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THE INFINITELY MANY SERVER QUEUE WITH SEMI-MARKOVIAN ARRIVALS AND CUSTOMER DEPENDENT EXPONENTIAL SEVICE TIMES

by

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Abstract:

An infinitely many server queue with N types of customers arriving according to a semi-Markov process is studied. The service times are independent exponential random variables with means $1/\mu_i$ where i is the type of customer being served. The stationary distribution of the embedded Markov chain and the limiting distribution of the continuous time process are obtained.

1. Introduction

In this note we consider an infinitely many server queue where the arrivals of customers form a semi-Markov process, S - MP, and the service times are independent exponential random variables with parameter μ_i which depends on the type of customer being served. This note is an extension of a recent paper by Neuts and Chen (1970) which considers the same process except that the exponential service times did not depend on the type of customer being served.

Single server queues with semi-Markovian arrivals have been investigated by Cinlar (1967a, 1967b), Neuts (1966) and others. The queueing system discussed in this note is a generalization of the GI/M/∞ queue which has been fully studied by Takács (1962). The results in this note follow from a straightforward extension of Takács' arguments. We will restrict ourselves to finding some stationary and limiting properties of the queueing system. However the time dependent properties of the queueing system can also be obtained by a suitable extension of the GI/M/∞ queueing results.

Consider a queueing system with finitely many types of customers, N. Let the random variable $X(t)$ be the type of the last customer to arrive before time t. We assume that $X(t)$ is a finite state S - MP. Let $\tau_0, \tau_1, \tau_2 \ldots$ be the arrival times of the customers and let $Y_n = \tau_n - \tau_{n-1}$ for $n \geq 1$. Let X_n be the type of the n-th customer, $X_n = X(t)$ for $\tau_n \leq t \leq \tau_{n+1}$. The S - MP $X(t)$ is defined by the set of defective distribution functions

$$G_{ij}(t) = \Pr(X_n = j \quad Y_n \leq t \mid X_{n-1} = i)$$

Let

$$F_i(t) = \sum_{j}^{N} G_{ij}(t). \tag{1}$$

In this note we assume that $F_i(\infty) = 1$ and that

$$(2) \qquad \alpha_i = \int_0^\infty t \, d\, F_i(t) < \infty.$$

We also assume that $F_i(t)$ for $i = 1,2,\ldots N$ is not a lattice distribution. Let $q_{ij}(s)$ denote the Laplace-Stieltjes transform of G_{ij},

$$(3) \qquad q_{ij}(s) = \int_0^\infty e^{-st} \, d\, G_{ij}(t)$$

and let $\mathcal{Q}(s) = [q_{ij}(s)]$ denote the matrix of Laplace-Stieltjes transforms. $P = \mathcal{Q}(0)$ is the one-step transition matrix for the embedded Markov chain, X_n. Throughout this note it is assumed that P is an irreducible stochastic matrix. Let $\Pi = (\Pi_1, \Pi_2 \ldots \Pi_N)$ be the vector of stationary probabilities,

$$(4) \qquad \Pi P = \Pi .$$

A customer starts service immediately upon arrival. The service times are independent random variables with distribution function $1 - e^{-\mu_i X}$, for $X \geq 0$, where i is the type of customer being served. Let $\eta_i(t)$ denote the number of customers of type i in service at time t, and let $\eta(t) = (\eta_1(t), \eta_2(t), \ldots \eta_N(t))$ be the vector process denoting the number of customers of each type in the queue at time t.

Although this model is described in terms of queueing theory terminology, it was suggested by a generalization of a stochastic model for an enzyme reaction in an open system (Smith (1971)). This application is too specialized to develop in detail here.

For another application of this model consider the following simplified model of a large multi-purpose computer communicating with many remote terminals. The machine processes N different types of jobs according to a semi-Markov process. After a job is finished its output is sent to a communicating buffer where it waits until the remote terminal has received the entire output. Assume that the transmission times of the output are independent exponential random variables with parameter μ_i where i is the type of job being transmitted. This system corresponds to the queueing system described above, where the number of messages in the buffer from jobs of type i at time t corresponds to $\eta_i(t)$, the number of customers of type i in service at time t.

Let η_n be the vector denoting the number of customers in service just before the n-th customer arrives, $\eta_n = \eta(\,_n-0)$. In Section 2 we investigate the stationary distribution of the embedded Markov chain $\{(X_n, \eta_n)\}$ and in Section 3 we find the limiting distribution of the process $\{(X(t), \eta(t))\}$.

2. Stationary Distribution of the Embedded Markov Chain

Since the service times for this queue are Markov, the imbedded process, $\{(X_n, \eta_n)\}$, is a homogeneous Markov chain. To simplify the computations for this vector-valued process we introduce the following notation: let ℓ, m, and r be integer valued row vectors of length N, define

(5) $$\binom{m}{r} = \begin{cases} \prod_{i=1}^{N} \binom{m_i}{r_i} & \text{if } r_i \le m_i, \quad i = 1,2,\ldots N \\ 0 & \text{otherwise} \end{cases}$$

and

(6) $$b(\ell;\, m,\, \mu t) = \prod_{i=1}^{N} \binom{m_i}{r_i} e^{-\mu_i t\, l_i} (1 - e^{-\mu_i t})^{m_i - l_i}$$

Let $p(k, \ell; j, m)$ denote the one-step transition matrix for the embedded chain

(7) $$p(k, \ell; j, m) = \Pr(X_{n+1} = k,\ \eta_{n+1} = \ell \mid X_n = j,\ \eta_n = m).$$

At time τ_n a new customer of type X_n starts service. Thus at time $\tau_n + 0$ the number of customers in the queue is $\eta_n + \delta(X_n)$, where $\delta(i)$ is a row vector of length N with a 1 in the i-th place and zeros everywhere else. Given that $\eta_n = \ell$ and no new customers arrive in the interval (τ_n, τ_n+t), the number of customers of each type are independent binomial random variables,

$$\Pr(\eta(\tau_n+t) = \ell \mid X_n = j,\ \eta_n = m,\ \tau_{n+1} - \tau_n > t)$$

$$= b(\ell;\ m + \delta(j),\ \mu\, t).$$

Thus the one-step transition probability is

$$(8) \qquad p(k,\ \ell;\ j,\ m) = \int_0^\infty b(\ell;\ m + \delta(j),\ \mu\, t)\ d\, G_{jk}(t).$$

Let $\Pi(k, \ell)$ denote the stationary distribution for the embedded Markov chain, $\Pi(k, \ell) \geq 0$,

$$(9) \qquad \Pi(k, \ell) = \sum_{j=1}^{N} \sum_{m} \Pi(j, m) p(k, \ell; j, m)$$

and

$$(10) \qquad \Pi_k = \sum_{\ell} \Pi(k, \ell),$$

where Π_k, $k = 1,2, \dots N$, is the stationary distribution of the Markov chain X_n. Throughout this note $\sum_m$ will denote the sum over all non-negative interger-valued vectors of length N.

We define a generalized binomial moment $B_i(r)$, $i = 1,2,\dots N$, for the stationary distribution,

$$(11) \qquad B_i(r) = \begin{cases} \sum_{\ell} \binom{\ell}{r} \Pi(k, \ell) & \text{if } r_j \geq 0, \text{ for } j = 1,2,\dots N. \\ 0 & \text{otherwise} \end{cases}$$

and let $\mathcal{B}(r)$ denote the vector

(12) $$\mathcal{B}(r) = [B_1(r), B_2(r) \ldots B_N(r)].$$

We can now state a result which gives an algorithm for obtaining the binomial moments of the stationary distribution of the Markov chain (X_n, η_n).

THEOREM 1:

The stationary binomial moments of the embedded Markov chain, $\{(X_n, \eta_n)\}$ satisfy

(13) $$\mathcal{B}(\lambda) = \overline{\mathcal{B}}(\lambda)\, \mathcal{Q}(\mu\cdot\lambda)[I - \mathcal{Q}(\mu\cdot\lambda)]^{-1} \quad \text{for } \lambda \neq 0$$

where

(14) $$\overline{\mathcal{B}}(\lambda) = (B_1(\lambda - \delta(1)), B_2(\lambda - \delta(2)), \ldots, B_j(\lambda - \delta(j)), \ldots, B_N(\lambda - \delta(N))$$

and

$$\mathcal{B}(0) = \Pi \ .$$

PROOF:

Multiplying (9) by $\binom{\ell}{\lambda}$ and summing over all ℓ one obtains

(15) $$B_i(\lambda) = \sum_{\ell}\sum_{j}\sum_{m} \binom{\ell}{\lambda} p(i, \ell; j, m)\, \Pi(j, m)$$

$$= \sum_{j}\sum_{m} \Pi(j, m) \int_0^\infty \sum_{\ell} \binom{\ell}{\lambda} b(\ell; m+\delta(j), \mu t)\, d\, G_{ji}(t)$$

$$= \sum_{j}\sum_{m} \Pi(j, m) \binom{m+\delta(j)}{\lambda} \int_0^\infty e^{-(\mu\cdot\lambda)t}\, d\, G_{ji}(t)$$

$$= \sum_{j}\sum_{m} \Pi(j, m) \left[\binom{m}{\lambda} + \binom{m}{\lambda - \delta(j)}\right] q_{ji}(\mu\cdot\lambda)$$

$$= \sum_{j} (B_j(\lambda) + B_j(\lambda - \delta(j))\, q_{ji}(\mu\cdot\lambda) \ .$$

Line two of (15) is obtained by substituting (8). Line three follows from a standard result for the binomial moments of a binomial random variable. Lines four and five follow from the definitions of q_{ji} and $B_i(\mathbf{r})$. In matrix form (15) becomes

$$(16) \qquad \mathcal{B}(\mathbf{r}) = [\mathcal{B}(\mathbf{r}) + \overline{\mathcal{B}}(\mathbf{r})]\, \mathcal{Q}(\mu\cdot\mathbf{r}),$$

which yields (13).

The first and second moments for the enbedded Markov chain can easily be computed from Theorem 2.1. The first moments are

$$\mathcal{B}(\delta(i)) = \delta(i)\ \Pi_i\ \mathcal{Q}(\mu_i)[I - \mathcal{Q}(\mu_i)]^{-1}.$$

For the second moments we have for $i \neq j$

$$\mathcal{B}(\delta(i) + \delta(j)) = [B_i(\delta(j))\cdot\delta(i) + B_j(\delta(i))\delta(j)]\ \mathcal{Q}(\mu_i+\mu_j)[I-\mathcal{Q}(\mu_i+\mu_j)]^{-1},$$

and for $i = j$

$$\mathcal{B}(2\ \delta(i)) = \delta_i B_i(\delta(i))\ \mathcal{Q}(2\ \mu_i)[I - \mathcal{Q}(2\ \mu_i)]^{-1}.$$

With a little effort it can be shown that for the special case $\mu_i = \mu$ for $i = 1,2 \ldots N$, these results correspond to the results given in Neuts and Chen (1970).

Because of the complexity of the equations for the binomial moments, in practice one would usually be satisfied with calculating the first few binomial moments. However, in theory, given all the multivariate binomial moments, one can use an extension of the univariate inversion formula for binomial moments to find the stationary probabilities $\Pi(i, \ell)$.

COROLLARY:

The stationary probabilities for the $\{(X_n, \eta_n)\}$ process are

$$\Pi(i, \ell) = \sum_{r} (-1)^{\sum_{j=1}^{N} l_j + \sum_{j=1}^{N} r_j} \binom{r}{\ell} B_i(r). \tag{17}$$

PROOF:

For $N = 1$ (17) is the binomial inversion formula for the univariate case, (Riordan 1968). The proof for the multivariate case follows from a straight forward inductive argument on N.

SECTION 3.

In this section the limiting distribution of the continuous time process $\{(X(t), \eta(t))\}$,

$$\Pi^*(i, k) = \lim_{t\to\infty} \Pr(X(t) = i, \eta(t) = k), \tag{18}$$

and its binomial moments

$$B_i^*(r) = \sum_{k} \binom{k}{r} \Pi^*(i, k) \tag{19}$$

are investigated by using the results for S - MP with auxiliary paths (Pyke and Schaufele, 1966). Let $B^*(r)$ denote the row vector whose i-th element is $B_i^*(r)$, then:

THEOREM 2.

The binomial moments for the limiting distribution of the $(X(t), \eta(t))$ process are,

$$B^*(r) = ((\mu\cdot r)\alpha)^{-1}\,\overline{B}(r)[I - Q(r\cdot\mu)]^{-1}[I - V(r\cdot\mu)], \tag{20}$$

where

(21) $$\alpha = \sum_{i=1}^{N} \alpha_i \Pi_i$$

(22) $$V_i(s) = \sum_{j=1}^{N} q_{ij}(s),$$

(23) $$V(s) = \mathrm{Diag}(V_1(s), V_2(s) \ldots V_n(s))$$

and $\overline{B}(h)$ is defined in (14).

PROOF:

Let $\eta^-(t)$ be the value of the $\eta(t)$ process just before the last transition in the $X(t)$ process, $\eta^-(t) = \eta_n$ if $\tau_{n+1} > t \geq \tau_n$. The $(X(t), \eta^-(t))$ process is a S - MP, and $\eta(t)$ can be considered as an auxiliary path of this process. Using a known result for the limiting distribution of auxiliary paths (Pyke and Schaufele (1966, page 1459)) we have

(24) $$\Pi^*(i,j) = \frac{1}{\alpha} \sum_k \Pi(i,k) \int_0^\infty \Pr(\eta(\tau_n+u) = j \mid \tau_{n+1}-\tau > u, X_n = i, \eta_n = k)\,du.$$

From the properties of the $\eta(t)$ process we have

(25) $$\Pi^*(i,j) = \frac{1}{\alpha} \sum_k \Pi(i,k) \int_0^\infty b(j;\ k+\delta(i), \mu t)(1-F_i(t))\,dt$$

where b and F_i are defined in (6) and (1) respectively. Multiplying by $\binom{j}{h}$ and summing over all j one obtains

(26) $$B_i^*(h) = \frac{1}{\alpha} \sum_k \Pi(i,k) \int_0^\infty \sum_j \binom{j}{h} b(j;\ k+\delta(i), \mu t)(1-F_i(t))\,dt$$

$$= \frac{1}{\alpha} \sum_k \Pi(i,k) \binom{k+\delta(i)}{h} \int_0^\infty e^{-\mu \cdot h t} (1-F_i(t))\,dt$$

$$= (\alpha(\mu\cdot\lambda))^{-1}(1-V_i(\mu\cdot\lambda))(B_i(\lambda)+B_i(r-\delta(i))).$$

Using matrix notation and equation (13) we obtain (20).

Again the limiting probabilities, $\Pi^*(i,k)$, can be found by applying the multivariate inversion formula, equation (17).

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