

THE INVARIANCE PRINCIPLE FOR ONE-SAMPLE  
RANK-ORDER STATISTICS

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# THE INVARIANCE PRINCIPLE FOR ONE-SAMPLE RANK-ORDER STATISTICS<sup>1</sup>

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Analogous to the Donsker theorem on partial cumulative sums of independent random variables, for a broad class of one-sample rank-order statistics, weak convergence to Brownian motion processes is studied here. A simple proof of the asymptotic normality of these statistics for random sample sizes is also presented.

1. Introduction and the main theorem. Let  $\{X_1, X_2, \dots\}$  be a sequence of independent and identically distributed random variables (iidrv) having a continuous distribution function  $F(x)$ ,  $x \in \mathbb{R}$ , the real line  $(-\infty, \infty)$ . Let  $c(u)$  be equal to 1 or 0 according as  $u$  is  $\geq$  or  $< 0$ , and for every  $n \geq 1$ , let

$$(1.1) \quad R_{ni} = \sum_{j=1}^n c(|X_i| - |X_j|), \quad 1 \leq i \leq n.$$

Consider then the usual one-sample rank order statistic

$$(1.2) \quad T_n = \sum_{i=1}^n c(X_i) J_n(R_{ni}/(n+1)), \quad n \geq 1,$$

where the rank-scores  $J_n(i/(n+1))$  are defined in either of the following two ways:

$$(1.3) \quad (a) \quad J_n\left(\frac{i}{n+1}\right) = EJ(U_{ni}) \quad \text{or} \quad (b) \quad J_n\left(\frac{i}{n+1}\right) = J\left(\frac{i}{n+1}\right) = J(EU_{ni}), \quad 1 \leq i \leq n,$$

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$U_{n1} \leq \dots \leq U_{nn}$  are the ordered random variables of a sample of size  $n$  from the rectangular  $(0,1)$  distribution, and the score-function  $J(u)$  is specified by

$$(1.4) \quad J(u) = J_{(1)}(u) - J_{(2)}(u) \quad \text{is } \uparrow \text{ in } u: \quad 0 < u < 1, \quad i=1,2,$$

$J(u)$  is absolutely continuous inside  $(0,1)$ , and

$$(1.5) \quad \int_0^1 \{ |J_{(1)}(u)| + |J_{(2)}(u)| \} \{u(1-u)\}^{-1/2} du < \infty.$$

The condition (1.5), due to Hoeffding (1968), is slightly more restrictive than the square integrability condition of Hájek (1968), and it implies that

$$(1.6) \quad A_i^2 = \int_0^1 J_{(i)}^2(u) du < \infty, \quad i=1,2.$$

Two well-known cases are (i) the Wilcoxon signed rank statistic for which  $J(u) = u: \quad 0 \leq u \leq 1$ , and (ii) the normal scores statistics, for which  $J(u)$  is the inverse of the chi distribution with one degree of freedom.

Let us define  $H(x) = F(x) - F(-x)$ ,  $x \geq 0$ , and let

$$(1.7) \quad \mu_J(F) = \int_0^\infty J(H(x)) dF(x), \quad \mu_J = \int_0^1 J(u) du, \quad \text{and} \quad A^2(J) = \int_0^1 J^2(u) du.$$

Note that  $|\mu_J(F)| \leq \int_0^1 |J(u)| du \leq A(J) < \infty$ . Asymptotic normality of  $n^{-1/2}(T_n - n\mu_J(F))$  were studied by Govindarajulu (1960), Pyke and Shorack (1968), Puri and Sen (1969), Sen (1970), and Hušková (1970), among others. In the present paper, we are primarily interested in the classical invariance principle or weak convergence to Brownian motion processes of  $\{T_n\}$ .

For every  $n \geq 1$ , let

$$(1.8) \quad Y_n(0) = 0, \quad Y_n\left(\frac{k}{n}\right) = [T_k - k\mu_J(F)] / (\sigma n^{1/2}), \quad k=1, \dots, n,$$

where  $\sigma^2$  is defined by (3.10), and it is assumed that  $0 < \sigma^2 < \infty$ . Consider then a stochastic process  $Y_n = \{Y_n(t) : t \in I\}$ ,  $I = \{t : 0 \leq t \leq 1\}$ , where for  $t \in [k/n, (k+1)/n]$ , we let

$$(1.9) \quad Y_n(t) = Y_n(k/n) + (nt - k)[Y_n((k+1)/n) - Y_n(k/n)], \quad k=0, \dots, n-1.$$

Then  $Y_n$  belongs to the space  $C[0,1]$  of all continuous functions on  $I$ , with which we associate the uniform topology

$$(1.10) \quad \rho(Y_n, Y_n^*) = \sup_{t \in I} \{|Y_n(t) - Y_n^*(t)| : Y_n, Y_n^* \in C\}.$$

Finally, let  $W = \{W_t : t \in I\}$  be a standard Brownian motion, so that

$$(1.11) \quad EW_t = 0 \quad \text{and} \quad E(W_s W_t) = \min(s, t); \quad s, t \in I.$$

Then, the main theorem of the paper is the following.

Theorem 1. If  $\sigma^2$ , defined by (3.10) is positive and finite, and (1.3)-(1.5) hold, then  $Y_n$  converges (as  $n \rightarrow \infty$ ) in distribution in the uniform topology on  $C[0,1]$  to a standard Brownian motion  $W$ .

We may remark that, in particular, the Wilcoxon signed rank statistic can be expressed as a von Mises' (1947) functional, and hence, the result follows from Miller and Sen (1971) who considered a similar theorem for Hoeffding's (1948) U-Statistics and von Mises' functionals. But, in general, this characterization is not possible for  $T_n$ , and hence, a different proof is needed. Our method of approach is based on a powerful polynomial approximation of  $J(u)$  by Hájek (1968), a subsequent follow up by Hoeffding (1968), a martingale theorem on  $T_n$ , and a

recent functional central limit theorem for martingales by Brown (1971). The martingale theorem is considered in Section 2. The theorem for  $J(u)$  having a bounded second deviative is proved in Section 3, while the general case is treated in the next section. The concluding section is devoted to a few applications having importance in the developing area of rank based sequential inference procedures.

2. A martingale theorem. For every  $n \geq 1$ , define the vectors

$$(2.1) \quad \underline{c}_n = (c(X_1), \dots, c(X_n))' \quad \text{and} \quad \underline{R}_n = (R_{n1}, \dots, R_{nn})'$$

of signs and ranks defined by (1.1). Note that for every  $F$ , the distribution of  $T_n$  is solely determined by the joint distribution of  $(\underline{c}_n, \underline{R}_n)$ . Let  $\mathcal{F}_n$  be the  $\sigma$ -field generated by  $(\underline{c}_n, \underline{R}_n)$ ,  $n \geq 1$ , so that  $\mathcal{F}_n$  is  $\uparrow$  in  $n$ . Also, let  $a_1 = E(T_1) = J_1(\frac{1}{2})P\{X_1 > 0\}$ , and for  $n \geq 2$ , let

$$(2.2) \quad a_n = \sum_{r=1}^n J_n\left(\frac{r}{n+1}\right) E\{F(X_{n-1,r}^*) - F(X_{n-1,r-1}^*)\},$$

where  $X_{n-1,0}^* = 0$ ,  $X_{n-1,n}^* = \infty$  and  $X_{n-1,1}^* \leq \dots \leq X_{n-1,n-1}^*$  are the ordered values of  $|X_1|, \dots, |X_{n-1}|$ ,  $n \geq 2$ . Since  $P\{X_n \in [X_{n-1,r-1}^*, X_{n-1,r}^*]\} = h_{n,r} = E\{F(X_{n-1,r}^*) - F(X_{n-1,r-1}^*)\}$  can be written as

$$(2.3) \quad h_{n,r} = \binom{n-1}{r-1} \int_0^\infty [H(x)]^{r-1} [1-H(x)]^{n-r} dF(x), \quad r=1, \dots, n,$$

we obtain that

$$(2.4) \quad a_n = \sum_{r=1}^n J_n(r/(n+1)) h_{n,r}, \quad n \geq 1.$$

For later use, we note that

$$(2.5) \quad h_{n,r} = n^{-1} E\{[dF(X_{n,r}^*)/dH(X_{n,r}^*)]\}; \quad 0 \leq h_{n,r} \leq n^{-1},$$

for all  $r=1, \dots, n$ . Finally, we define

$$(2.6) \quad T_n^* = T_n - a_n^*; \quad a_n^* = \sum_{k=1}^n a_k.$$

Then, we have the following theorem which extends theorem 4.5 of Sen and Ghosh (1971) to underlying df's  $\{F\}$ , not necessarily symmetric about zero.

Theorem 2.1. If  $\int_0^1 |J(u)| du < \infty$  and the scores are defined by (a) in (1.3), then  $\{T_n^*, \mathfrak{F}_n; n \geq 1\}$  is a martingale.

Proof. By (1.2) and (2.6), for every  $n \geq 2$ ,

$$\begin{aligned} E\{T_n^* | \mathfrak{F}_{n-1}\} &= \sum_{i=1}^{n-1} c(X_i) E\{J_n(R_{ni}/(n+1)) | \mathfrak{F}_{n-1}\} - a_n^* \\ &\quad + E\{c(X_n) J_n(R_{nn}/(n+1)) | \mathfrak{F}_{n-1}\} - a_n, \end{aligned}$$

as  $c_{n-1}$  is held fixed under  $\mathfrak{F}_{n-1}$ . Also, given  $R_{n-1}$ ,  $R_{ni}$  can assume the two values  $R_{n-1i}$  and  $(R_{n-1i} + 1)$  with respective conditional probabilities,  $(1 - n^{-1} R_{n-1i})$  and  $n^{-1} R_{n-1i}$ ,  $1 \leq i \leq n-1$ . Hence,

$$\begin{aligned} E\{J_n((n+1)^{-1} R_{ni}) | \mathfrak{F}_{n-1}\} &= (n^{-1} R_{n-1i}) J_n\left(\frac{R_{n-1i} + 1}{n+1}\right) + (1 - n^{-1} R_{n-1i}) J_n\left(\frac{R_{n-1i}}{n+1}\right) \\ (2.8) \quad &= J_{n-1}(n^{-1} R_{n-1i}), \quad 1 \leq i \leq n-1, \end{aligned}$$

where the last equation follows from a well-known recursion relation among the expected order statistics (cf. [15, p. 198]). Thus, by (2.7) and (2.8),

$$(2.9) \quad E\{T_n^* | \mathfrak{F}_{n-1}\} = T_{n-1}^* + \{E[c(X_n) J_n((n+1)^{-1} R_{nn}) | \mathfrak{F}_{n-1}] - a_n\}.$$

Now  $c(X_n)J_n((n+1)^{-1}R_{nn})$  is either 0 (when  $X_n < 0$ ), or equal to  $J_n(r/(n+1))$ , when  $X_n \in [X_{n-1,r-1}^*, X_{n-1,r}^*]$ ,  $r=1, \dots, n$ . Thus, by (2.2) and (2.4),

$$(2.10) \quad E\{c(X_n)J_n((n+1)^{-1}R_{nn}) | \mathfrak{F}_{n-1}\} = \sum_{r=1}^n J_n\left(\frac{r}{n+1}\right)h_{n,r} = a_n.$$

Hence, from (2.9) and (2.10),

$$(2.11) \quad E(T_n^* | \mathfrak{F}_{n-1}) = T_{n-1}^*, \quad n \geq 2.$$

Also,  $E(T_1^*)=0$ . Hence the theorem follows.

Corollary 2.1.  $E(T_n) = a_n^*$  for all  $n \geq 1$ .

Proof. The result follows directly by noting that  $E(T_n^*)=0$ ,  $\forall n \geq 1$ , and that  $a_n$ ,  $n \geq 1$ , are all non-stochastic constants.

Remark. If, in particular,  $F(x)+F(-x)=1$ ,  $\forall x \geq 0$ ,  $dF(x)=\frac{1}{2}dH(x)$ , then  $h_{n,r} = \frac{1}{2n}$ , and hence,  $a_n = \{\sum_{i=1}^n J_n(i/(n+1))\}/2n = \frac{1}{2} \int_0^1 J(u)du = \frac{1}{2}\mu_J$ . Thus,  $\{T_n - \frac{n}{2}\mu_J, \mathfrak{F}_n; n \geq 1\}$  forms a martingale; this was already observed by Sen and Ghosh (1971).

3. Proof of the theorem when  $J''$  is bounded inside  $I$ . We define  $\{T_n^*\}$  as in (2.6) with the scores defined by (a) in (1.3), and let  $T_0^*=0$ . Let us then define two processes  $\xi_n = \{\xi_n(t): t \in I\}$  and  $\xi_n^* = \{\xi_n^*(t): t \in I\}$  by

$$(3.1) \quad \xi_n(t) = v_n^{-1}\{T_k^* + (T_{k+1}^* - T_k^*)(tv_n^2 - v_k^2)/(v_{k+1}^2 - v_k^2)\}, \quad (t \in I),$$

for  $v_k^2 \leq tv_n^2 \leq v_{k+1}^2$ ,  $k=0, \dots, n$ , where

$$(3.2) \quad v_k^2 = E\{T_k^*\}^2, \quad k \geq 0, \quad \text{so that } v_0^2 = 0;$$

$$(3.3) \quad \xi_n^*(t) = \{T_k^* + (nt-k)(T_{k+1}^* - T_k^*)\} / \{\sigma_n^{1/2}\} \text{ for } t \in \left(\frac{k}{n}, \frac{k+1}{n}\right),$$

$k=0,1,\dots,n-1$ , where  $\sigma^2$  is defined by (3.10). We shall approximate  $Y_n$  by  $\xi_n^*$  and subsequently by  $\xi_n$ , and the theorem will be proved for  $\xi_n$ . For this purpose, we let  $V_k = T_k^* - T_{k-1}^*$ ,  $k \geq 1$ , and define

$$(3.4) \quad q_1^2 = EV_1^2, \quad q_k^2 = E\{V_k^2 | \mathcal{F}_{k-1}\}, \quad k \geq 2;$$

$$(3.5) \quad Q_n = \sum_{k=1}^n q_k^2, \quad n \geq 1.$$

Since  $\{T_k^*, \mathcal{F}_k; k \geq 1\}$  has been shown in theorem 3.1 to be a martingale, by theorem 3 of Brown (1971), we obtain that

$$(3.6) \quad \xi_n \xrightarrow{\mathcal{D}} W \text{ as } n \rightarrow \infty,$$

provided that

$$(3.7) \quad Q_n^2 / V_n^2 \xrightarrow{P} 1 \text{ as } n \rightarrow \infty,$$

and for every  $\varepsilon > 0$ ,

$$(3.8) \quad V_n^{-2} \left\{ \sum_{i=1}^n E[V_i^2 I(|V_i| > \varepsilon V_n)] \right\} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where  $I(A)$  stands for the indicator function of the set  $A$ , and  $\xrightarrow{\mathcal{D}}$  for convergence in distribution.

Now, by (2.6) and Corollary 2.1, for every  $n \geq 1$ ,

$$(3.9) \quad V_n^2 = E(T_n^*)^2 = \text{Var}(T_n).$$

Let us also define

$$\begin{aligned}
(3.10) \quad \sigma^2 &= \int_0^{\infty} J^2(H(x))dF(x) - \left(\int_0^{\infty} J(H(x))dF(x)\right)^2 + \\
&2\left[\iint_{0 < x < y < \infty} H(x)[1-H(y)]J'(H(x))J'(H(y))dF(x)dF(y) - \right. \\
&\iint_{0 < x < y < \infty} H(x)J'(H(x))J(H(y))dF(x)dF(y) + \\
&\left.\iint_{0 < x < y < \infty} J(H(x))[1-H(y)]J'(H(y))dF(x)dF(y)\right].
\end{aligned}$$

Then by lemma 2 of Hušková (1970), it follows that

$$(3.11) \quad [0 < \sigma < \infty] \Rightarrow [\text{Var}(T_n)/n\sigma^2] \rightarrow 1 \text{ as } n \rightarrow \infty,$$

and hence, from (3.9) through (3.11), we obtain that

$$(3.12) \quad [0 < \sigma < \infty] \Rightarrow v_n / (n\sigma^2) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Thus, in the proof of (3.8), we may replace  $v_n$  by  $\sigma n^{\frac{1}{2}}$ . Now, by (2.6)

$$(3.13) \quad |v_n| \leq \sum_{i=1}^{n-1} \left| J_n\left(\frac{R_{ni}}{n+1}\right) - J_{n-1}\left(\frac{R_{n-1i}}{n}\right) \right| + \left| J_n\left(\frac{R_{nn}}{n+1}\right) \right| + |a_n|,$$

where by (2.4) and (2.5),  $|a_n| < A(J)$  for all  $n \geq 1$ . Also, by (1.4),

$$(3.14) \quad \left| J_n\left(\frac{R_{nn}}{n+1}\right) \right| \leq \sum_{s=1}^2 \left\{ \max_{1 \leq i \leq n} \left| J_{(s)n}\left(\frac{i}{n+1}\right) \right| \right\},$$

where  $J_{(s)n}(i/(n+1)) = EJ_{(s)}(U_{ni})$ ,  $1 \leq i \leq n$ ,  $s=1,2$ . Further, noting that for  $1 \leq i \leq n-1$ ,  $R_{n-1i} \leq R_{ni} \leq R_{n-1i} + 1$ , and by (1.4), the  $J_{(s)}$  are monotonic, we obtain by using (2.8) that

$$\begin{aligned}
(3.15) \quad \sum_{i=1}^{n-1} \left| J_n \left( \frac{R_{ni}}{n+1} \right) - J_{n-1} \left( \frac{R_{n-1i}}{n} \right) \right| &\leq \sum_{s=1}^2 \left( \sum_{i=1}^{n-1} \left| J_{(s)n} \left( \frac{i+1}{n+1} \right) - J_{(s)n} \left( \frac{i}{n+1} \right) \right| \right) \\
&= \left| J_{(1)n} \left( \frac{n}{n+1} \right) - J_{(1)n} \left( \frac{1}{n+1} \right) \right| + \left| J_{(2)n} \left( \frac{n}{n+1} \right) - J_{(2)n} \left( \frac{1}{n+1} \right) \right|.
\end{aligned}$$

Now, (1.5) insures that  $[J_{(s)}(u)(1-u)^{\frac{1}{2}}] \rightarrow 0$  as  $u \rightarrow 1$ , and hence,

$$(3.16) \quad J_{(s)} \left( \frac{n}{n+1} \right) = o(n^{\frac{1}{2}}), \quad \text{for } s=1,2.$$

Also, noting that  $J''$  is bounded inside  $I$ , we have

$$\begin{aligned}
(3.17) \quad \left| J_{(s)n} \left( \frac{n}{n+1} \right) \right| &= \left| E J_{(s)}(U_{nn}) \right| = \left| J_{(s)}(EU_{nn}) + E(U_{nn} - EU_{nn}) J'_{(s)}(EU_{nn}) \right. \\
&\quad \left. + \frac{1}{2} E(U_{nn} - EU_{nn})^2 K \right| \quad (\text{where } |K| < \infty) \\
&= \left| J_{(s)} \left( \frac{n}{n+1} \right) \right| + O(n^{-2}), \quad s=1,2,
\end{aligned}$$

as  $E(U_{nn} - EU_{nn})^2 = n/(n+1)^2(n+2) = O(n^{-2})$ . Thus, by (3.16) and (3.17),

$$(3.18) \quad \left| J_{(s)n} \left( \frac{n}{n+1} \right) \right| = o(n^{\frac{1}{2}}), \quad s=1,2,$$

and a similar case holds for  $\left| J_{(s)n} \left( \frac{1}{n+1} \right) \right|$ ,  $s=1,2$ . Hence, from (3.13) through

(3.18), we conclude that

$$(3.19) \quad |V_n| = o(n^{\frac{1}{2}}),$$

that is, for every  $\varepsilon > 0$  and  $0 < \sigma < \infty$ , there exists an  $n_0(\varepsilon, \sigma)$ , such that

$$(3.20) \quad |V_n| < \varepsilon \sigma \sqrt{n} \quad \text{for all } n \geq n_0(\varepsilon, \sigma).$$

On the other hand, for every  $k > 1$ ,

$$(3.21) \quad v_n^{-2} \sum_{i=1}^k E\{V_i^2 I(|V_i| > \epsilon v_n)\} \leq v_n^{-2} \sum_{i=1}^k E(V_i^2) \\ = v_k^2 / v_n^2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, (3.8) readily follows from (3.12), (3.20) and (3.21).

We now proceed to the proof of (3.7). By (2.6), we have

$$(3.22) \quad q_n^2 = \sum_{i=1}^{n-1} c^2(X_i) E\{[J_n(\frac{R_{ni}}{n+1}) - J_{n-1}(\frac{R_{n-1i}}{n})]^2 | \mathcal{F}_{n-1}\} \\ + \sum_{i \neq j=1}^{n-1} c(X_i) c(X_j) E\{[J_n(\frac{R_{ni}}{n+1}) - J_{n-1}(\frac{R_{n-1i}}{n})][J_n(\frac{R_{nj}}{n+1}) - J_{n-1}(\frac{R_{n-1j}}{n})] | \mathcal{F}_{n-1}\} \\ + [E\{[c(X_n) J_n(\frac{R_{nn}}{n+1})]^2 | \mathcal{F}_{n-1}\} - a_n^2] + \\ 2 \sum_{i=1}^{n-1} c(X_i) E\{[c(X_n) J_n(\frac{R_{nn}}{n+1})][J_n(\frac{R_{ni}}{n+1}) - J_{n-1}(\frac{R_{n-1i}}{n})] | \mathcal{F}_{n-1}\}.$$

Proceeding as in the proof of theorem 2.1, the first term on the right hand side (rhs) of (3.22) can be written as

$$(3.23) \quad \sum_{i=1}^{n-1} c(X_i) \{R_{n-1i} (n - R_{n-1i}) / n^2\} \{J_n(\frac{R_{n-1i}+1}{n+1}) - J_n(\frac{R_{n-1i}}{n+1})\}^2 \\ \leq \sum_{i=1}^{n-1} \{i(n-i) / n^2\} \{J_n(\frac{i+1}{n+1}) - J_n(\frac{i}{n+1})\}^2 \\ \leq \{ \max_{1 \leq i \leq n-1} [i^{\frac{1}{2}}(n-i)^{\frac{1}{2}} / n] |J_n(\frac{i+1}{n+1}) - J_n(\frac{i}{n+1})| \} \{ \sum_{i=1}^{n-1} \frac{i^{\frac{1}{2}}(n-i)^{\frac{1}{2}}}{n} |J_n(\frac{i+1}{n+1}) - J_n(\frac{i}{n+1})| \}.$$

Let us now denote by

$$(3.24) \quad K_1 = \sup_{0 < t < 1} |J'(u)| \quad \text{and} \quad K_2 = \sup_{0 < t < 1} |J''(u)|.$$

By our assumption, both  $K_1$  and  $K_2$  are finite, positive constants (depending only on  $J$ ). Then for every  $i$ :  $1 \leq i \leq n-1$

$$(3.25) \quad \left| J_n\left(\frac{i+1}{n+1}\right) - J_n\left(\frac{i}{n+1}\right) \right| = \left| J\left(\frac{i+1}{n+1}\right) - J\left(\frac{i}{n+1}\right) \right| + o(n^{-1}) \\ (\leq K_1(n+1)^{-1} + o(n^{-1})).$$

Consequently, the first factor on the right hand side of (3.23) is  $o(n^{-1})$ , while the second factor is bounded by  $CK_1 \int_0^1 [x(1-x)]^{\frac{1}{2}} dx < \infty$ , where  $C < \infty$ . Thus, (3.23) converges to zero as  $n \rightarrow \infty$ , with probability one.

Let us now define for every  $n \geq 1$ ,

$$(3.26) \quad F_n^*(x) = \frac{1}{n+1} \sum_{i=1}^n c(x - X_i) \quad (-\infty < x < \infty), \quad H_n^*(x) = F_n^*(x) - F_n^*(-x-), \quad x \geq 0.$$

As  $n/(n+1) \rightarrow 1$  with  $n \rightarrow \infty$ , by the Glivenko-Cantelli theorem, as  $n \rightarrow \infty$

$$(3.27) \quad \sup_{-\infty < x < \infty} |F_n^*(x) - F(x)| \rightarrow 0, \quad \sup_{x \geq 0} |H_n^*(x) - H(x)| \rightarrow 0,$$

almost surely (a.s.). For the second term on the rhs of (3.22), we note that for  $R_{n-1i} < R_{n-1j}$ , given  $\mathcal{F}_{n-1}$ ,  $(R_{ni}, R_{nj})$  can assume the values  $(R_{n-1i}, R_{n-1j})$ ,  $(R_{n-1i}, R_{n-1j+1})$  and  $(R_{n-1i+1}, R_{n-1j+1})$  with respective conditional probabilities  $(n - R_{n-1j})/n$ ,  $(R_{n-1j} - R_{n-1i})/n$  and  $R_{n-1i}/n$ . A similar case follows for  $R_{n-1i} > R_{n-1j}$ . Hence, by some simple steps, the second term on the rhs of (3.22) can be expressed in the integral form as

$$(3.28) \quad (2n^2/(n+1)^2) \left[ \iint_{0 < x < y < \infty} H_{n-1}^*(x) [1 - H_{n-1}^*(y)] \left\{ (n+1) \left[ J_n \left( \frac{nH_{n-1}^*(x)+1}{n+1} \right) - J_n \left( \frac{nH_{n-1}^*(x)}{n+1} \right) \right] \right\} \right. \\ \left. \left\{ (n+1) \left[ J_n \left( \frac{nH_{n-1}^*(y)+1}{n+1} \right) - J_n \left( \frac{nH_{n-1}^*(y)}{n+1} \right) \right] \right\} dF_{n-1}^*(x) dF_{n-1}^*(y) \right].$$

Now, by (3.24) and (3.25), the integrand in (3.28) is bounded (in absolute value), for all  $0 < x < y < \infty$ , by  $\frac{1}{4}(K_1 + \frac{1}{2}K_2)^2$ , and it converges a.s. [by (3.27)] to  $H(x)[1-H(y)]J'(H(x))J'(H(y))$  as  $n \rightarrow \infty$ . Consequently, we may write (3.28) as

$$(3.29) \quad 2 \iint_{0 < x < y < \infty} H(x)[1-H(y)]J'(H(x))J'(H(y))dF_{n-1}^*(x)dF_{n-1}^*(y) + o(1), \text{ a.s.},$$

as  $n \rightarrow \infty$ . Since, by (3.24), the integrand in (3.29) is bounded (in absolute value) by  $\frac{1}{4}K_1^2$  for all  $0 < x < y < \infty$ , and (3.27) holds, (3.29) converges a.s. to

$$(3.30) \quad 2 \iint_{0 < x < y < \infty} H(x)[1-H(y)]J'(H(x))J'(H(y))dF(x)dF(y), \text{ as } n \rightarrow \infty.$$

Proceeding as in the proof of theorem 2.1, the third term on the rhs of (3.22) can be shown to be equal to

$$(3.31) \quad \sum_{r=1}^n J_n^2(r/(n+1))h_{n,r} - a_n^2.$$

Now, by the same method of proof as in lemma 2 of Hušková (1970), it follows that under (3.24), as  $n \rightarrow \infty$ ,

$$(3.32) \quad |a_n - \mu_J(F)| \rightarrow 0, \quad \left| \sum_{r=1}^n J_n^2\left(\frac{r}{n+1}\right)h_{n,r} - \int_0^{\infty} J^2(H(x))dF(x) \right| \rightarrow 0.$$

Consequently, (3.31) converges (as  $n \rightarrow \infty$ ) to

$$(3.33) \quad \int_0^{\infty} J^2(H(x))dF(x) - \left[ \int_0^{\infty} J(H(x))dF(x) \right]^2.$$

Finally, noting that under  $\mathcal{F}_{n-1}$ ,  $(R_{nn}, R_{ni})$  ( $i \leq n-1$ ) can be either  $(r, R_{n-1i}+1)$ ,  $1 \leq r \leq R_{n-1i}$  or  $(r, R_{n-1i})$ ,  $R_{n-1i} < r \leq n$ , with respective conditional probability  $h_{n,r}$ ,  $1 \leq r \leq n$ , the last term on the rhs of (3.22) can be shown to be equal to

$$(3.34) \quad \begin{aligned} & 2 \sum_{i=1}^{n-1} c(X_i) \left\{ \sum_{j=1}^{R_{n-1i}} J_n \left( \frac{r}{n+1} \right) \right\} \left\{ J_n \left( \frac{R_{n-1i}+1}{n+1} \right) - J_{n-1} \left( \frac{R_{n-1i}}{n} \right) \right\} h_{n,r} \\ & + 2 \sum_{i=1}^{n-1} c(X_i) \left\{ \sum_{j=R_{n-1i}+1}^n J_n \left( \frac{r}{n+1} \right) \right\} \left\{ J_n \left( \frac{R_{n-1i}}{n+1} \right) - J_{n-1} \left( \frac{R_{n-1i}}{n} \right) \right\} h_{n,r} \\ & = 2 \sum_{i=1}^{n-1} c(X_i) \left\{ \left[ \sum_{j=1}^{R_{n-1i}} J_n \left( \frac{r}{n+1} \right) h_{n,r} \right] \left( \frac{n-R_{n-1i}}{n} \right) \left[ J_n \left( \frac{R_{n-1i}+1}{n+1} \right) - J_n \left( \frac{R_{n-1i}}{n+1} \right) \right] \right\} \\ & - 2 \sum_{i=1}^{n-1} c(X_i) \left\{ \left[ \sum_{j=R_{n-1i}+1}^n J_n \left( \frac{r}{n+1} \right) h_{n,r} \right] \frac{R_{n-1i}}{n} \left[ J_n \left( \frac{R_{n-1i}+1}{n+1} \right) - J_n \left( \frac{R_{n-1i}}{n+1} \right) \right] \right\}, \end{aligned}$$

by using (2.8). Again writing the above in the integral form [on using (3.24) - (3.26)], it follows by using (3.27) that as  $n \rightarrow \infty$ , it converges a.s. to

$$(3.35) \quad \begin{aligned} & 2 \iint_{0 < x < y < \infty} J(H(x)) [1-H(y)] J'(H(y)) dF(x) dF(y) \\ & - 2 \iint_{0 < x < y < \infty} H(x) J'(H(x)) J(H(y)) dF(x) dF(y). \end{aligned}$$

Thus, from (3.10), (3.22), (3.23), (3.30), (3.33) and (3.35), it follows that

$$(3.36) \quad [0 < \sigma < \infty] \Rightarrow q_n^2 / \sigma^2 \rightarrow 1 \text{ a.s., as } n \rightarrow \infty.$$

Consequently, by (3.5), (3.12) and (3.7),

$$(3.37) \quad Q_n^2 / (n\sigma^2) \xrightarrow{P} 1 \text{ as } n \rightarrow \infty,$$

which proves (3.7), and hence, (3.6) holds.

Now, by the tightness property of  $\xi_n$  [cf. Billingsley (1968, p. 56)], for every  $\varepsilon > 0$  and  $\eta > 0$  there exist a  $\delta > 0$  and an  $n_0 = n_0(\varepsilon, \eta)$ , such that

$$(3.38) \quad P\left\{ \sup_{|t-s| < \delta} |\xi_n(t) - \xi_n(s)| > \varepsilon \right\} < \eta \text{ for } n \geq n_0.$$

Hence, by (3.1), (3.3), (3.12) and (3.38), we obtain that

$$(3.39) \quad \rho(\xi_n, \xi_n^*) \xrightarrow{P} 0 \text{ as } n \rightarrow \infty,$$

so that by (3.6) and (3.39),

$$(3.40) \quad \xi_n^* \xrightarrow{D} W \text{ as } n \rightarrow \infty.$$

Let us now define another process in  $C[0,1]$  by

$$(3.41) \quad \tilde{\xi}_n^* = \{\tilde{\xi}_n^*(t) : t \in I\}, \quad \tilde{\xi}_n^*(t) = (v_n / n\sigma^2) \xi_n^*(t), \quad t \in I.$$

Then, by (3.12), (3.40) and a well-known theorem in Cramér (1946, p. 254), we obtain that

$$(3.42) \quad \tilde{\xi}_n^* \xrightarrow{D} W \text{ as } n \rightarrow \infty.$$

Finally, by (1.9), (3.3), (3.41) and corollary 2.1,

$$\begin{aligned}
(3.43) \quad \rho(Y_n, \tilde{\xi}_n^*) &= \sup_{t \in I} |Y_n(t) - \tilde{\xi}_n^*(t)| \\
&\leq \left\{ \max_{1 \leq k \leq n} |a_k^* - k\mu_J(F)| / \sigma n^{\frac{1}{2}} \right\} = \left\{ \max_{1 \leq k \leq n} |ET_k - k\mu_J(F)| / \sigma n^{\frac{1}{2}} \right\} \\
&\leq Mn^{-\frac{1}{2}}, \quad \text{where } M(\infty) \text{ depends only on } J,
\end{aligned}$$

and the last inequality follows from lemma 2 of Hušková (1970). Hence the proof is completed for scores defined by (a) in (1.3). If the scores are defined by (b) in (1.3), we note that

$$\begin{aligned}
&n^{-\frac{1}{2}} \left| \left\{ \sum_{i=1}^k c(X_i) J(R_{ki} / (k+1)) - \sum_{i=1}^k c(X_i) EJ(U_{ki}) \right\} \right| \\
&\leq n^{-\frac{1}{2}} \sum_{i=1}^k |J(i/(k+1)) - EJ(U_{ki})| \\
(3.44) \quad &= n^{-\frac{1}{2}} \sum_{i=1}^k \left| J\left(\frac{i}{k+1}\right) - J(EU_{ki}) - E(U_{ki} - EU_{ki}) J'(EU_{ki}) - \right. \\
&\quad \left. - \frac{1}{2} E\{(U_{ki} - EU_{ki})^2 J''(\theta U_{ki} + (1-\theta)EU_{ki})\} \right| \quad (0 < \theta < 1) \\
&\leq \frac{1}{2} n^{-\frac{1}{2}} K_2 k(k+1)^{-1} \quad \text{for all } k \leq n, \quad \text{by (3.24)}.
\end{aligned}$$

Hence, the metric  $\rho(Y_n, Y_n^*)$ , defined by (1.10) for the two processes with the  $T_k$  defined by (a) and (b) in (1.3) is  $O(n^{-\frac{1}{2}})$ , and thereby tends to zero as  $n \rightarrow \infty$ . The proof of the theorem for bounded  $J''$  is thus complete.

4. ~~The proof for the general case.~~ We now use the Hájek (1968) polynomial approximation of  $J(u)$ , as further studied by Hoeffding (1968). By lemma 4 of Hoeffding (1968), under (1.4) and (1.5), for every  $\alpha > 0$ , there exists a decomposition

$$(4.1) \quad J(u) = \psi(u) + \phi^{(1)}(u) - \phi^{(2)}(u), \quad 0 < u < 1,$$

such that  $\psi$  is a polynomial,  $\phi^{(1)}$  and  $\phi^{(2)}$  are non-decreasing, and

$$(4.2) \quad \int_0^1 \{|\phi^{(1)}(u)| + |\phi^{(2)}(u)|\} \{u(1-u)\}^{-\frac{1}{2}} du < \alpha;$$

the last inequality, in turn, implies that

$$(4.3) \quad \int_0^1 [\{\phi^{(1)}(u)\}^2 + \{\phi^{(2)}(u)\}^2] du < \alpha.$$

In (1.2), on replacing  $J$  by  $\psi$ ,  $\phi^{(1)}$  and  $\phi^{(2)}$ , we define  $T_n(\psi)$ ,  $T_n^{(1)}$  and  $T_n^{(2)}$ , respectively, so that

$$(4.4) \quad T_n = T_n(\psi) + T_n^{(1)} - T_n^{(2)}, \quad n \geq 1;$$

the corresponding processes, defined by (1.8) and (1.9), are denoted by  $Y_n(\psi)$ ,  $Y_n^{(1)}$  and  $Y_n^{(2)}$ , so that

$$(4.5) \quad Y_n = Y_n(\psi) + Y_n^{(1)} - Y_n^{(2)}.$$

Note that in (4.5), all the processes have  $(\sigma n^{\frac{1}{2}})$ , in the denominator, defined in (1.8).

Now from Hájek (1968) and Hušková (1970), along with (3.12), it follows that for every  $\varepsilon > 0$ , there exists an  $\alpha > 0$ , such that (4.2) holds, and

$$(4.6) \quad |1 - \{\text{Var}(T_n(\psi))\}^{\frac{1}{2}} / (\sigma n^{\frac{1}{2}})| < \varepsilon \quad \text{for } n \geq n_0(\varepsilon).$$

Since  $\psi$  is a polynomial and (4.6) holds, by the results of section 3,

$$(4.7) \quad Y_n(\psi) \xrightarrow{\mathcal{L}} W, \quad \text{as } n \rightarrow \infty.$$

Consequently, it suffices to show that for every  $\varepsilon > 0$  and  $\eta > 0$ , there exists a choice of  $\alpha > 0$  in (4.2), such that with probability  $> 1 - \eta$ ,

$$(4.8) \quad \sup_{t \in I} \{ |Y_n^{(1)}(t)| + |Y_n^{(2)}(t)| \} < \varepsilon \quad \text{for } n \geq n_0(\varepsilon, \eta).$$

For each  $i (= 1, 2)$ , we define  $a_{n(i)}$  as in (2.4) (for  $J = \phi^{(i)}$ ), and let

$$(4.9) \quad T_n^{(i)*} = T_n^{(i)} - a_{n(i)}^*; \quad a_{n(i)}^* = \sum_{k=1}^n a_{k(i)}, \quad n \geq 1.$$

Then

$$(4.10) \quad \sup_{t \in I} |Y_n^{(i)}(t)| \leq (\sigma n^{1/2})^{-1} \{ \max_{1 \leq j \leq n} |T_j^{(i)*}| + \max_{1 \leq j \leq n} |a_{j(i)}^* - j \mu_J^{(i)}(F)| \},$$

where

$$(4.11) \quad \mu_J^{(i)}(F) = \int_0^\infty \phi^{(i)}(H(x)) dF(x), \quad i=1,2.$$

Let us first consider the case where the scores are defined by (a) in (1.3).

Then, by theorem 2.1,  $\{T_k^{(i)*}, \mathcal{F}_k; k \geq 1\}$  is a martingale, and hence, by the Kolmogorov inequality for martingales [viz., Feller (1965, p. 235)], for every  $\eta > 0$ ,

$$\begin{aligned}
 (4.12) \quad P\{\max_{1 \leq k \leq n} |T_k^{*(i)}| > \eta \sigma n^{\frac{1}{2}}\} &\leq \text{Var}(T_n^{(i)*}) / (\eta^2 \sigma^2) \\
 &= [\text{Var}(T_n^{(i)})] / (\eta^2 \sigma^2), \text{ as } ET_n^{(i)*} = 0.
 \end{aligned}$$

Also, by theorem 4 of Hušková (1970), for each  $i(=1,2)$

$$(4.13) \quad \text{Var}(T_n^{(i)}) \leq 10 \sum_{k=1}^n [\phi_n^{(i)}(\frac{k}{n+1})]^2 \leq 10 n \int_0^1 \{\phi^{(i)}(u)\}^2 du.$$

Consequently, by (4.3) and (4.13), (4.12) can be made arbitrarily small by proper choice of  $\alpha(>0)$ .

Now, under (1.5), we obtain on using our corollary 2.1 and proceeding as in proposition 2 of Hoeffding (1968) that

$$\begin{aligned}
 (4.14) \quad |a_{k(i)}^{*-k\mu_J^{(i)}}(F)| &= |ET_k^{(i)*-k\mu_J^{(i)}}(F)| \\
 &\leq C k^{\frac{1}{2}} \alpha, \quad k \geq 1, \quad C < \infty,
 \end{aligned}$$

where  $C$  does not depend on  $\alpha$  or the  $\phi^{(i)}$ . Consequently,

$$(4.15) \quad \max_{1 \leq k \leq n} |a_{k(i)}^{*-k\mu_J^{(i)}}(F)| / (\sigma n^{\frac{1}{2}}) \leq (C/\sigma) \alpha, \quad i=1,2,$$

and (4.15) can be made adequately small by proper choice of  $\alpha(>0)$  in (4.2).

Thus (4.8) holds, and the proof is completed for scores defined by (a) in (1.3).

We complete the proof of the theorem by considering the scores defined by (b) in (1.3). Since in (4.1),  $\psi$  is the polynomial component on which the results of section 3 apply, all we need to show is that on defining

$$(4.16) \quad T_n^{(i)} = \sum_{i=1}^n c(X_i) \phi^{(i)} \left( \frac{R_{ni}}{n+1} \right), \quad \tilde{T}_n^{(i)} = \sum_{i=1}^n c(X_i) E \phi^{(i)} (U_{nR_{ni}}), \quad n \geq 1$$

( $i=1,2$ ), that for every  $\varepsilon > 0$  and  $\eta > 0$ , there exists a  $\alpha > 0$  in (4.2), such that

$$(4.17) \quad P\{ \max_{1 \leq k \leq n} |T_k^{(i)} - \tilde{T}_k^{(i)}| / (\sigma n^{1/2}) > \varepsilon \} < \eta, \quad i=1,2,$$

for all  $n \geq n_0(\varepsilon, \eta)$ . Now, by our (4.16) and proposition 1 of Hoeffding (1968),

$$(4.18) \quad |T_k^{(i)} - \tilde{T}_k^{(i)}| \leq \sum_{i=1}^k |\phi^{(i)}(i/(k+1)) - E \phi^{(i)}(U_{ki})| \\ \leq C_1 k^{1/2} \int_0^1 |\phi^{(i)}(u)| \{u(1-u)\}^{-1/2} du, \quad i=1,2, \quad C_1 < \infty,$$

where  $C_1$  does not depend on  $\phi^{(i)}$ . Hence, (4.17) readily follows from (4.18) and (4.2) by proper choice of  $\alpha > 0$ , and the proof is terminated.

Remark. If  $F$  is symmetric about 0,  $F(x) + F(-x) = 1$ ,  $\forall x \geq 0$ , so that  $a_{k(i)} = \mu_J^{(i)}(F) = \frac{1}{2} \int_0^1 \phi^{(i)}(u) du$ , for  $i=1,2, k \geq 1$ . Thus, we do not require (4.14) and (4.15), and hence, for scores defined by (a) in (1.3), the square integrability condition (1.6) and (1.4) suffice our purpose. However, in general, for arbitrary  $F$  or scores defined by (b) in (1.3), we are not in a position to replace (1.5) by (1.6).

5. Weak convergence for random sample sizes. For every  $t > 0$ , consider now a positive integer-valued random variable  $N_t$ , and for  $N_t = n (> 1)$  define  $T_n$  and  $Y_n$  as in (1.2), (1.8) and (1.9). If  $N_t$  satisfies the condition

$$(5.1) \quad \lim_{t \rightarrow \infty} (N_t/t) = \theta: \quad 0 < \theta < \infty,$$

then analogous to theorem 1, we have under (1.3)-(1.5)

$$(5.2) \quad Y_{N_t} \xrightarrow{\mathcal{L}} W, \text{ as } t \rightarrow \infty.$$

The proof is quite similar to theorem 17.1 of Billingsley (1968, p. 146), and follows as a corollary to our main theorem in section 1. For brevity, the details are omitted. In particular, we have from (5.2) that

$$(5.3) \quad \mathcal{L}(N_t^{-1/2}(T_{N_t} - N_t \mu_J(F))/\sigma) \rightarrow \mathcal{N}(0,1) \text{ as } t \rightarrow \infty;$$

for a different proof of (5.3) under slightly different regularity conditions, we may refer to Pyke and Shorack (1968).

We conclude this section by the following problem. As in section 2, define  $T_n^* = T_n - a_n^*$ ,  $n \geq 1$ . For every  $\tau > 0$ , define a positive number  $K_\tau$ , such that

$$(5.4) \quad \lim_{\tau \rightarrow \infty} \tau^{-1/2} K_\tau = K^*: \quad 0 < K^* < \infty.$$

Let then

$$(5.5) \quad T_\tau^+ = \inf\{n: T_n^* \geq K_\tau\}, \quad \tau > 0,$$

i.e.,  $T_\tau^+$  is the first time (n),  $T_n^*$  exceeds or reaches  $K_\tau$ . We want to find an expression for

$$(5.6) \quad P\{T_\tau^+ \leq t_\tau\} \quad \text{for } t_\tau > 0.$$

Note that on denoting by  $[s]$  the greatest integer contained in  $s$ ,

$$\begin{aligned}
 (5.7) \quad P\{T_{\tau}^+ < t_{\tau}\} &= P\{T_n^* > K_{\tau} \text{ for at least one } n: 1 \leq n \leq [t_{\tau}]\} \\
 &= P\{\max_{1 \leq n \leq [t_{\tau}]} T_n^* / \sigma [t_{\tau}]^{1/2} \geq K_{\tau} / \sigma [t_{\tau}]^{1/2}\},
 \end{aligned}$$

where  $\sigma^2$  is defined by (3.10). Thus, if

$$(5.8) \quad t_{\tau} = c^2 \tau + o(\tau), \quad 0 < c < \infty,$$

from (5.4), (5.7) and (5.8), we obtain on using theorem 1 that

$$(5.9) \quad \lim_{\tau \rightarrow \infty} P\{T_{\tau}^+ < t_{\tau}\} = P\{\sup_{u \in I} W_u \geq (K^*/c\sigma)\}$$

when  $W = \{W_u : u \in J\}$  is standard Brownian motion on  $I$ . Hence, by a well-known result on  $W$ , we have from (5.9)

$$(5.10) \quad \lim_{\tau \rightarrow \infty} P\{T_{\tau}^+ < t_{\tau}\} = 2 \left\{ \frac{1}{\sqrt{2\pi}} \int_{\beta}^{\infty} e^{-\frac{1}{2}u^2} du \right\}; \quad \beta = K^*/c\sigma.$$

Similarly, if

$$(5.11) \quad T_{\tau}^* = \inf\{n: |T_n^*| > K_{\tau}\}, \quad \tau > 0,$$

it can be shown that

$$(5.12) \quad \lim_{\tau \rightarrow \infty} P\{T_{\tau}^* < t_{\tau}\} = \sum_{k=-\infty}^{\infty} (-1)^k \int_{(2k-1)\beta}^{(2k+1)\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du, \quad \beta = K^*/c\sigma.$$

These results are of importance in sequential tests based on  $\{T_n\}$ .

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