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# LINEAR REPRESENTATIONS OF DERIVED SHEARS PLANES*

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## Abstract

A linear representation is determined for any semitranslation plane derived from a shears plane $\delta T$, which generalizes that determined in a special case by Bose and Barlotti. The connection is made between the geometry of the representation of $\delta T$ in the case that $\delta T$ is derivable and the conditions imposed on coordinatizations of $\delta T$ determined by Ostrom. Finally, we note that a family of translation planes discovered by Knuth are derivable. The derived planes are not strict semitranslation planes but they fall in general outside the class covered by Bose and Barlotti in their representation.

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## §1. Preliminaries

*Definition:* A *right quasifield* $Q = (Q, +, \circ)$ is a set containing distinct elements 0, 1 and admitting 2 binary operations "+" and "$\circ$" called addition and multiplication and satisfying

1) $(Q,+)$ is an abelian group with null element $0$.
2) $0 \circ a = a \circ 0 = 0$ for all $a$ in $Q$.
3) $(Q\backslash\{0\}, \circ)$ is a loop with identity 1.
4) $(a+b) \circ c = a \circ c + b \circ c$ for all $a, b, c$ in $Q$.
5) for $r, s, d$ in $Q$, $r \neq s$ there is a unique $x$ in $Q$ so that $x \circ r - x \circ s = d$.

To each right quasifield, we associate a translation plane as follows: points are ordered pairs of elements of $Q$, lines are sets of the form $\{(x,y)\colon x = c, c\in Q\}$ or $\{(x,y)\colon y = x\circ m+b\}$ and incidence is given by containment.

The *kernel* of a right quasifield is the set of all $k$ such that $k\circ(x+y) = k\circ x + k\circ y$ and $k\circ(x\circ y) = (k\circ x)\circ y$ for all $x, y$ in $Q$.

The kernel is in general a skew field but since all planes and quasifields in this paper are finite, the kernel will be a field.

If $H$ is a subfield of the kernel of $Q$, then $Q$ is a left vector space of dimension $r$ over $H$ for some $r$. The translation plane $T$ associated with $Q$ may be represented in $S_{2r}$, the $2r$ dimensional projective space over $H$, as follows: Let $\Sigma$ be a hyperplane in $S_{2r}$ and write the elements of $Q$ as r-vectors with components from $H$. Let $s_\infty = \{(0,x)\colon x \text{ in } Q\}$ and for $m$ in $Q$ write $s_m = \{(x,x\circ m)\colon x \text{ in } Q\}$. Let $\mathcal{S} = \{s_\infty\} \cup \{s_m\colon m \text{ in } Q\}$. $\mathcal{S}$ consists of $r-1$ spaces and these elements of $\mathcal{S}$ partition the points of $\Sigma$. Such a set $\mathcal{S}$ is called an $r-1$ spread. The points of $T$ are represented by the points of $S_{2r}$ not in $\Sigma$ and the lines of $T$ are those $r$ spaces meeting $\Sigma$ in a member of $\mathcal{S}$. Incidence is given by containment.

T may be extended to a projective plane by adjoining $\Sigma$ and the members of $S$ as the line and points at infinity respectively.

T may be described by means of the spread $S$ and without reference to a specific quasifield. Such a representation of T will be called the usual representation of T.

A *shears plane* is the dual of a translation plane and may be represented in $S_{2r}$ by interchanging the definition of point and line. The shears plane may be given an affine representation by deleting one member $s'$ of the spread $S$ as the line at infinity and all points on it, namely $\Sigma$ and all r spaces meeting $\Sigma$ in $s'$.

If T is coordinatized by Q then the dual of T, denoted by $\delta T$ (here we require that $s' = s_\infty$) may be coordinatized by the left quasifield $Q^*$ with the same addition as in Q but with $a*b = b \circ a$ where $*$ denotes multiplication in $Q^*$.

In a projective plane P of order $m^2$, a subplane of order m is called a Baer subplane.

In a projective plane of order $m^2$, a *derivation set* is a set $\mathcal{D}$ of m+1 points on some line W such that for any pair of points x, y not on W, if the line joining x to y meets $\mathcal{D}$ in a point then $\langle \mathcal{D} \cup \{x,y\} \rangle$, the subplane generated by x and y, is a Baer subplane.

If P is made into an affine plane $\bar{P}$ by deleting W, then a new affine plane may be defined: $\Delta P$ points are the points of $\bar{P}$, and $\Delta P$ lines are i) the lines of $\bar{P}$ meeting W in a point not in $\mathcal{D}$, and ii) those Baer subplanes whose points at infinity are precisely those elements in $\mathcal{D}$.

The Bose-Barlotti representation: Let T be a translation plane of degree 2 over its kernel. Represent T in the usual way in projective 4 space over its kernel. Let $\Omega$ be a 3 space meeting $\Sigma$ in a plane $\Pi$ which contains the line $s'$ of the spread $S$. In the shears plane $\delta T$, we define the derived plane $\Delta\delta T$: $\Delta\delta T$ points are those planes meeting $\Sigma$ in a line of $\mathcal{S}$ distinct from $s'$, $\Delta\delta T$ lines are i) the points of $S_4$ not in $\Sigma$ or $\Omega$, and ii) the planes in $\Omega$ meeting $\Pi$ in a line distinct from $s'$. Incidence of a point and a line of type i) is given by containment and a point and a line of type ii) are incident just when the planes representing the point and the line meet in a line of $S_4$.

The Baer subplanes in the derivation of $\delta T$ are desarguesian, since they are merely planes in $S_4$. The derivation set $\mathcal{D}$ is the 3 space $\Sigma$ and those planes in $\Omega$ meeting $\Sigma$ in $s'$.

## §2. Linear representations of derivable shears planes

Let $T$ be a translation plane of order $p^r$, $r = 2s$, $p$ a prime, such that its dual $\delta T$ is derivable. Represent $T$ in the usual manner in the projective space $S_{2r}$ of dimension $2r$ over $GF(p)$.

*Theorem 1.* An r-space $R$ meeting $\Sigma$ in an $r-1$ space represents a Baer subplane of $T$ if and only if for each $s$ in $\mathcal{S}$ either $R \cap s = \emptyset$ or $\dim R \cap s = s-1$.

*Proof:* Let $R$ represent a Baer subplane $B$ of $T$. Then $B$ has $p^s+1$ ideal points. Let $x$ be a point in $R$ but not in $\Sigma$ and let $L$ be a line of $B$ containing $x$. The T-line $L = \langle s,x \rangle$ for some $s$ in $\mathcal{S}$ must meet $R$ in $p^s$ points not in $\Sigma$ so that $\dim L \cap R = s$. Thus $L \cap R \cap \Sigma = R \cap s$ is of dimension $s-1$. For each ideal point of $B$, we find an $s$ such that $R \cap s$ has dimension $s-1$. But these intersections exhaust the points of $R \cap \Sigma$, so that if $s'$ does not represent an ideal point of $B$, then $R \cap s' = \emptyset$. Conversely, there will be precisely $p^s+1$ members of $\mathcal{S}$ meeting $R$ in an $s-1$ space. $R$ will contain $p^r = (p^s)^2$ points not in $\Sigma$. Each T-line through one of these $p^s+1$ spread elements containing a point of R not in $\Sigma$ will meet R in an s-space and so each line contains $p^s$ points of R not in $\Sigma$. Further for x, y in R but not in $\Sigma$, there is a unique T-line containing them and this T-line meets R in $p^s$ points so that by Dembowski (p.139 (4.b)) the points of R lie in an affine plane of order $p^s$ and so R represents a Baer subplane of T.

*Theorem 2.* If B is a Baer subplane of T having more than one point on $\Sigma$, then B is represented by an r-space R.

*Lemma 2.1.* If $\sigma$ is a collineation of T fixing each member of $S$ setwise and if $\sigma$ maps the affine points x, y of B to points $x^\sigma$, $y^\sigma$ also in B, then $B^\sigma = B$ and so $\sigma$ induces a collineation of B.

*Proof:* Let E be the set of ideal points of B. Then

$$B = \langle E \cup \{x,y\} \rangle = \langle E \cup \{x^\sigma, y^\sigma\} \rangle = \langle E^\sigma \cup \{x^\sigma, y^\sigma\} \rangle$$
$$= \langle E \cup \{x,y\} \rangle^\sigma = B^\sigma.$$

*Lemma 2.2.* If x, y are affine points of B and there is a translation of B mapping x to y, then the affine points of the line $\langle x,y \rangle$ also lie in B.

*Proof:* Such a translation has order p, the line $\langle x,y \rangle$ has p affine points on it and each of the p images of x under repeated application of this translation must also lie in B. Thus all the affine points of $\langle x,y \rangle$ lie in B.

This lemma need not hold if the plane T is represented over some field larger than GF(p). We shall see examples in §4 where Theorem 2 will not hold for larger fields.

*Lemma 2.3.* If to any pair $x, y$ of affine points of $B$ there is a translation of $B$ mapping $x$ to $y$ then the affine points of $B$ lie in an r-space.

*Proof:* Let $x, y$ be affine points of $B$. Let $G_1 = \langle\sigma_1\rangle$ with $\sigma_1$ the translation mapping $x$ to $y$. If $z$ is not on the line joining $x$ to $y$ and $z$ is in $B$, let $\sigma_2$ map $x$ to $z$. Then $G_2 = \langle\sigma_1, \sigma_2\rangle$ is a collineation group of $B$ of order $p^2$ and $x$ is mapped to all points of the affine plane $\langle x,y,z\rangle$, so this plane lies in $B$. Continuing this procedure, after $r$ steps the group $G_r = \langle\sigma_1, \sigma_2, \ldots, \sigma_r\rangle$ maps $x$ to each of the points in an affine r-space and this exhausts the affine points of $B$.

*Lemma 2.4.* Let $x, y$ be 2 affine points of $B$. Then there is a translation of $B$ mapping $x$ to $y$.

*Proof:* For $p = 2$, the translation $\sigma$ of $T$ mapping $x$ to $y$ maps $y$ to $x$ and so with $y = x^\sigma$, $x = y^\sigma$, Lemma 2.1 gives the desired result. For $p > 2$, introduce homogeneous coordinates $(\xi_1 \ldots \xi_{2r}, \eta)$ for $S_{2r}$ and let $\Sigma$ have equation $\eta = 0$. We may assume that $x:(0,0,\ldots,0,1)$ and $y:(1,0,\ldots,0,1)$. Let $\tau$ be the mapping $(\xi_1 \ldots \xi_{2r}, \eta)^\tau = (\xi_1 - \eta, \xi_2 \ldots \xi_{2r}, -\eta)$. Then $\tau$ fixes all points of $\Sigma$, exchanges $x$ with $y$ and so is a collineation of $B$. Further $\tau$ fixes the point $z:(\frac{1}{2}, 0 \ldots 0, 1)$. Let $w$ be an affine point of $B$ not on the T-line containing $x$ and $y$. Then $w^\tau$ is also in $B$ and the T-line through $w, w^\tau$ meets the T-line through $x, y$ in $z$ which is therefore also in $B$. Let $\sigma$ be the mapping $(\xi_1 \ldots \xi_{2r}, \eta)^\sigma = (\xi_1 + \frac{1}{2}\eta, \xi_2 \ldots \xi_{2r}, \eta)$. Then $\sigma$ is a translation of $T$ mapping $x$ to $z$ and $z$ to $y$ so $\sigma$ induces a translation of $B$. Now $\sigma^2$ is the desired translation.

Lemmas 2.4 and 2.3 now yield the conclusion of the theorem.

Note that in $\delta T$, $\delta B$ is a Baer subplane containing the $\delta T$ point $\Sigma$ so Theorem 2 implies that any Baer subplane of $\delta T$ containing $\Sigma$ is given by an r-space.

Let $s_\infty$ in $S$ be a line of the shears plane $\delta T$ and let $\mathcal{D} = \{\Sigma, A_1 \ldots A_{p^s}\}$ be a derivation set on $s_\infty$. We assume explicitly that $\Sigma$ is in $\mathcal{D}$ so that Theorem 2 will apply. The $A_i$ are r-spaces meeting $\Sigma$ in $s_\infty$.

*Lemma 2.5.* The spaces $A_1 \ldots A_{p^s}$ lie in a space $\Omega$ of dimension $3s$.

*Proof:* Let $s_1 \neq s_\infty \neq s_2$ be members of $S$. Let $q_1$ be a point in $A_1$. Let $L_1 = \langle s_1, q_1 \rangle$, $L_2 = \langle s_2, q_2 \rangle$. Then $L_1$, $L_2$ are $\delta T$ points whose $\delta T$ joining line is $q_1$ which meets $\mathcal{D}$ in the $\delta T$ point $A_1$. Since $\mathcal{D}$ is a derivation set $\langle \mathcal{D}, L_1, L_2 \rangle$ is a Baer subplane of $\delta T$, represented by an r space R. Let $\Omega = \langle R, s_\infty \rangle$. From Theorem 1, $dim R \cap s_\infty = s-1$ and so $dim \Omega = r+r-1-(s-1) = 2r-s = 3s$. Further, $L_1$ meets $A_i$ in a point $q_i$ representing a $\delta T$ line of $\langle \mathcal{D}, L_1, L_2 \rangle$ and so

$$A_i = \langle s_\infty, q_i \rangle \subset \langle s_\infty, R \rangle = \Omega.$$

Thus the spaces $A_1 \ldots A_{p^s}$ lie in $\Omega$. Note that the $A_i$ partition the points of $\Omega$ not in $\Sigma$. Thus $\Omega$ is determined by the set $\{A_1, \ldots, A_{p^s}\}$ and does not depend on the elements of the spread used in the proof of Lemma 2.5.

Let $\Pi = \Omega \cap \Sigma$. Then $dim \Pi = 3s-1$. For each $s \neq s_\infty$ in $S$, let $\bar{s} = s \cap \Pi$. Now $s_1 \cap R = s_1 \cap \Pi = \bar{s}_1$ so that from Theorem 1 we have that $dim \bar{s}_1 = s-1$. But any $s \neq s_\infty$ could have been used so that for each $\bar{s}$, $dim \bar{s} = s-1$.

There are $p^r$ members of $S$ distinct from $s_\infty$ and so $s_\infty$ and $\{\bar{s}: s \text{ in } S, s \neq s_\infty\}$ partition the points of $\Pi$.

*Lemma 2.6.* Let $\bar{s}_i \neq \bar{s}_j$. Then i) $dim\langle\bar{s}_i,\bar{s}_j\rangle = r-1$, ii) if we denote by $\bar{R}$ the space $\langle\bar{s}_i,\bar{s}_j\rangle$, then $\bar{R}$ is the intersection with $\Pi$ of a Baer subplane of $\delta T$ containing $\mathcal{D}$ and the $\delta T$ lines $s_i$ and $s_j$, iii) $\bar{s}_k \cap \bar{R} = \emptyset$ or else $\bar{s}_k \subset \bar{R}$ for all $s_k \neq s_\infty$, iv) $dim\bar{R}\cap s_\infty = S-1$.

*Proof:* i) follows from the dimension of $\bar{s}_i$ and $\bar{s}_j$ and the fact that $\bar{s}_i \cap \bar{s}_j = \emptyset$. Let $M_1 = \langle s_i,q_1\rangle$, $M_2 = \langle s_j,q_1\rangle$. Since $\mathcal{D}$ is a derivation set $\langle\mathcal{D},M_1,M_2\rangle$ is a Baer subplane of $\delta T$ containing $\Sigma$, $M_1$ and $M_2$ and hence the linear subspace R representing this subplane contains $\bar{s}_i$, $\bar{s}_j$ and so $\langle\bar{s}_i,\bar{s}_j\rangle = R\cap\Pi = \bar{R}$. iii) and iv) now follow from Theorem 1.

*Lemma 2.7.* The subspaces of $\Pi$ generated by pairs $\langle\bar{s}_i,\bar{s}_j\rangle$ partition $s_\infty$ into a spread of s-1 spaces.

*Proof:* If $\bar{R} \neq \bar{R}'$ where $\bar{R}$ and $\bar{R}'$ are subspaces of $\Pi$ generated by $\langle\bar{s}_i,\bar{s}_j\rangle$, $\langle\bar{s}_m,\bar{s}_n\rangle$ respectively, then $dim\bar{R}\cap\bar{R}' = s-1$. For $\bar{R}'$ contains some $\bar{s}_k$ disjoint from $\bar{R}$ by Lemma 2.6 (iii). Thus $dim\Pi = 3s-1 \geq dim\langle\bar{R},\bar{R}'\rangle \geq dim\langle\bar{R},\bar{s}_k\rangle = r-1+s-1-(-1) = 3s-1$. Thus $dim\langle\bar{R},\bar{R}'\rangle = 3s-1$ and so $dim\bar{R}\cap\bar{R}' = s-1$. From Lemma 2.6 iii) and iv), either $\bar{R}\cap\bar{R}' = \bar{s}_\ell$ for some $s_\ell$ or else $\bar{R}\cap s_\infty = \bar{R}\cap\bar{R}' = \bar{R}'\cap s_\infty$. These last equalities follow from the fact that the dimension of each intersection is s-1. To complete the proof of the lemma, it must be shown that each point of $s_\infty$ is in some $\bar{R}$. Let $q\in s_\infty$. Let q' be in $\Pi$ but not in $s_\infty$, and let $\ell$ be the line $\langle q,q'\rangle$. Let $\bar{s}_j$, $\bar{s}_k$ each contain one point of $\ell$[$\bar{s}_j$ contains at most one point of $\ell$ since $\bar{s}_j \cap s_\infty = \emptyset$]. Let $\bar{R} = \langle\bar{s}_j,\bar{s}_k\rangle$. Then $\bar{R}$ contains $\ell$ and hence q.

Let the (s-1)-spread in $s_\infty$ contain the spaces $t_0,t_1\ldots t_{p^s}$. We define an incidence structure P in $\Pi$. The P points are $\{t_0\ldots t_{p^s}, \bar{s}_1,\ldots,\bar{s}_{p^r}\}$. The P-lines are $s_\infty$ and the subspaces $\langle\bar{s}_i,\bar{s}_j\rangle$ where $\bar{s}_i \neq \bar{s}_j$. Incidence is given by containment.

*Theorem 3.* P is a desarguesian projective plane of order $p^s$.

*Proof:* There are $(p^s)^2+p^s+1$ P-points. Each P-line contains $p^s+1$ P-points since each P-line is an r-1 space and the P-points on it form an (s-1) spread on the P-line. To show that 2 P-points are on a unique P-line, we have only to consider the P-points $t_i$, $\bar{s}_j$ as the other 2 cases are obvious. Let $q \in t_i$, $q' \in \bar{s}_j$ and let $\bar{s}_k \neq \bar{s}_j$ be such that $\bar{s}_k$ contains a point of the line $\langle q,q'\rangle$. Then $\bar{s}_j \subset \langle \bar{s}_j,\bar{s}_k\rangle$ and $q$ is in $\langle \bar{s}_j \cap \bar{s}_k\rangle \cap t_i$. Thus from Lemma 2.7, $t_i \subset \langle \bar{s}_j,\bar{s}_k\rangle$ and this is the only r-1 space containing $t_i$ and $\bar{s}_j$ since $\langle t_i,\bar{s}_j\rangle = \langle \bar{s}_j,\bar{s}_k\rangle$. Thus by Dembowski (p.138,3.c) P is a projective plane of order $p^s$.

In the language of Bruck and Bose [2], this representation of P is rigid for every point of P. The representation is called rigid at x if the P-line joining x to y is represented by the subspace $\langle x,y\rangle$ for every $y \neq x$. But from [2] p.134 Corollary to Theorem 6.1, it suffices that P be rigidly represented for 2 points for P to be desarguesian. Thus P is desarguesian and the theorem is proved.

*Corollary:* Each of the Baer subplanes of $\delta T$ having ideal points precisely the points of $\mathcal{D}$, is desarguesian.

*Proof:* Let R represent such a plane and let $\bar{R} = R \cap \Pi = \langle \bar{s}_i,\bar{s}_j\rangle$. Taking $\bar{R}$ as an ideal line in P, the P-points on $\bar{R}$ are just the ideal points of the affine plane $P^{\bar{R}}$. We give a representation of $P^{\bar{R}}$ as a translation plane in the usual manner as follows: Let $\phi$ be an r-space in $\Pi$ containing $\bar{R}$. Then any P-point not on $\bar{R}$ meets $\phi$ in a point since $\mathit{dim}(\bar{s}_k \cap \phi) = s-1+r-(3s-1) = 0$. Further each P-line $\bar{R}'$ of $P^{\bar{R}}$ meets $\phi$ in an s-space since $\mathit{dim}(\bar{R}' \cap \phi) = r-1+r-(3s-1) = s$ and this s space meets $\bar{R}$ in a space $\bar{s}_k \subset \bar{R}$. Thus the spread in $\bar{R}$ is the spread for a desarguesian plane and does not depend on which r-space containing $\bar{R}$ is used to represent the points.

Thus the r-space R itself represents a desarguesian plane which is a Baer subplane of $\delta T$.

The proof could be facilitated somewhat by arguing that the spread in $\bar{R}$ is regular, but then the conclusion would not be valid for $p = 2$.

*Theorem 4:* The derived plane $\Delta\delta T$ can be described as follows: $\Delta$-points are the r-spaces of $S_{2r}$ metting $\Sigma$ in a member of $\mathcal{S}$ different from $s_\infty$, $\Delta$-lines are (i) the points of $S_{2r}$ in neither $\Sigma$ nor $\Omega$ and (ii) the r spaces in $\Omega$ meeting $\Pi$ in a P-line different from $s_\infty$. A $\Delta$ line of type (i) is incident with a $\Delta$ point if it is contained in the $\Delta$ point and a $\Delta$-line of type (ii) is incident with a $\Delta$ point if their representing r-spaces meet in an s-space.

Theorem 4 is merely a restatement of the general process of derivation in terms of the representation of $\delta T$.

The description of Theorem 4 will often be possible over a field larger than $GF(p)$ but we shall give some examples in §4 where the underlying field of the projective space $S_{2r}$ must then be a proper subfield of the kernel of T.

*Theorem 5.* If T of order $p^r$ is represented in the usual manner in $S_{2r}$ over $GF(p)$, then $\delta T$ is derivable if and only if there is some $3s-1$ dimensional space $\Pi \subset \Sigma$ such that $\Pi$ contains some member $s_\infty$ of $\mathcal{S}$ and the other members of $\mathcal{S}$ meet $\Pi$ in the affine points of a rigidly represented desarguesian plane.

*Proof:* One direction has been proved in Theorem 3. For the converse, let such a $\Pi$ exist. Let $\Omega$ meet $\Sigma$ in $\Pi$. We show that $\mathcal{D} = \{\Sigma, A_1 \ldots A_{p^s}\}$ forms a derivation set where the $A_i$ are T-lines containing $s_\infty$ and contained in $\Omega$. Suppose $L_1$, $L_2$ are 2 $\delta T$ points whose $\delta T$ line $q$ is in some $A_i$. Then $L_1 = \langle q,s_1\rangle$, $L_2 = \langle q,s_2\rangle$ where $s_1 \neq s_2$ and $s_1$, $s_2$ are in $S$. Let $\bar{s}_1 = s_1 \cap \Pi$, $\bar{s}_2 = s_2 \cap \Pi$ and let $R = \langle q,\bar{s}_1,\bar{s}_2\rangle$. Then $R$ is an r-space and by the definition of rigidity satisfies the conditions of Theorem 1 so that $R$ represents a Baer subplane of $\delta T$. This subplane is just $\langle \mathcal{D} \cup \{L_1,L_2\}\rangle$. If $L_1$, $L_2$ are $\delta T$ points whose joining $\delta T$-line meets $\mathcal{D}$ in $\Sigma$, then $L_1 \cap \Sigma = s = L_2 \cap \Sigma$. Let $q_1 = A_1 \cap L_1$, $q_2 = A_2 \cap L_2$ and let $M$ be the $\delta T$ point representing the intersection of the $\delta T$ lines $q_1$ and $q_2$. $M$ meets $\Sigma$ in a space $s' \neq s$ and $\langle \mathcal{D} \cap \{L_1,L_2\}\rangle = \langle \mathcal{D} \cap \{L_1,M\}\rangle$ so we have returned to the first case. Thus $\mathcal{D}$ is a derivation set.

## §3. Coordinatization of Derivable Shears Planes

Let $\Sigma$ be given the homogeneous coordinates $(\eta_1,\ldots,\eta_S, \zeta_1,\ldots,\zeta_S, \rho_1,\ldots,\rho_S, \sigma_1,\ldots,\sigma_S)$ with components elements of $GF(p)$ not all of which are 0. Let $s_\infty$ be given by the equations $\eta_1 = \ldots = \eta_S = \zeta_1 = \ldots = \zeta_S = 0$. Let $t_0$, $t_1$ be P-points on $s_\infty$ given by the equations $\sigma_1 = \ldots = \sigma_S = 0$ and $\rho_1 = \ldots = \rho_S = 0$ respectively. Let $\bar{s}_1$ have defining equations $\zeta_1 = \ldots = \zeta_S = \rho_1 = \ldots = \rho_S = \sigma_1 = \ldots = \sigma_S = 0$. Then $\Pi = \langle s_\infty,\bar{s}_1\rangle$ and $\Pi$ is given by the equations $\zeta_1 = \ldots = \zeta_S = 0$. Let $\bar{s}_2$ be disjoint from $\langle t_0,\bar{s}_1\rangle$ and from $\langle t_1,\bar{s}_1\rangle$. The field $F = GF(p^S)$ may be considered as an s-dimensional vector space over $GF(p)$ and so choose $\{\sigma_1\ldots\sigma_S\}$, $\{\rho_1\ldots\rho_S\}$, and $\{\eta_1\ldots\eta_S\}$ as bases for copies of $F$ such that the members of $\bar{s}_2$ are of the form $(x,0,x,x)$ where $0 \neq x \in F$ and the second component in the vector denotes the s-vector $(0,0,\ldots 0)$ over $GF(p)$ corresponding to the null element of $F$.

The 4 P-points $t_0,t_1,\bar{s}_1,\bar{s}_2$ form a quadrangle in $P$ with coordinates $F(0,0,1,0)$, $F(0,0,0,1)$, $F(1,0,0,0)$ and $F(1,0,1,1)$ respectively. Thus each of the $\bar{s}_j$ have coordinates over $F$ of the form $F(1,0,m_1,m_2)$.

Let $Q$ be a quasifield coordinatizing $T$ so that the spread elements $s_m \neq s_\infty$ are given by $\{(x, x\circ m): x \text{ in } Q\}$. We associate to each element of $Q$ an ordered pair of elements from $F$. For $\bar{s}_m = F(1,0,m_1,m_2)$ write $m = (m_1,m_2)$.

If $0 \neq f \in F$ and $(1,0,m_1,m_2) \in \bar{s}_m$, then so is $(f,0,fm_1,fm_2)$. On the other hand, $((f,0),(f,0)\circ(m_1,m_2)) \in s_m$. But the difference of these 2 vectors is in $s_\infty$ since the first 2 components vanish. Thus the last 2 components must vanish also and we have $(f,0)\circ(m_1,m_2) = (fm_1,fm_2)$. (1)

*Theorem 6.* Identifying $(f,0)$ with $f$, for $f \in F$, $F$ is a subfield of $Q$ and $Q$ is a left vector space of dimension 2 over $F$.

*Proof:* Since $(f,0)\circ(g,0) = (fg,0)$ and $(f,0)+(g,0) = (f+g,0)$ the isomorphism between $F$ and $\{(f,0): f \in F\}$ is clear.

Now $f\circ(g\circ(m_1,m_2)) = f\circ(gm_1,gm_2) = (fgm_1,fgm_2) = fg\circ(m_1,m_2) = (f\circ g)\circ(m_1,m_2)$, and $f\circ((m_1,m_2)+(n_1,n_2)) = f\circ(m_1+n_1,m_2+n_2) = (f(m_1+n_1),f(m_2+n_2)) = (fm_1+fn_1,\ fm_2+fn_2) = f\circ(m_1,m_2)+f\circ(n_1,n_2)$.

Thus $Q$ is a left vector space over $F$ and the dimension is clearly 2.

*Theorem 7.* If $Q$ is a right quasifield and if $Q$ contains a subfield $F$ such that $Q$ is a left vector space of dimension 2 over $F$, then the dual of the translation plane $T$ coordinatized by $Q$ is derivable.

*Proof:* Let $s_\infty = \{(0,x): x \in F\}$, and $s_m = \{(x,x\circ m): x \in Q\}$. Restricting the values of $x$ to those of $F$, we define $\bar{s}_m = \{((f,0),(f,0)\circ(m_1,m_2)): f \in F\}$ where $m = (m_1,m_2)$, $m_1$, $m_2$ in $F$. The $\bar{s}_m$ all lie in $\Pi = \{(f_1,0,f_2,f_3): f_1,f_2,f_3 \in F\}$ and identifying $\bar{s}_m = F(1,0,m_1,m_2)$ with the pair $(m_1,m_2)$ we see that the set of $\bar{s}_m$ form the set of points of the affine plane over $F$. Thus the conditions of Theorem 5 are satisfied so that $\delta T$ is derivable.

Theorems 6 and 7 yield an algebraic characterization of derivable shears planes first proved by Ostrom [6].

## §4. EXAMPLES OF DERIVABLE SHEARS PLANES

A semifield is a right quasifield satisfying the left distributive law. Thus if T is coordinatized by a semifield Q, then $\delta T$ may be coordinatized by Q* which will also be a translation plane. In a semifield define the left, middle and right nucleus of Q as

$$N_\ell = \{a:\ a\circ(b\circ c) = (a\circ b)\circ c \text{ for all } b,c\in Q\}$$

$$N_m = \{b:\ a\circ(b\circ c) = (a\circ b)\circ c \text{ for all } a,c\in Q\}$$

$$N_r = \{c:\ a\circ(b\circ c) = (a\circ b)\circ c \text{ for all } a,b\in Q\}.$$

Since the left distributive law is always valid, the kernel of Q is the left nucleus and the kernel of Q* is the right nucleus of Q.

A weak nucleus of Q is a field F such that the equation $a\circ(b\circ c) = (a\circ b)\circ c$ holds whenever at least 2 of a,b,c are in F.

Knuth [4] has defined a class of semifields of degree 2 over their weak nucleus. Since both Q and Q* are thus 2 dimensional vector spaces over their weak nucleus the planes $\delta T$ and T defined over Q* and Q respectively are derivable. An investigation of the derived planes, which will be translation planes [Dembowski, p.224] and thus not strict semitranslation planes, will be undertaken at a later time. In this present paper, we shall define one subset of Knuth's planes and make some observations about the representation of the derived planes.

Let $F = GF(p^m)$ p an odd prime, $m > 1$. Let f be a nonsquare in F and let $\alpha,\beta,\sigma$ be automorphisms of F not all of which are the identity. On the set of ordered pairs of elements of F define:

$$(a,b) + (c,d) = (a+c,\ b+d)$$

$$(a,b) \circ (c,d) = (ac+b^\alpha d^\beta f,\ a^\sigma d+bc).$$

Then this defines a semifield $Q$ of dimension 2 over its weak nucleus $\{(f,0): f \in F\} \simeq F$ [Knuth, p.213].

We next determine the left and right nucleus of $Q$.

Suppose $(m,n)$ is in the right nucleus of $Q$. Then for all $(a,b)$, $(g,h)$ in $Q$, we have $((a,b)\circ(g,h))\circ(m,n) = (a,b)\circ((g,h)\circ(m,n))$.

Thus

1) $((a,b)\circ(g,h))\circ(m,n) = (ag+b^{\alpha}h^{\beta}f,\ a^{\sigma}h+bg)\circ(m,n) =$

$$((ag+b^{\alpha}h^{\beta}f)m+(a^{\sigma}h+bg)^{\alpha}n^{\beta}f,\ (ag+b^{\alpha}h^{\beta}f)^{\sigma}n+(a^{\sigma}h+bg)m)$$

and

2) $(a,b)\circ((g,h)\circ(m,n)) = (a,b)\bullet[gm+h^{\alpha}n^{\beta}f,\ g^{\sigma}n+hm] =$

$$(a(gm+h^{\alpha}n^{\beta}f)+b^{\alpha}(g^{\sigma}n+hm)^{\beta}f,\ a^{\sigma}(g^{\sigma}n+hm)+b(gm+h^{\alpha}n^{\beta}f)).$$

Equating the first components of 1) and 2), we find

3) $b^{\alpha}h^{\beta}mf + a^{\sigma\alpha}h^{\alpha}n^{\beta}f + b^{\alpha}g^{\alpha}n^{\beta}f = ah^{\alpha}n^{\beta}f + b^{\alpha}g^{\sigma\beta}n^{\beta}f + b^{\alpha}h^{\beta}m^{\beta}f.$

In 3), setting $h = 0$ we find $b^{\alpha}g^{\alpha}n^{\beta}f = b^{\alpha}g^{\sigma\beta}n^{\beta}f$.

4) For this to hold for all $b$, $g$, we have either $n = 0$ or $\alpha = \sigma\beta$.

In 3) setting $g = a = 0$, we find $b^{\alpha}h^{\beta}mf = b^{\alpha}h^{\beta}m^{\beta}f$.

5) For this to hold for all $b$, $h$, we must have $m = m^{\beta}$.

In 3) setting $b = 0$, we find $a^{\sigma\alpha}h^{\alpha}n^{\beta}f = ah^{\alpha}n^{\beta}f$

6) so that either $n = 0$ or else $\sigma\alpha = 1$.

Thus the first components are equal if $m^{\beta} = m$ and either $n = 0$ or else $\alpha = \sigma\beta$ and $\sigma\alpha = 1$. The converse is easily verified.

Equating the second components of 1) and 2), we obtain

7) $b^{\alpha\sigma}h^{\beta\sigma}nf^{\sigma} = bh^{\alpha}n^{\beta}f.$

This equation will hold for all $b$, $h$ if and only if either $n = 0$ or $nf^{\sigma} = n^{\beta}f$ and $\alpha\sigma = 1$, $\beta\sigma = \alpha$.

*Lemma 4.1.* The right nucleus of $Q$ is $\{(m,0): m^{\beta}=m\}$ unless $\beta = \alpha^2$ and $\sigma = \alpha^{-1}$ and then the right nucleus of $Q$ is $\{(m,n): m^{\beta}=m,\text{ and } nf^{\sigma}=n^{\beta}f\}$.

Note that the second condition does not guarantee that there will be a nonzero $n$ satisfying the given equation.

As an example, let $\theta$ be a primitive element of the field $GF(p^{12}) = F$ $\alpha: x \to x^{p^3}$, $\beta: x \to x^{p^6}$, $\sigma: x \to x^{p^9}$. We may write $f = \theta^{2j+1}$ for some $j$ since $f$ is a non-square in $F$. Write $n = \theta^i$ if $n \neq 0$. Then $nf^\sigma = n^\beta f$ becomes $\theta^i \cdot \theta^{(2j+1)p^9} = \theta^{i \cdot p^6}\, \theta^{2j+1}$ or $i + p^9(2j+1) \equiv p^6 i + 2j+1 \mod (p^{12}-1)$ or $(p^9-1)(2j+1) \equiv (p^6-1)i \mod (p^{12}-1)$ and this congruence has a solution $i$ if and only if $p^6-1$ divides $(p^9-1)(2j+1)$ which holds if and only if $p^3+1$ divides $(p^6+p^3+1)(2j+1)$. But $p^3+1$ is even and the other term is odd. So the only solution to $nf^\sigma = n^\beta f$ is $n = 0$.

A similar argument from equations 3) and 7) assuming $a$ and $b$ to be fixed yields

*Lemma 4.2.* The left nucleus of $Q$ is $\{(a,0): a^{\sigma\alpha} = a\}$ unless $\beta = 1$, $\alpha = \sigma$ and then the left nucleus of $Q$ is

$$\{(a,b): \ a^{\sigma^2} = a \ \text{ and } \ b^{\sigma^2} f^\sigma = bf\}.$$

*Examples:* Let $F = GF(p^6)$: $\alpha: x \to x^{p^4}$, $\beta: x \to x^{p^2}$, $\sigma: x \to x^{p^2}$.

Then the left nucleus of $Q$ is $\{(f,0): f \in F\}$ and the right nucleus of $Q$ is isomorphic to $GF(p^4)$ with $N_r \cap F = \{(f,0): f^{p^2} = f\} \simeq GF(p^2)$.

Thus the translation planes $T$ and $\delta T$ coordinatized by $Q$ and $Q^*$ respectively can be represented in the usual manner in 4 space over $F$ and in 6 space over $N_r$.

There are 2 extremes in choosing the space over which to represent a given translation plane. One is to represent it over the kernel so that the projective space is of lowest possible dimension. The other is to represent it over the prime field. The advantages of the latter method occur in studying collineations of the plane, for then all collineations are given by linear transformations. Another advantage is found in Theorem 2 and the enunciation of Theorem 5.

In order to preserve the representation of Baer subplanes as linear spaces, and so preserve the representation of Theorem 5, we must represent the plane over a subfield of the kernel intersect F. Thus $\Delta\delta T$ may be represented as a semitranslation plane in 4 space since $F = N_{\ell}$. But $\Delta T = \Delta\delta\delta T$ must be represented in 12 space over $GF(p^2)$. Note that $\delta T$ is of degree 3 over its kernel and there are no subspaces of 6 space over $GF(p^4)$ with $p^6$ points so that the subplane of $\delta T$ coordinatized by $\{(f,0): f \in F\}$ cannot be represented by a linear subspace in this representation of $\delta T$.

The author is not aware of any strict semitranslation planes derived from shears planes of degree greater than 2 over their kernel.

*References:*

[1] Bose, R.C. and Barlotti, A.: "Linear representation of a class of projective planes in 4 dimensional projective space." *Ann. Mat. Pura. Appl.*, 88, (1971) 9-32.

[2] Bruck, R.H. and Bose, R.C.: "Linear representations of projective planes in projective spaces." *J. Alg. 4* (1969) 85-102.

[3] Dembowski, P.: *Finite Geometries*, Springer Verlag, New York/Berlin 1968.

[4] Knuth, D.: "Finite semifields and projective planes." *J. Alg. 2* (1965) 182-217.

[5] Ostrom, T.G.: "Semitranslation planes." *Trans. A.M.S. 111* (1964) 1-18.

[6] _____: "Vector spaces and construction of finite projective planes." *Arch. Math. 19* (1968) 1-25.