t_u) \to 0$.

If $-r''(t)/r'(t) \to C > 0$, $\le \infty$, we observe that for $\varepsilon > 0$, we have $\mathit{lim\ inf}_{u\to\infty} |ur'(T-\varepsilon)| \ge M > 0$, so that for large u we have $T \le K_1\, u^{1/\gamma} + K_2$. This in turn implies that $t_u - T \to \infty$ and consequently that $P(\tau_u > t_u) \to 0$ even in this case.

Now let $t_- \to \infty$ so that $2\sigma^2(t) \le (2+\delta)\lambda_2$ for $t \ge t_-$ and

$$\lim_{u\to\infty} P(\mathit{sup}_{0<t<t_u} |\delta(t)| < x_u) \;=\; \lim_{u\to\infty} P(\mathit{sup}_{t_-<t<t_u} |\delta(t)| < x_u).$$

The probability of no crossings in (t_-, t_u) is bounded by the expected number of crossings which, in turn, is not more than

$$K_3 \int_{t_-}^{t_u} \phi(x_u/\sigma)\; \Psi\!\left(\frac{\mu}{\sqrt{1-\mu^2}} \cdot \frac{x_u}{\sigma}\right) dt$$

$$\le\; K_4\, t_u x_u\, \mathit{exp}(-x_u^{\,2}/(2+\delta)\lambda_2) \;=\; K_5\, u^{1/\gamma'} \cdot \sqrt{\mathit{log} u} \cdot \mathit{exp}(-\mathit{log} u)$$

which tends to zero if $u \to \infty$. Therefore

$$P(sup_{0<t<t_u} |\delta(t)|>x_u) \to 0.$$

Since, furthermore, $u^{-1} t_u x_u \to 0$, the theorem is proved. □

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