

* *This research was partially supported by the Office of Naval Research under Contract No. N00014-67-A-0321-0002.*

** *University of North Carolina, Chapel Hill, and University of Lund, Sweden.*

DISCRETE WAVE-ANALYSIS OF CONTINUOUS STOCHASTIC PROCESSES*

by

Georg Lindgren**

*Department of Statistics
University of North Carolina at Chapel Hill
Institute of Statistics Mimeo Series No. 761*

June, 1971

DISCRETE WAVE-ANALYSIS OF CONTINUOUS STOCHASTIC PROCESSES*

By Georg Lindgren, *University of North Carolina, Chapel Hill*
and *University of Lund, Sweden*

1. Introduction. In the theory of stationary stochastic processes, much effort has been devoted to the problem of finding the statistical distribution of wave-characteristics such as the time between successive crossings of a level and the time and the vertical distance between a local maximum and the following minimum, which are all important for the physical appearance of the process paths.

In a very few cases, exact solutions are known; see Slepian [9] and Wong [12]. In others, one has to rely on series approximations whose accuracy are highly dependent on the covariance structure of the process; see Longuet-Higgins [7] and Lindgren [6].

One fruitful approach, founded on Kac and Slepian's horizontal window concept [3], was introduced by Slepian [10], who derived an explicit expression for the conditional process in which a zero upcrossing occurs at time zero. The same technique is used by Lindgren [4] to describe the process after a local maximum with a prescribed height u . In both cases, the choice of conditions has to be justified by delicate ergodic arguments.

The aim of this paper is to derive similar representations considering only a sampled version of the process. It is also shown that, if the continuous process is sufficiently regular, then these discrete representations and those in [4] and [10] coincide in the limit as the sampling interval tends to zero.

* *This research was partially supported by the Office of Naval Research under Contract No. N00014-67-A-0321-0002.*

Numerical illustrations are given in Section 4. Previously, Tick and Shaman [11] have shown that the expected number of crossings, maxima etc. per time unit are approximately the same for the continuous and the discrete version if the process is known to be of the highly regular band-pass spectral type. One minor difference is that extremely high maxima are likely to be overlooked which causes an excess of moderate height maxima in the sampled process.

In general, the discrete approach has two main advantages. Firstly, regardless of the regularity of the process, it is more close to the practical situation in which crossings, maxima, etc., are identified in a sampled process. Secondly, and more important in the non-regular case, it contains no reference to the "infinitesimal" properties of the random process. With a continuous approach, the covariance derivatives $r(0)$, $r''(0)$, $r^{IV}(0)$, etc., - or equivalently the spectral moments $\lambda_0, \lambda_2, \lambda_4, \dots$ - play a heavy role. These quantities are often difficult to estimate from a record of a realization of the process which is either sampled or affected by the impact of a filtering device. In both cases, one has to make far reaching assumptions in order to extrapolate to the behaviour of the covariance function at the origin or the spectral function at infinity.

This vague statement can be exemplified as follows. Suppose the continuous process $\xi(t)$ has mean zero and the covariance function

$$r(t) = E(\xi(t) \cdot \xi(0))$$

with the spectral density

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} r(t) \exp(-i\lambda t) dt.$$

Then, sampling at discrete time points $\{kd, k=0, \pm 1, \pm 2, \dots\}$ yields accurate estimates of the discrete covariance function $\{r(kd), k=0, \pm 1, \pm 2, \dots\}$, while a linear filter yields estimates of $f(\lambda) \cdot |g(\lambda)|^2$ where g is the "gain" function of the filter. In both cases, it is possible to extrapolate to the values

of $r^{(2n)}(0)$ or $\lambda_{2n} = \int \lambda^{2n} f(\lambda) d\lambda$ only if one imposes severe constraints on r or f . To use these quantities to make probability statements about the number of crossings, the wave-height, etc., for the continuous process can be practically meaningless.

In the irregular case when r is not differentiable at the origin ($\lambda_2 = \infty$) the continuous approach breaks down. The sampling technique can still yield useful results as is illustrated in Example 4 in Section 4.

2. Discrete and continuous conditional processes. Let $\{\xi(t), t \in \mathbb{R}\}$ be a stationary, zero-mean, Gaussian process with the covariance function r , i.e., $r(t) = \text{COV}(\xi(s), \xi(s+t))$ for all s , and let $\{\xi_k, k=0, \pm 1, \pm 2, \dots\}$ be the same process observed at the discrete time points $\dots -2d, -d, 0, d, 2d, \dots$ with a sampling interval $d > 0$; thus $\xi_k = \xi(kd)$. Also write $r_k = r(kd) = \text{COV}(\xi_1, \xi_{1+k})$.

Then we say that a zero upcrossing occurs in the sampled process at time k if

$$\xi_{k-1} < 0 < \xi_k.$$

Similarly, we say that a maximum with height in $[u, u+h]$ occurs if

$$\xi_{k-1} < \xi_k, \quad \xi_{k+1} < \xi_k, \quad u \leq \xi_k \leq u+h.$$

Then, for any set of integers k_1, \dots, k_n , we can compute the conditional distributions of $(\xi_k, k=k_1, \dots, k_n)$ given say, a zero upcrossing or a maximum with height in $[u, u+h]$ for $k = 0$. Considering the limiting case $h \searrow 0$, we say that we have the conditional distributions given a maximum with height u at time 0_s

The representations in the following two theorems describe explicitly what happens with the process ξ_k after a zero upcrossing or a maximum.

THEOREM 1. The conditional process $\{\xi_k | \text{zero upcrossing for } k = 0\}$ has the same distributions functions as the process

$$(2.1) \quad \xi^d \cdot \frac{r_k + r_{k+1}}{1 + r_1} - \eta^d \cdot \frac{d}{2} \cdot \frac{r_{k+1} - r_k}{1 - r_1} + \kappa_k, \quad k = 0, \pm 1, \pm 2, \dots$$

where $\{\kappa_k, k=0, \pm 1, \pm 2, \dots\}$ is a nonstationary Gaussian sequence with mean zero and the covariance function

$$R_{i,j} = \text{Cov}(\kappa_i, \kappa_j) = r_{i-j} - \frac{1}{1-r_1} \left\{ r_i r_j - r_1 r_{i+1} r_j - r_1 r_i r_{j+1} + r_{i+1} r_{j+1} \right\},$$

and ξ^d and η^d are random variables, independent of the process κ and with the joint density

$$f_{\xi^d, \eta^d}(x, y) = \begin{cases} d \cdot \exp\left\{-\frac{x^2}{1+r_1} - \frac{d^2 y^2}{4(1-r_1)}\right\} / \sqrt{1-r_1} \arccos r_1 & \text{for } 2|x| \leq dy \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let $\xi = \frac{1}{2}(\xi_0 + \xi_{-1})$, $\eta = d^{-1}(\xi_0 - \xi_{-1})$. Then the random variable $(\xi, \eta, \xi_{k_1}, \dots, \xi_{k_n})$ is normal with mean zero and the covariances

$$\text{Cov}(\xi, \eta) = 0, \quad V(\xi) = \frac{1}{2}(1+r_1), \quad V(\eta) = 2d^{-2}(1-r_1),$$

$$\text{Cov}(\xi, \xi_{k_i}) = \frac{1}{2}(r_{k_i} + r_{k_i+1}), \quad \text{Cov}(\eta, \xi_{k_i}) = d^{-1}(r_{k_i} - r_{k_i+1}),$$

$$\text{Cov}(\xi_{k_i}, \xi_{k_j}) = r_{k_i - k_j}.$$

Let us first find out what happens if ξ and η assume the values x and y .

Then the conditional n -variate normal distribution of $(\xi_{k_1}, \dots, \xi_{k_n})$ has mean

and covariances (see Rao [8], p. 441)

$$\begin{aligned}
 E(\xi_{k_i} | \xi = x, \eta = y) \\
 &= (\frac{1}{2}(r_{k_i} + r_{k_i+1}), d^{-1}(r_{k_i} - r_{k_i+1})) \begin{pmatrix} \frac{1}{2}(1+r_1) & 0 \\ 0 & 2d^{-2}(1-r_1) \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \\
 &= x \cdot \frac{r_{k_i} + r_{k_i+1}}{1+r_1} - y \cdot \frac{d}{2} \cdot \frac{r_{k_i+1} - r_{k_i}}{1-r_1},
 \end{aligned}$$

$$\begin{aligned}
 \text{Cov}(\xi_{k_i}, \xi_{k_j} | \xi = x, \eta = y) &= r_{k_i - k_j} \\
 &- (\frac{1}{2}(r_{k_i} + r_{k_i+1}), d^{-1}(r_{k_i} - r_{k_i+1})) \begin{pmatrix} \frac{1}{2}(1+r_1) & 0 \\ 0 & 2d^{-2}(1-r_1) \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2}(r_{k_j} + r_{k_j+1}) \\ d^{-1}(r_{k_j} - r_{k_j+1}) \end{pmatrix} \\
 &= R_{k_i, k_j}.
 \end{aligned}$$

Thus, if κ_k is a sequence with the covariance function R then

$(\xi_k, k=k_1, \dots, k_n | \xi = x, \eta = y)$ has the same distribution as

$$x \cdot \frac{r_k + r_{k+1}}{1+r_1} - y \cdot \frac{d}{2} \cdot \frac{r_{k+1} - r_k}{1-r_1} + \kappa_k, \quad k = k_1, \dots, k_n.$$

Since $\xi_{-1} < 0 < \xi_0$ if and only if $2|\xi| < d\eta$, we have now only to give x and y free with the conditional distribution of $(\xi, \eta | 2|\xi| < d\eta)$ which has the density

$$f_{\xi, \eta}^{d, d}(x, y) = \text{constant} \cdot \exp\left\{-\frac{x^2}{1+r_1} - \frac{d^2 y^2}{4(1-r_1)}\right\} \text{ for } 2|x| < dy,$$

where the constant by integration is found to be $d/\sqrt{1-r_1}^2 \arccos r_1$. \square

Before we can state a similar theorem for the maximum case, we need some definitions. Let

$$\Sigma_{11} = \begin{pmatrix} 1 & 0 & 2d^{-2}(1-r_1) \\ 0 & \frac{1}{2}d^{-2}(1-r_2) & 0 \\ 2d^{-2}(1-r_1) & 0 & d^{-4}(6-8r_1+2r_2) \end{pmatrix}$$

$$\Sigma_{22} = \begin{pmatrix} 1 & r_{k_1-k_2} & \cdots & r_{k_1-k_n} \\ r_{k_1-k_2} & 1 & \cdots & r_{k_2-k_n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k_1-k_n} & r_{k_2-k_n} & \cdots & 1 \end{pmatrix}$$

$$\Sigma_{12} = \begin{pmatrix} r_{k_1} & \cdots & r_{k_n} \\ \frac{1}{2}d^{-1}(r_{k_1-1}-r_{k_1+1}) & \cdots & \frac{1}{2}d^{-1}(r_{k_n-1}-r_{k_n+1}) \\ d^{-2}(2r_{k_1}-r_{k_1-1}-r_{k_1+1}) & \cdots & d^{-2}(2r_{k_n}-r_{k_n-1}-r_{k_n+1}) \end{pmatrix},$$

$$\Sigma_{21} = \Sigma_{12}',$$

and write $\Sigma_{2 \cdot 1} = \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{21}'$. Also write $\Gamma_1, \Gamma_2, \Gamma_3$ for the three columns in $\Sigma_{21} \Sigma_{11}^{-1}$. For example, the i :th element in Γ_1 is equal to

$$\frac{d^{-6}}{\det \Sigma_{11}} \left\{ r_{k_i} (1-r_2) (3-4r_1+r_2) - (2r_{k_i}-r_{k_i-1}-r_{k_i+1}) (1-r_1) (1-r_2) \right\}.$$

If we denote this expression by A_{k_i} then $(A_k)_{k=-\infty}^{\infty}$ is a real sequence such that $(\Gamma_1)_i = A_{k_i}$. In the same way, we can define sequences $(B_k^y)_{k=-\infty}^{\infty}$, $(B_k^z)_{k=-\infty}^{\infty}$ and $(C_{i,j})_{i,j=-\infty}^{\infty}$ such that $(\Gamma_2)_i = B_{k_i}^y$, $(\Gamma_3)_i = -B_{k_i}^z$, $(\Sigma_{2 \cdot 1})_{ij} = C_{k_i, k_j}$. The reason why we chose the minus sign for B^z will be apparent later on. Also write $m(u) = u \cdot 2d^{-2}(1-r_1)$, $\sigma_1^2 = \frac{1}{2}d^{-2}(1-r_2)$, $\sigma_2^2 = d^{-4}(2-4r_1+2r_2)$.

THEOREM 2. The conditional process $\{\xi_k | \text{maximum with height } u \text{ for } k=0\}$ has the same distribution functions as the process

$$(2.2) \quad uA_k + \eta_u^d B_k y - \zeta_u^d B_k z + \Delta_k, \quad k = 0, \pm 1, \pm 2, \dots$$

where $\{\Delta_k, k = 0, \pm 1, \pm 2, \dots\}$ is a non-stationary, zero mean, Gaussian sequence with the covariance function $\text{Cov}(\Delta_i, \Delta_j) = C_{i,j}$, and η_u^d and ζ_u^d are random variables, independent of the process Δ and with the joint density

$$f_{\eta_u^d, \zeta_u^d}(y, z) = \begin{cases} \exp\{-\frac{1}{2}y^2/\sigma_1^2 - \frac{1}{2}(z-m(u))^2/\sigma_2^2\}/k_u & \text{for } 2|y| \leq dz \\ 0 & \text{otherwise.} \end{cases}$$

Here k_u is $2\pi\sigma_1 \int_0^\infty \{2\Phi(dz/2\sigma_1) - 1\} \phi((z-m(u))/\sigma_2) dz$ and ϕ and Φ the standardized normal density and distribution functions.

PROOF. If we define $\xi' = \xi_0$, $\eta' = \frac{1}{2}d^{-1}(\xi_1 - \xi_{-1})$, $\zeta' = d^{-2}(2\xi_0 - \xi_1 - \xi_{-1})$, then the matrix

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

is the covariance matrix of the $(n+3)$ -variate normal variable

$(\xi', \eta', \zeta', \xi_{k_1}, \dots, \xi_{k_n})$ whereas $\Sigma_{21} \Sigma_{11}^{-1}(u, y, z)'$ and $\Sigma_{2 \cdot 1} = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$ are the conditional mean and covariance matrices respectively for ξ_k ,

$k = k_1, \dots, k_n$ given $\xi' = u$, $\eta' = y$, $\zeta' = z$. But

$$(2.3) \quad (\Sigma_{21} \Sigma_{11}^{-1}(u, y, z)')_i = uA_{k_i} + yB_{k_i} y - zB_{k_i} z,$$

and we can proceed as in Theorem 1 and define the sequence Δ_k with the covariance function $C_{i,j}$. To find the proper distribution of the coefficients in

(2.3), we observe that the event $\xi_{-1} < \xi_0$, $\xi_1 < \xi_0$, $\xi_0 = u$ is equivalent to $\xi' = u$, $2|\eta'| < d\zeta'$. Since the conditional distribution of η' and ζ' ,

given $\xi' = u$ are independent and normal with mean 0 and $m(u)$ and variances

σ_1^2 and σ_2^2 we get that the distribution of $\eta', \zeta' | \xi' = u$, $2|\eta'| < d\zeta'$ has just the density

$$\begin{aligned} & \text{constant} \cdot \frac{1}{\sqrt{2\pi} \sigma_1} \exp(-y^2/2\sigma_1^2) \cdot \frac{1}{\sqrt{2\pi} \sigma_2} \exp\{-(z-m(u))^2/2\sigma_2^2\} \\ & = f_{\eta_u^d, \zeta_u^d}(y, z) \quad \text{for } 2|y| < dz \end{aligned}$$

and the theorem is proved. □

It is convenient to introduce the processes ξ_{*}^d and ξ_u^d so that $\xi_{*}^d(t)$ and $\xi_u^d(t)$ are defined by (2.1) and (2.2) respectively for $t = dk$, $k = 0, \pm 1, \pm 2, \dots$ and by linear approximation for other t -values. To stick to the discrete approach, we still define crossings, maxima, etc., as before, i.e., if $\xi_{*}^d(d(k-1)) < 0 < \xi_{*}^d(dk)$ then ξ_{*}^d is said to have a zero upcrossing at time dk although the crossing actually occurs somewhere in the interval $(d(k-1), dk)$.

We shall now present the results of Slepian and Lindgren concerning conditional continuous processes. Consider the continuous time process $\{\xi(t), t \in \mathbb{R}\}$. We say that ξ has a zero upcrossing at t_0 if, for some $\epsilon > 0$, $\xi(t) \leq 0$ for $t \in (t_0 - \epsilon, t_0)$ and $\xi(t) \geq 0$ for $t \in (t_0, t_0 + \epsilon)$.

We also have that ξ has a local maximum in $(0, h')$ with height in $[u, u+h]$ if the derivative ξ' has a zero downcrossing at some $t \in (0, h')$ for which $\xi(t) \in [u, u+h]$.

As in the discrete case we can then, for any set of times t_1, \dots, t_n , define the conditional distribution of $(\xi(t_i), i=1, \dots, n)$ given, say a zero upcrossing in $(0, h')$ (resp. maximum in $(0, h')$ with height in $[u, u+h]$). Again considering the limiting cases $h, h' \downarrow 0$ we arrive at "the conditional distributions of the process at times t_1, \dots, t_n given a zero upcrossing (resp. local maximum with height u) for $t = 0$ ".

THEOREM 3. (Slepian). If $-r''(0) = \lambda_2 < \infty$ then the conditional process $\{\xi(t) | \text{zero upcrossing for } t=0\}$ has the same distribution functions as the process

$$(2.4) \quad \xi_{\mathbf{x}}(t) = -\eta r'(t)/\lambda_2 + \kappa(t)$$

where $\{\kappa(t), t \in \mathbb{R}\}$ is a non-stationary, zero-mean, Gaussian process with the covariance function

$$R(s,t) = \text{COV}(\kappa(s), \kappa(t)) = r(s-t) - r(s)r(t) - r'(s)r'(t)/\lambda_2,$$

and η is a random variable, independent of the process κ and with the probability density

$$f_{\eta}(y) = \lambda_2^{-1} y \exp(-y^2/2\lambda_2), \quad y > 0.$$

For a similar maximum theorem, assume $-r''(0) = \lambda_2$, $r^{IV}(0) = \lambda_4 < \infty$ and define

$$A(t) = (\lambda_4 r(t) + \lambda_2 r''(t))/(\lambda_4 - \lambda_2^2),$$

$$B(t) = (\lambda_2 r(t) + r''(t))/(\lambda_4 - \lambda_2^2),$$

$$C(s,t) = r(s-t) - [\lambda_2(\lambda_4 - \lambda_2^2)]^{-1} \{ \lambda_2 \lambda_4 r(s)r(t) + \lambda_2^2 r(s)r''(t) \\ + (\lambda_4 - \lambda_2^2) r'(s)r'(t) + \lambda_2^2 r''(s)r(t) + \lambda_2 r''(s)r''(t) \}.$$

THEOREM 4. If $-r''(0) = \lambda_2$, $r^{IV}(0) = \lambda_4 < \infty$, then the conditional process $\{\xi(t) | \text{maximum with height } u \text{ for } t=0\}$ has the same distribution functions as the process

$$(2.5) \quad \xi_u(t) = uA(t) - \zeta_u B(t) + \Delta(t),$$

where $\{\Delta(t), t \in \mathbb{R}\}$ is a non-stationary, zero-mean, Gaussian process with the covariance function $C(s,t) = \text{COV}(\Delta(s), \Delta(t))$, and ζ_u is a random variable,

independent of the process Δ and with the probability density

$$f_{\zeta_u}(z) = z \exp(-(z-\lambda_2 u)^2 / 2(\lambda_4 - \lambda_2^2)) / k(u), \quad z > 0.$$

Here $k(u) = \int_0^\infty z \exp(-(z-\lambda_2 u)^2 / 2(\lambda_4 - \lambda_2^2)) dz$ can be expressed in terms of ϕ and Φ .

3. Discrete versus continuous approach. As we have seen, the discrete conditional processes $\xi_{\mathfrak{x}}^d$ and ξ_u^d are of elementary character and their distributions are well suited for quite obvious frequency interpretations (see Remark 3 below). Their logical simplicity is, however, coupled with a fairly complex probabilistic structure, including two dependent random variables and a random sequence, and this makes them less suitable for numerical work. This drawback does not apply to the continuous processes $\xi_{\mathfrak{x}}$ and ξ_u which contain only one random variable and a stochastic process. The real meaning of $\xi_{\mathfrak{x}}$ and ξ_u is, however, not clear from their definition, and, therefore, it is desirable to derive them directly from $\xi_{\mathfrak{x}}^d$ and ξ_u^d . This is done in the following two theorems; see also Appendix. Thus $\xi_{\mathfrak{x}}$ and ξ_u , defined by (2.4) and (2.5), can be regarded as limiting cases of the easily comprehensible processes $\xi_{\mathfrak{x}}^d$ and ξ_u^d .

THEOREM 5. If $-r''(0) - \lambda_2 < \infty$ and if $dk_i \rightarrow t_i$, $i = 1, \dots, n$ as $d \rightarrow 0$ then

$$P(\xi_{\mathfrak{x}}^d(dk_i) \leq x_i, i=1, \dots, n) \rightarrow P(\xi_{\mathfrak{x}}(t_i) \leq x_i, i=1, \dots, n).$$

PROOF. Let $\xi_m \xrightarrow{L} \xi$ denote convergence in law. By expanding in Taylor series, it is directly shown that, as $d \rightarrow 0$

$$\text{Cov}(\kappa_{k_i}, \kappa_{k_j}) = R_{k_i, k_j} \rightarrow R(t_i, t_j) = \text{Cov}(\kappa(t_i), \kappa(t_j)),$$

which in turn implies that $(\kappa_{k_i}, i=1, \dots, n) \xrightarrow{L} (\kappa(t_i), i=1, \dots, n)$. Also

$$\frac{r_{k_i} + r_{k_i+1}}{1 + r_1} \rightarrow r(t_i) \quad \text{and} \quad \frac{d}{2} \cdot \frac{r_{k_i+1} - r_{k_i}}{1 - r_1} \rightarrow r'(t_i)/\lambda_2.$$

The marginal distribution of η^d has the density

$$\begin{aligned} f_{\eta^d}(y) &= \int_{-\frac{1}{2}dy}^{+\frac{1}{2}dy} f_{\xi^d, \eta^d}(x, y) dx \\ &\sim \frac{d^2 y}{\sqrt{1-r_1^2} \cdot \arccos r_1} \exp\{-d^2 y^2 / 4(1-r_1)\} \\ &\sim \frac{d^2 y}{\sqrt{\lambda_2 d^2} \cdot \sqrt{\lambda_2} d} \exp\{-d^2 y^2 / 4(1-r_1)\} \\ &\rightarrow \frac{y}{\lambda_2} \exp\{-y^2 / 2\lambda_2\} = f_{\eta}(y) \quad \text{as } d \rightarrow 0. \end{aligned}$$

The convergence is easily seen to be dominated so that actually $\eta^d \xrightarrow{L} \eta$. Since $|\xi^d| \leq \frac{1}{2} d \eta^d$ we also get that $\xi^d \xrightarrow{L} 0$. The theorem is proved if we apply the n -variate Cramér-Slutsky theorem. \square

REMARK 1. The density $f_{\eta}(y) = \lambda_2^{-1} y \exp(-y^2/\lambda_2)$, $y > 0$, can be regarded as "the density of the derivative at a randomly chosen zero upcrossing".

REMARK 2. The normed average $d^{-1} \xi^d$ has the limiting density $(2\pi/\lambda_2)^{\frac{1}{2}} \{1 - \Phi(2|x|/\lambda_2)\}$.

REMARK 3. There are two possible frequency interpretations of $P(\xi_{x^*}^d(dk_i) \leq x_i, i=1, \dots, n)$. Firstly, it is equal to a certain relative frequency in the classical situation in which one, from several independent realizations of ξ , picks out those for which $\xi_{-1} < 0 < \xi_0$, and looks at their values at dk_1, \dots, dk_n . The second interpretation is just as natural, namely when one has one single very long realization and picks out all zero upcrossings of ξ_k and

looks at the values of the process after additional k_1, \dots, k_n sampling steps; the process is supposed to be ergodic.

THEOREM 6. If $-r''(0) = \lambda_2$, $r^{IV}(0) = \lambda_4 < \infty$ and if $dk_i \rightarrow t_i$, $i = 1, \dots, n$ as $d \rightarrow 0$ then

$$P(\xi_u^d(dk_i) \leq x_i, i=1, \dots, n) \rightarrow P(\xi_u(t_i) \leq x_i, i=1, \dots, n).$$

The proof proceeds exactly as the previous one and is omitted. \square

4. Numerical illustrations. In this section, we give some examples of how the sampled processes $\xi_{\mathfrak{X}}^d$ and ξ_u^d can be used to describe the behaviour of the continuous processes $\xi_{\mathfrak{X}}$ and ξ_u and vice versa, with respect to the zero distance $\tau_{\mathfrak{X}}$ and the wave-length τ_u :

$$\begin{aligned} \tau_{\mathfrak{X}}^d &= \text{the time for the first zero downcrossing by } \xi_{\mathfrak{X}}^d \\ \tau_u^d &= \text{the time for the first local minimum of } \xi_u^d. \end{aligned}$$

We use the technique with moment approximations which has proved useful in many cases (cf. [7] and [6]).

Let us start with $\tau_{\mathfrak{X}}$ and define

$$N_k^d = \text{the number of zero downcrossings by } \xi_{\mathfrak{X}}^d(dj) \text{ for } j = 1, \dots, k \\ N_x^d \quad \xi_{\mathfrak{X}}(t) \text{ for } 0 < t \leq x.$$

Also define $\chi_k^d = 1$ if $\xi_{\mathfrak{X}}^d(d(k-1)) > 0 > \xi_{\mathfrak{X}}^d(dk)$ and write $p_{\mathfrak{X}}^d(k) = P(\chi_k^d = 1)$. Then

$$\begin{aligned}
P(\tau_{\mathfrak{x}}^d \leq dk) &= P(N_k^d \geq 1) \\
&\leq E(N_k^d) = \sum_{j=1}^k E(\chi_j^d) = \sum_{j=1}^k p_{\mathfrak{x}}^d(j) \\
&\geq E(N_k^d) - \frac{1}{2}E(N_k^d(N_k^d - 1)) = \sum_{j=1}^k p_{\mathfrak{x}}^d(j) - R_k^d.
\end{aligned}$$

We see that $p_{\mathfrak{x}}^d(k)$ is a good approximation for $P(\tau_{\mathfrak{x}}^d = dk)$ as long as $R_k^d = \frac{1}{2}E(N_k^d(N_k^d - 1))$ remains small, (which also means that the probability of more than one downcrossing in $(0, dk)$ is small). The probability $p_{\mathfrak{x}}^d(k)$ can be easily calculated by the formula

$$\iint_{2|x| \leq dy} f_{\xi^d, \eta^d}(x, y) P(x\alpha_{k-1} - y\beta_{k-1} + \kappa_{k-1} > 0 > x\alpha_k - y\beta_k + \kappa_k) dx dy,$$

where α_k and β_k are the factors for ξ^d and η^d in $\xi_{\mathfrak{x}}^d(k)$.

Similar relations hold in the continuous case:

$$\begin{aligned}
P(\tau_{\mathfrak{x}} \leq x) &\leq E(N_{\mathfrak{x}}) = \int_0^x f_{1\mathfrak{x}}(t) dt \\
&\geq E(N_{\mathfrak{x}}) - \frac{1}{2}E(N_{\mathfrak{x}}(N_{\mathfrak{x}} - 1)) = \int_0^x f_{2\mathfrak{x}}(t) dt \quad \text{say.}
\end{aligned}$$

Here $f_{1\mathfrak{x}}$ and $f_{2\mathfrak{x}}$ can be used to approximate the density $f_{\mathfrak{x}}$ of $\tau_{\mathfrak{x}}$ and they are also easy to compute. We recapitulate the formulas for them, given by Longuet-Higgins [7]; let

$$\begin{aligned}
\sigma_0^2 &= \sigma_t^2 = \lambda_2 - r'(t)^2 / (1 - r^2(t)) \\
\rho_t &= \left\{ -r''(t) - r(t)r'(t)^2 / (1 - r^2(t)) \right\} / \sqrt{\sigma_0^2 \sigma_t^2}
\end{aligned}$$

be the conditional variances and correlation coefficient for $\xi'(0)$, $\xi'(t)$, given that $\xi(0) = \xi(t) = 0$. Then

$$f_{1\mathfrak{x}}(t) = \frac{1}{2\pi\sqrt{\lambda_2}} \sqrt{\frac{\sigma_0^2 \sigma_t^2}{1 - r^2(t)}} \left\{ \sqrt{1 - \rho_t^2} - \rho_t \arccos \rho_t \right\}.$$

Similarly, let $\Sigma_{\tau t}$ be the covariance matrix of $\xi(0)$, $\xi(\tau)$, $\xi(t)$ and let μ_0^2 , μ_{τ}^2 , μ_t^2 , $V = (v_{ij})$ be the conditional variances and the conditional

correlation matrix for $\xi'(0)$, $\xi'(\tau)$, $\xi'(t)$ given that $\xi(0) = \xi(\tau) = \xi(t) = 0$. Also let

$$s_1 = \arccos \frac{v_{12}v_{13} - v_{23}}{\sqrt{(1-v_{12}^2)(1-v_{13}^2)}}, \quad a_1 = v_{12}v_{13} - v_{23},$$

and define s_2, s_3, a_2, a_3 by cyclical permutation of indices. Then

$$f_{2\mathbb{K}}(t) = f_{1\mathbb{K}}(t) - \frac{1}{4\pi^2\sqrt{\lambda_2}} \int_{\tau=0}^t \sqrt{\frac{\mu_0^2 \mu_\tau^2 \mu_t^2}{\det \Sigma_{\tau t}}} \cdot \left\{ \sqrt{\det v} + s_1 a_1 + (s_2 - \pi) a_2 + (s_3 - \pi) a_3 \right\} d\tau.$$

REMARK 4. The discrete probability $p_{\mathbb{K}}^d$ is the analog of $f_{1\mathbb{K}}$. To facilitate comparisons $p_{\mathbb{K}}^d(k)$ is represented in the diagrams as the area of a rectangle over $dk \pm \frac{1}{2}d$ with height $d^{-1}p_{\mathbb{K}}^d(k)$.

EXAMPLE 1. In Figure 1, we compare $p_{\mathbb{K}}^d$ with $f_{1\mathbb{K}}$ (and $f_{2\mathbb{K}}$) for different d -values for the covariance function

$$r_1(t) = \exp(-\frac{1}{2}t^2).$$

The agreement between the two approaches is striking even for such big sampling intervals as $d = 0.5$.

EXAMPLE 2. Figure 2 illustrates the results for the covariance function

$$r_2(t) = \exp(-\sqrt{0.6}|t|) \cdot \left\{ 1 + \sqrt{0.6}|t| - 0.2t^2 - 0.4\sqrt{0.6}|t|^3 + 0.04t^4 \right\}.$$

The characteristic angle in $f_{1\mathbb{K}}$ for $t \approx 0.3$ is clearly seen for $d = 0.25$.

(See Figures 1 and 2 at the end of this section.)

We now turn to the wave-lengths τ_u^d and τ_u after a maximum with height u . If $p_u^d(k) = P(\xi_u^d(d(k-1)) > \xi_u^d(dk), \xi_u^d(d(k+1)) > \xi_u^d(dk))$ and M_k^d (resp. M_x) is the number of local minimas of $\xi_u^d(dj)$ for $j = 1, \dots, k$ (resp. of ξ_u for $0 < t \leq x$) then, as before

$$P(\tau_u^d \leq dk) \leq \sum_{j=1}^k p_u^d(j)$$

$$P(\tau_u^d \leq dk) \geq \sum_{j=1}^k p_u^d(j) - S_k^d.$$

Thus $p_u^d(k)$ is a good approximation for $P(\tau_u^d = dk)$ if $S_k^d = \frac{1}{2}E(M_k^d(M_k^d - 1))$ is small.

Similarly

$$P(\tau_u \leq x) \leq E(M_x) = \int_0^x f_{1u}(t) dt$$

$$P(\tau_u \leq x) \geq E(M_x) - \frac{1}{2}E(M_x(M_x - 1)) = \int_0^x f_{2u}(t) dt.$$

The functions f_{1u} and f_{2u} , which approximate the density f_u of τ_u , cannot be as explicitly expressed as f_{1x} and f_{2x} and we refer the reader to [6] in which defining integrals can be found.

EXAMPLE 3. In Figure 3a-c, we compare p_u^d with f_{1u} (and f_{2u}) for the covariance function

$$r_3(t) = \frac{\sin \sqrt{3} t}{\sqrt{3} t},$$

representing a band-pass spectrum over $(0, \sqrt{3})$. For the moderate maximum height $u = 1$, a sampling distance of $d = 0.25$ seems quite sufficient but for higher u -values ($u = 3$) a smaller d is needed to catch the narrow distribution of τ_u .

(See Figures 3a, 3b, 3c at the end of this section.)

The covariance functions so far have all been of the regular type with a finite λ_2 and λ_4 respectively, and we have seen a good agreement between the continuous and the sampled version. We now turn to the non-regular case in which the covariance function fails to be differentiable at the origin.

EXAMPLE 4. The covariance function

$$r_4(t) = (1-c) \frac{\sin \sqrt{3} t}{\sqrt{3} t} + c \exp(-|t|)$$

represents a process of the band-pass type with an independent superposed Markov process. For small c the function r_4 is hardly distinguishable from r_3 , at least when t is bounded away from zero.

Since r_4 has an infinite second derivative at the origin the expected number of level-crossings per time unit is infinite. In every neighbourhood of a time t at which $\xi(t) = u$ the process ξ crosses the level u infinitely often, and thus both the zero-crossing distance and the wave-length degenerate to zero in the continuous approach.

In the sampled version, however, the concepts of zero-crossing distance and wave-length still make sense, and we can calculate the probabilities $p_{\mathbf{x}}^d$ and p_u^d as before. Figure 4 shows p_u^d for $c = 0.02$, $u = 1$ and it is also compared with f_{1u} (f_{2u}) for the pure band-pass process with covariance function r_3 . The effect of the "disturbing" Markov process is easily seen for $d = 0.25$ but is hardly visible for bigger sampling distances.

The evident conclusion is that in the presence of an irregular disturbance over a regular process one can "filter out" the disturbance by sampling with an appropriate sampling interval and still get a good idea of the appearance of the "pure" process.

This again emphasizes the need for careful appraisal of the model when dealing with practical crossing problems and estimated covariance functions.

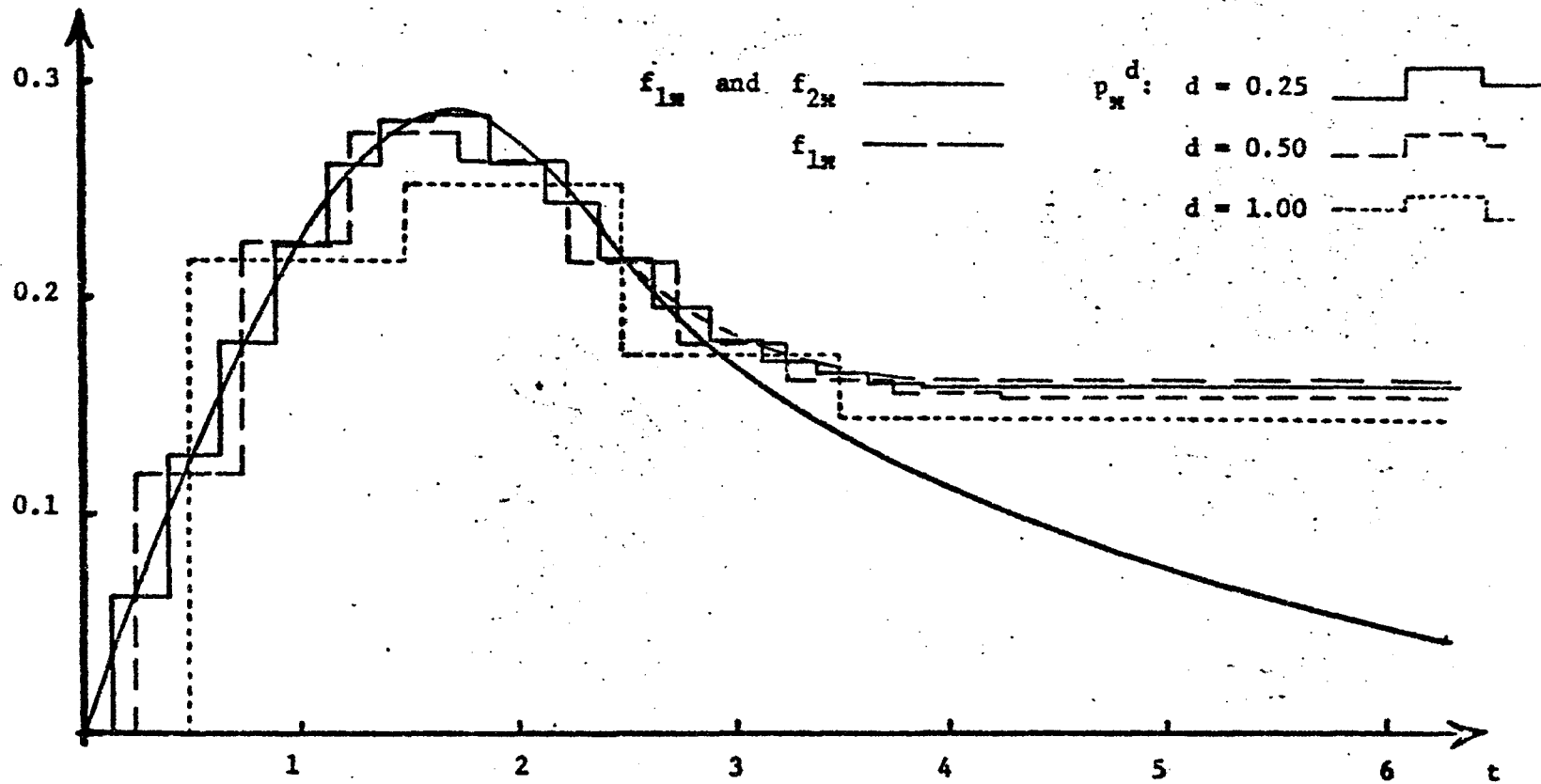


Figure 1. Discrete and continuous zero-distance for r_1 .

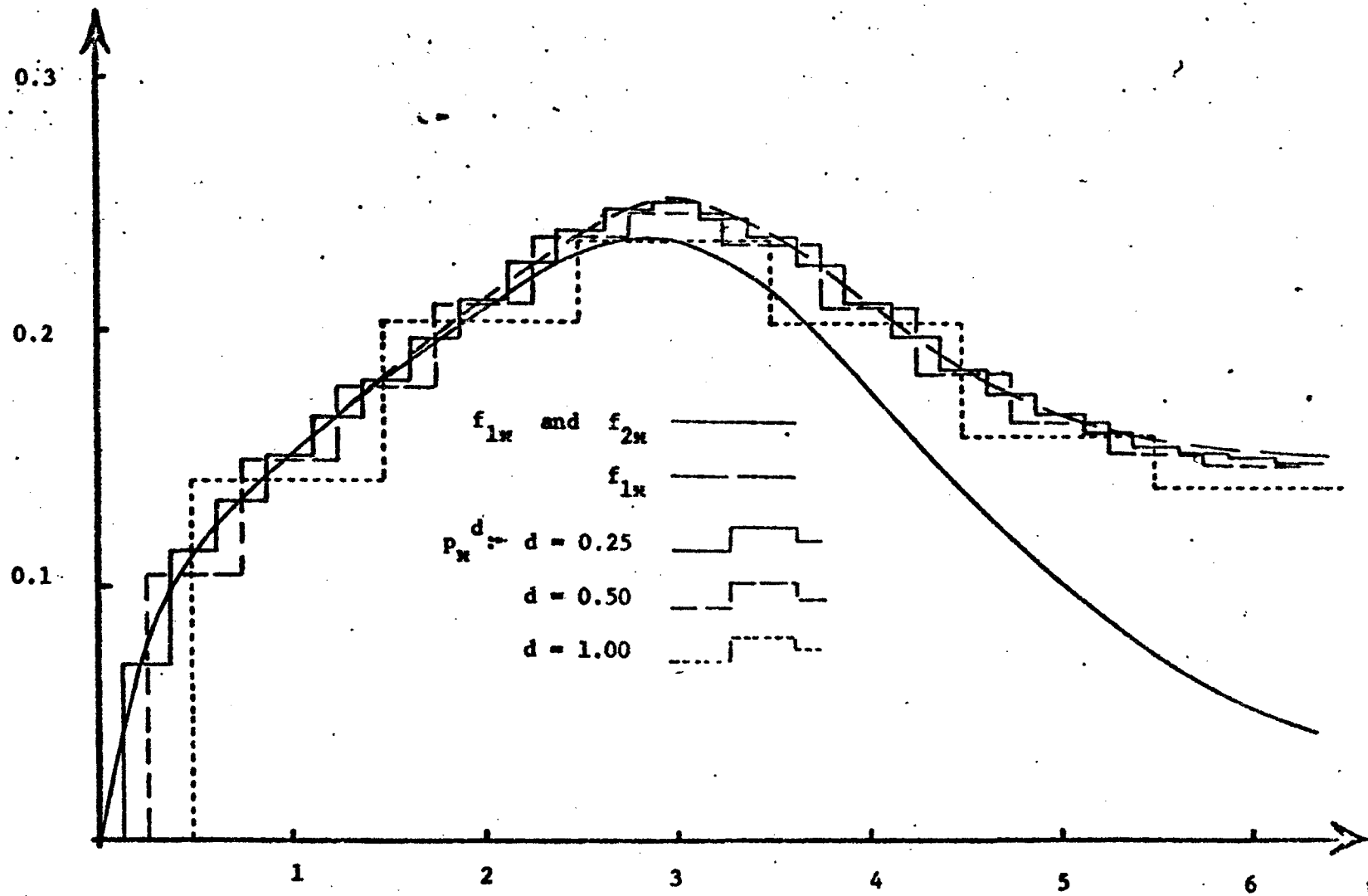


Figure 2. Discrete and continuous zero-distance for r_2 .

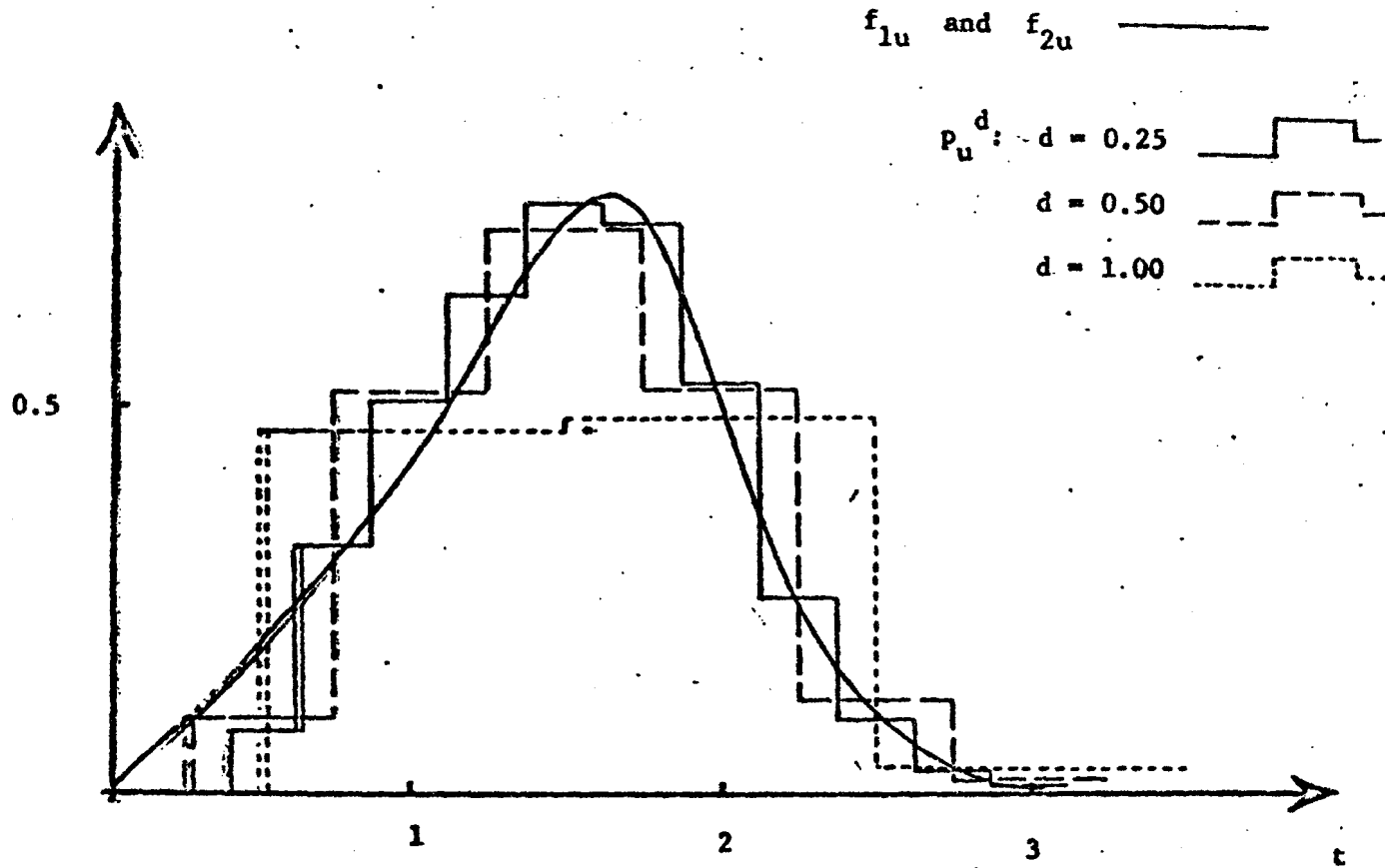


Figure 3a. Discrete and continuous wave-length for r_3 ;
 max-height $u = -1$.

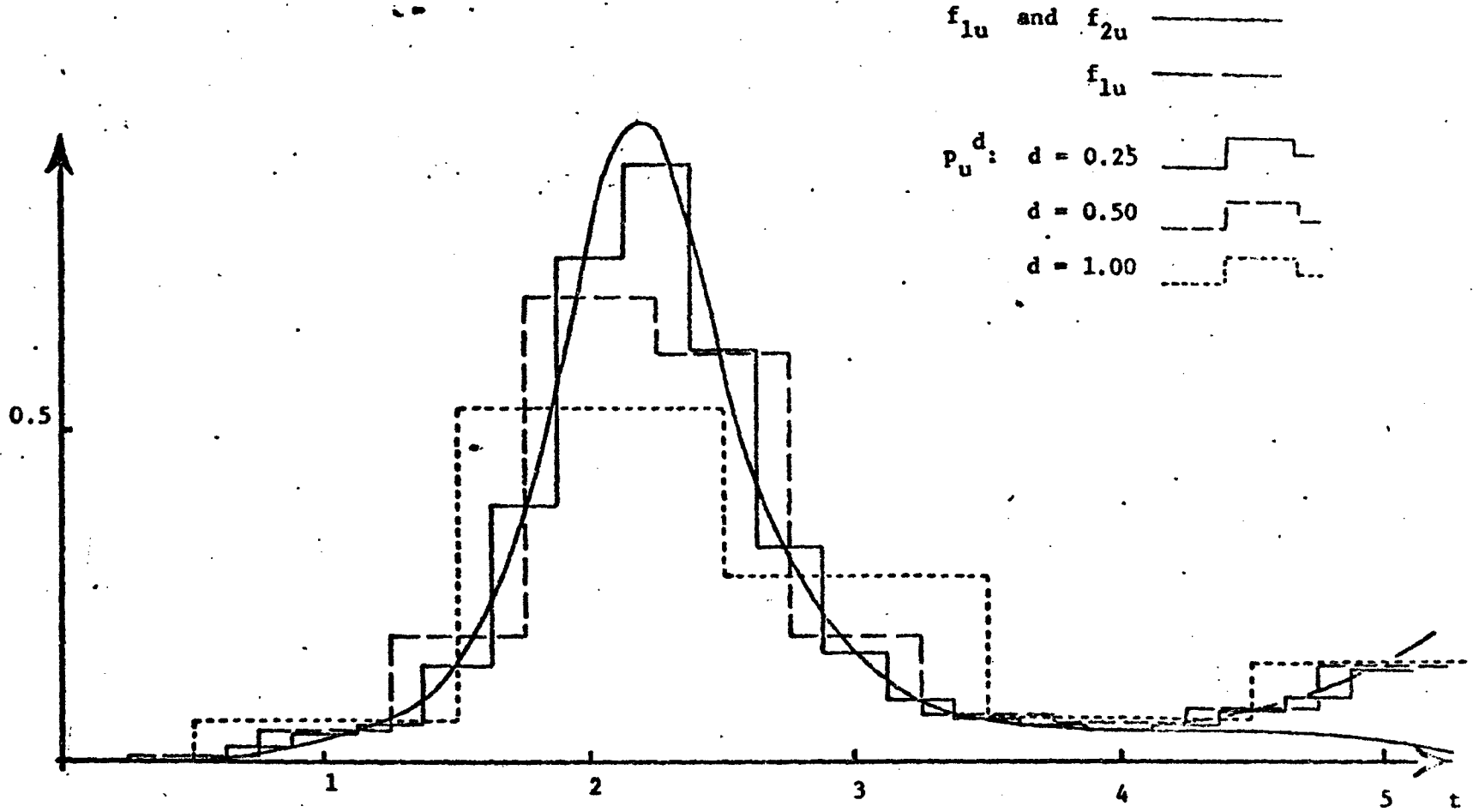


Figure 3b. Discrete and continuous wave-length for r_3 ; max-height $u = 1$.

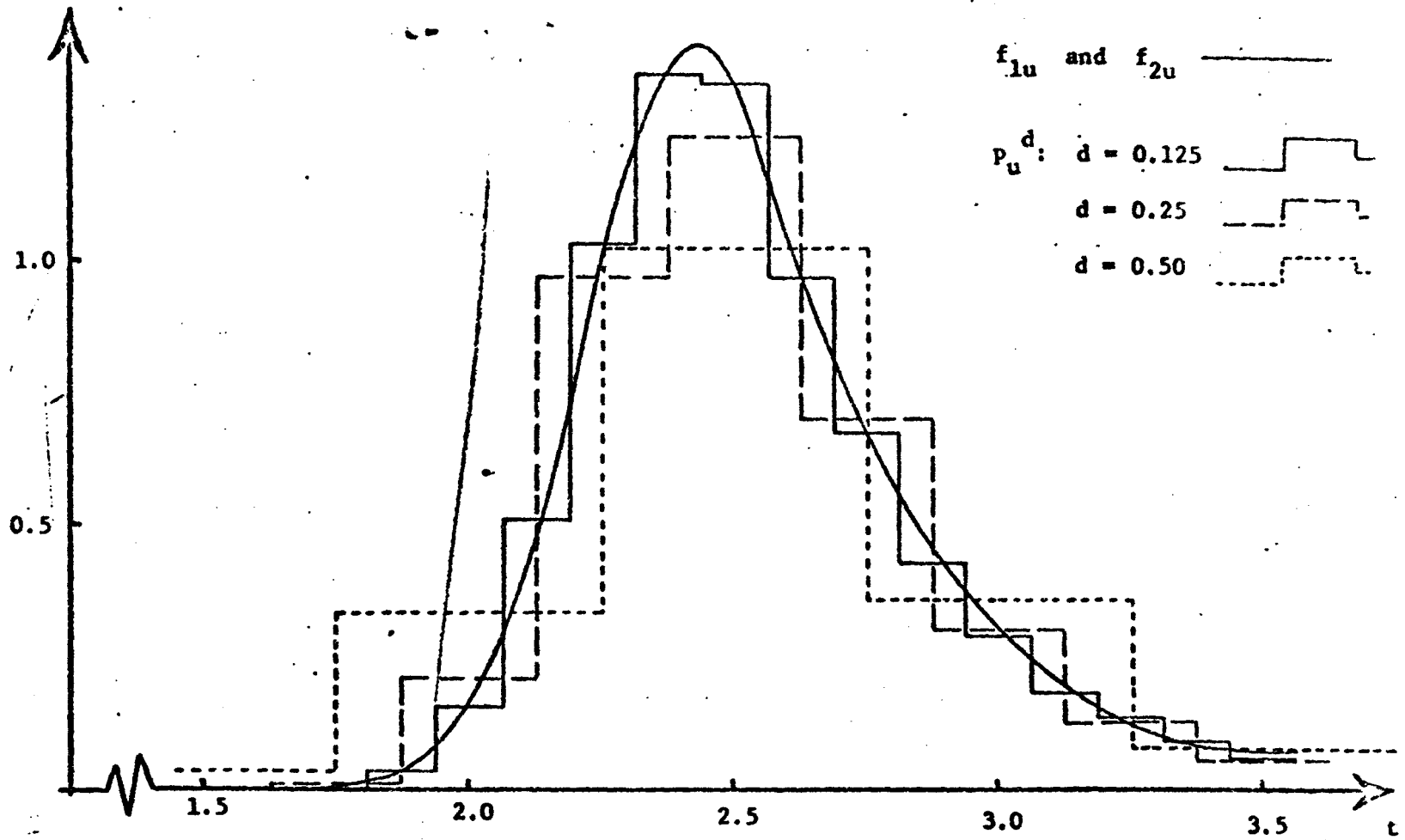


Figure 3c. Discrete and continuous wave-length for r_3 ; max-height $u = 3$.

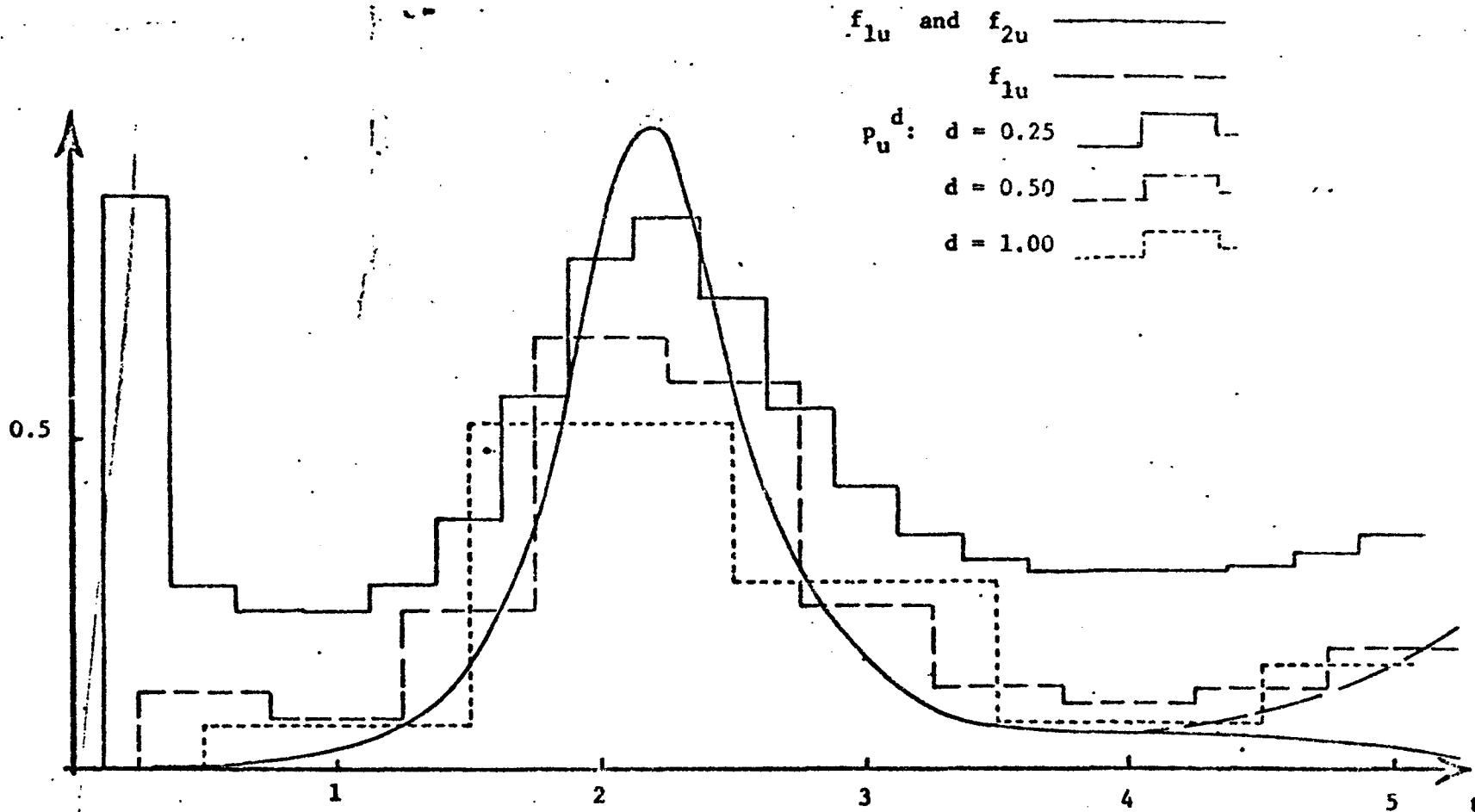


Figure 4. Discrete wave-length for r_4 ($c = 0.02$) compared with continuous wave-length for r_3 ; max-height $u = 1$.

APPENDIX: Weak convergence of probability measures.

In Section 3, we proved by elementary methods the convergence of the finite dimensional distributions of the sampled processes $\xi_{\mathbf{x}}^d$ and ξ_u^d to those of $\xi_{\mathbf{x}}$ and ξ_u . Now we will show some more sophisticated statements about weak convergence of probability measures which will enable us to prove the convergence in law of the zero-crossing distance $\tau_{\mathbf{x}}^d$ and the wave-length τ_u^d to $\tau_{\mathbf{x}}$ and τ_u .

We start with $\xi_{\mathbf{x}}^d$ and $\tau_{\mathbf{x}}^d$. Let C be the metric space of continuous functions on the interval $[0, T]$ with the distance $d(\omega, \tilde{\omega}) = \sup_{[0, T]} |\omega(t) - \tilde{\omega}(t)|$, and let \mathcal{C} be the smallest σ -algebra that contains all open sets. Since $\xi_{\mathbf{x}}^d$ obviously has continuously sample paths with probability one, and the same is true for $\xi_{\mathbf{x}}$ under the given assumptions (they are actually continuously differentiable, cf., Lemma 1.1 in [5]), we can define unique probability measures $\mu_{\mathbf{x}}^d$ and $\mu_{\mathbf{x}}$ on $\{C, \mathcal{C}\}$ such that

$$\mu_{\mathbf{x}}^d\{\omega \in C; \omega(t_i) \leq x_i, i=1, \dots, n\} = P(\xi_{\mathbf{x}}^d(t_i) \leq x_i, i=1, \dots, n)$$

and with a similar relation holding for $\mu_{\mathbf{x}}$ and $\xi_{\mathbf{x}}$.

A set A in C is called $\mu_{\mathbf{x}}$ -continuous if $\mu_{\mathbf{x}}(\partial A) = 0$ where ∂A is the boundary of A . Then the definition of weak convergence on $\{C, \mathcal{C}\}$ can be stated as follows: $\mu_{\mathbf{x}}^d \Rightarrow \mu_{\mathbf{x}}$ ($\mu_{\mathbf{x}}^d$ converges weakly to $\mu_{\mathbf{x}}$) as $d \rightarrow 0$ if $\mu_{\mathbf{x}}^d(A) \rightarrow \mu_{\mathbf{x}}(A)$ for every $\mu_{\mathbf{x}}$ -continuous set A . When $\mu_{\mathbf{x}}^d$ and $\mu_{\mathbf{x}}$ are measures for stochastic processes $\xi_{\mathbf{x}}^d$ and $\xi_{\mathbf{x}}$ we have the following sufficient conditions for weak convergence (see Billingsley [1, p.95]):

- 1) The finite dimensional distributions of $\xi_{\mathbf{x}}^d$ converge to those of $\xi_{\mathbf{x}}$.
- 2a) $\sup_d E|\xi_{\mathbf{x}}^d(0)|^2 < \infty$,

2b) There is a K such that for any $t, t+h \in [0, T]$

$$E|\xi_{\mathbb{X}}^d(t) - \xi_{\mathbb{X}}^d(t+h)|^2 \leq Kh^2.$$

THEOREM A1. If $-r''(0) = \lambda_2 < \infty$ then $\mu_{\mathbb{X}}^d \Rightarrow \mu_{\mathbb{X}}$ on $\{C, C\}$ as $d \rightarrow 0$.

PROOF. 1) Since all involved functions are continuous, Theorem 5 gives the convergence of the finite dimensional distributions.

2) Since $E(\kappa_0) = 0, V(\kappa_0) = R_{00} = 0$, we have that $\kappa_0 = 0$ and $\xi_{\mathbb{X}}^d(0) = \xi_{\mathbb{X}}^d + \eta^d \cdot \frac{d}{2}$ almost surely, so that 2a) is fulfilled.

It suffices to show that 2b) holds for $t, t+h$ of the form $t = dk, t+h = d(k+1)$.

To see this, we observe that if $dk \leq t < d(k+1), d(k+1) \leq t+h < d(k+2)$ then the difference $|\xi_{\mathbb{X}}^d(t) - \xi_{\mathbb{X}}^d(t+h)|$ is bounded by the maximum of four differences of the type $|\xi_{\mathbb{X}}^d(dk) - \xi_{\mathbb{X}}^d(d(k+1))|$. Furthermore, for any random variables ξ_1, \dots, ξ_4 it holds $E(\max(\xi_1^2, \dots, \xi_4^2)) \leq 4 \max(E(\xi_1^2), \dots, E(\xi_4^2))$.

But for $t, t+h$ of the said type we have, if we write α_k and β_u for the coefficients for ξ^d and η^d in (2.1)

$$\begin{aligned} E|\xi_{\mathbb{X}}^d(t) - \xi_{\mathbb{X}}^d(t+h)|^2 &= V + E^2 = V(\xi^d(\alpha_k - \alpha_{k+1}) - \eta^d(\beta_k - \beta_{k+1})) \\ &\quad + V(\kappa_k - \kappa_{k+1}) + E^2(\eta^d) \cdot (\beta_k - \beta_{k+1})^2 = A + B + C, \text{ say.} \end{aligned}$$

Then

$$\begin{aligned} A &\leq 4 \max\{(\alpha_k - \alpha_{k+1})^2 V(\xi^d), (\beta_k - \beta_{k+1})^2 V(\eta^d)\} \leq K_1 (di)^2 = K_1 h^2, \\ B &= 2(1-r_1) + \frac{2}{1+r_1} (r_k - r_{k+1})(r_{k+1} - r_{k+2}) - \frac{1}{1-r_1} (r_k - r_{k+1} - r_{k+2} + r_{k+3})^2 \\ &\leq K_2 h^2, \\ C &\leq K_3 h^2. \end{aligned}$$

These estimates are all based on the fact that if $-r''(0) < \infty$ then $r''(t)$ exists for all t . Thus 2b) holds and the theorem is proved. \square

Now we can apply the weak convergence theorem to several important random variables, e.g., the first hitting time of a level u : $\inf\{t \geq 0; \xi_{\mathbb{X}}^d(t) \geq u\}$. However, if we try to use it on the probability $P(\tau_{\mathbb{X}}^d > x) = P(\xi_{\mathbb{X}}^d(t) > 0 \text{ for } 0 < t \leq x)$ then we have to consider the set $A_{\mathbb{X}} = \{\omega \in C; \omega(t) > 0 \text{ for } 0 < t \leq x\}$, which has the boundary $\partial A_{\mathbb{X}} = \{\omega \in C; \inf_{0 \leq t \leq x} \omega(t) = 0\}$. This set has a positive $\mu_{\mathbb{X}}$ -measure and thus $A_{\mathbb{X}}$ is not $\mu_{\mathbb{X}}$ -continuous. We, therefore, introduce a new concept for which the convergence still holds.

DEFINITION. A set A in C is uniformly almost $\mu_{\mathbb{X}}$ -continuous, relative the family $\{\mu_{\mathbb{X}}^d\}$, (u.a. $\mu_{\mathbb{X}}$ -continuous, $\{\mu_{\mathbb{X}}^d\}$), if, for each $\epsilon > 0$ there is a $\mu_{\mathbb{X}}$ -continuous set A_{ϵ} such that

$$\mu_{\mathbb{X}}(A \Delta A_{\epsilon}) \leq \epsilon, \quad \limsup_{d \rightarrow 0} \mu_{\mathbb{X}}^d(A \Delta A_{\epsilon}) \leq \epsilon,$$

where $A \Delta A_{\epsilon}$ is the symmetric difference: $(A - A_{\epsilon}) \cup (A_{\epsilon} - A)$.

LEMMA A1. If A is u.a. $\mu_{\mathbb{X}}$ -continuous, $\{\mu_{\mathbb{X}}^d\}$, and $\mu_{\mathbb{X}}^d \Rightarrow \mu_{\mathbb{X}}$ then $\mu_{\mathbb{X}}^d(A) \rightarrow \mu_{\mathbb{X}}(A)$ as $d \rightarrow 0$.

PROOF. Take $\epsilon > 0$ and an approximating $\mu_{\mathbb{X}}$ -continuous set A_{ϵ} . Then

$$\mu_{\mathbb{X}}^d(A_{\epsilon}) - \mu_{\mathbb{X}}^d(A_{\epsilon} - A) \leq \mu_{\mathbb{X}}^d(A) \leq \mu_{\mathbb{X}}^d(A_{\epsilon}) + \mu_{\mathbb{X}}^d(A - A_{\epsilon}),$$

so that

$$\mu_{\mathbb{X}}(A) - 2\epsilon \leq \mu_{\mathbb{X}}(A_{\epsilon}) - \epsilon \leq \lim \mu_{\mathbb{X}}^d(A) \leq \mu_{\mathbb{X}}(A_{\epsilon}) + \epsilon \leq \mu_{\mathbb{X}}(A) + 2\epsilon.$$

Since ϵ is arbitrary, the lemma is proved. \square

LEMMA A2. If $-r''(t) = \lambda_2 - O(|\log|t||^{-a})$ as $t \rightarrow 0$ for some $a > 1$ then the set $A_{\mathbb{X}} = \{\omega \in C; \omega(t) > 0 \text{ for } 0 < t \leq x\}$ is u.a. $\mu_{\mathbb{X}}$ -continuous, $\{\mu_{\mathbb{X}}^d\}$.

PROOF. We show that for small δ the set $A_{\delta x} = \{\omega \in C; \omega(t) > 0 \text{ for } \delta \leq t \leq x\}$ fulfills the requirements for an approximating set.

a) $A_{\delta x}$ is $\mu_{\mathbb{X}}$ -continuous: Its boundary set is $\{\omega \in C; \inf_{\delta \leq t \leq x} \omega(t) = 0\}$ and by a theorem by Ylvisaker [13] this has $\mu_{\mathbb{X}}$ -measure zero. Note that

$V(\xi_{\mathbb{X}}(t)) \geq \delta' > 0$ for $t > \delta$, but $V(\xi_{\mathbb{X}}(0)) = 0$.

b) $A_{\delta\mathbb{X}}$ is uniformly close to $A_{\mathbb{X}}$ for small δ : The difference $A_{\mathbb{X}} \Delta A_{\delta\mathbb{X}} = \{\omega \in C; \omega(t) \leq 0 \text{ for some } t \in (0, \delta)\}$ differs from CA_{δ} only by a set of $\mu_{\mathbb{X}}$ -measure zero so that

$$\begin{aligned} \mu_{\mathbb{X}}(A_{\mathbb{X}} \Delta A_{\delta\mathbb{X}}) &= 1 - \mu_{\mathbb{X}}(A_{\delta\mathbb{X}}) = 1 - P(\xi_{\mathbb{X}}(t) > 0 \text{ for } 0 < t \leq \delta) \\ &= 1 - P(\kappa(t) > \eta r'(t) / \lambda_2 \text{ for } 0 < t \leq \delta). \end{aligned}$$

But if $-r''(t) = \lambda_2 - 0(|\log|t||^{-a})$ with $a > 1$ then κ has (or can be redefined to have) continuously differentiable sample paths (cf., Lemma 1.1 in [5]).

Since $P(\kappa(0)=0) = 1$ we therefore have

$$\begin{aligned} P(\kappa(t) > \eta r'(t) / \lambda_2 \text{ for } 0 < t \leq \delta) &\geq P(\kappa'(t) > \eta r''(t) / \lambda_2 \text{ for } 0 < t \leq \delta) \\ &\geq \int_0^{\infty} f_{\eta}(y) P(\inf_{0 < t \leq \delta} \kappa'(t) > y \sup_{0 < t \leq \delta} r''(t) / \lambda_2) dy \\ &\geq \int_0^{\infty} f_{\eta}(y) P(\inf_{0 < t \leq \delta} \kappa'(t) > -\frac{1}{2}y) dy \end{aligned}$$

if δ is small. Since $E(\kappa'(0)) = 0$, $V(\kappa'(0)) = -\partial^2 R(s, t) / \partial s \partial t \Big|_{s=t=0} = 0$ we have that $P(\kappa'(0)=0) = 1$ and $P(\kappa'(t) \rightarrow 0 \text{ as } t \rightarrow 0) = 1$. This gives that $P(\inf_{0 < t \leq \delta} \kappa'(t) > -\frac{1}{2}y) \rightarrow 1$ as $\delta \rightarrow 0$ for $y > 0$, and it follows that $\mu_{\mathbb{X}}(A_{\mathbb{X}} \Delta A_{\delta\mathbb{X}}) \rightarrow 0$ as $\delta \rightarrow 0$.

To get a similar relation for $\mu_{\mathbb{X}}^d$, let $K_{\delta} = [d^{-1}\delta]$. Then $\mu_{\mathbb{X}}^d(A_{\delta}) = P(\xi_{\mathbb{X}}^d(t) > 0 \text{ for } 0 < t \leq \delta) \geq P(\xi_{\mathbb{X}}^d(t) > 0 \text{ for } 0 < t \leq d(K_{\delta}+1))$. If we recall the definition of $\xi_{\mathbb{X}}^d$ as a conditional process, we see that this probability is nothing but

$$\begin{aligned} &P(\xi_k > 0 \text{ for } k=1, \dots, K_{\delta}+1 \mid \xi_{-1} < 0 < \xi_0) \\ (A1) \quad &= 1 - \frac{P(\xi_{-1} < 0 < \xi_0, \xi_k > 0 > \xi_{k+1} \text{ for some } k=0, 1, \dots, K_{\delta})}{P(\xi_{-1} < 0 < \xi_0)}. \end{aligned}$$

If N_d and $N_{\delta d}$ denote the number of times ξ crosses the zero level in the intervals $[-d, 0]$ and $[0, d(K_\delta + 1)]$ respectively, then the event in the nominator in (A1) implies that $N_d \geq 1$, $N_{\delta d} \geq 1$ and the nominator is less or equal to $P(N_d \geq 1, N_{\delta d} \geq 1) \leq E(N_d \cdot N_{\delta d})$. Now

$$\begin{aligned} E(N_d \cdot N_{\delta d}) &= \int_{t_1=-d}^0 \int_{t_2=0}^{d(K_\delta+1)} \Psi(t_2-t_1) dt_2 dt_1 \leq \int_{t_1=-d}^0 \int_{\tau=0}^{\delta+2d} \Psi(\tau) d\tau dt_1 \\ &= d \int_0^{\delta+2d} \Psi(\tau) d\tau, \end{aligned}$$

where

$$\Psi(\tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |y_1 y_2| p_\tau(0, 0; y_1, y_2) dy_1 dy_2,$$

$$p_\tau(x_1, x_2; y_1, y_2) = \text{the density of } \xi(0), \xi(\tau), \xi'(0), \xi'(\tau).$$

For the denominator in (A1), we have $P(\xi_{-1} < 0 < \xi_0) = (2\pi)^{-1} \arccos r_1 \sim d\sqrt{\lambda_2}/(2\pi)$ as $d \rightarrow 0$. In total, we see that there is an M such that

$$\mu_{\mathfrak{K}}^d(A_{\mathfrak{X}} \Delta A_{\delta \mathfrak{X}}) \leq M d^{-1} \cdot d \int_0^{\delta+2d} \Psi(\tau) d\tau.$$

Now the condition $-r''(t) = \lambda_2 - 0(|\log|t||^{-a})$ implies that Ψ is integrable over an interval near zero, (cf. [2], p.210). Therefore,

$$\limsup_{d \rightarrow 0} \mu_{\mathfrak{K}}^d(A_{\mathfrak{X}} \Delta A_{\delta \mathfrak{X}}) \leq M \int_0^{\delta} \Psi(\tau) d\tau \rightarrow 0 \text{ as } \delta \rightarrow 0$$

which finally proves that $A_{\delta \mathfrak{X}}$ is uniformly approximating.

By combining the two lemmas, we get the desired theorem for $\tau_{\mathfrak{K}}^d$:

THEOREM A2. If $-r''(t) = \lambda_2 - 0(|\log|t||^{-a})$ as $t \rightarrow 0$ for some $a > 1$ then $P(\tau_{\mathfrak{K}}^d > x) \rightarrow P(\tau_{\mathfrak{K}} > x)$ as $d \rightarrow 0$.

We now turn to ξ_u^d and τ_u^d . One possibility is to proceed as for ξ_x^d and show weak convergence on $\{C, C\}$ which will give us the convergence in law for all random variables depending continuously on uniformly small changes of the sample paths. However, the wave-length τ_u depends on the derivative of ξ_u and can be greatly affected even by small changes of $\xi_u(t)$. The remedy for this is to look at, not the process ξ_u^d but its "derivative", $\dot{\xi}_u^d$, which is defined by $\dot{\xi}_u^d(t) = d^{-1}(\xi_u^d(t) - \xi_u^d(t-d))$ for $t = dk$, $k = 0, \pm 1, \dots$, and by linear approximation for other t -values. The derivative $\xi_u'(t) = uA'(t) - \tau_u B'(t) + \Delta'(t)$ exists and is continuous under the assumed conditions, and τ_u^d and τ_u can be defined as

$$\tau_u^d = \text{the time for the first zero upcrossing by } \dot{\xi}_u^d \\ \tau_u \quad \xi_u'$$

Now we can proceed as before and define measures $\dot{\mu}_u^d$, μ_u' on $\{C, C\}$ for the processes $\dot{\xi}_u^d$, ξ_u' and finally arrive at the following theorems, the proofs of which are left out.

THEOREM A3. If $r^{IV}(0) = \lambda_4 < \infty$, then $\dot{\mu}_u^d \Rightarrow \mu_u'$ as $d \rightarrow 0$.

THEOREM A4. If $r^{IV}(t) = \lambda_4 - O(|\log|t||^{-a})$ as $t \rightarrow 0$ for some $a > 1$ then $P(\tau_u^d > x) \rightarrow P(\tau_u > x)$ as $d \rightarrow 0$.

Theorem A3 can be restated as follows. Let D be the metric space of continuously differentiable functions on $[0, T]$ with $\omega(0) = 0$ and with the distance $d(\omega, \tilde{\omega}) = \sup_{[0, T]} |\omega'(t) - \tilde{\omega}'(t)|$, and define an isometry σ between D and C such that $\omega \xrightarrow{\sigma} \sigma(\omega) = \omega' \in C$. If $\mu_u(A) = \mu_u'(\sigma(A))$, $\mu_u^d(A) = \dot{\mu}_u^d(\sigma(A))$, then μ_u gives the distributions for ξ_u and we have:

THEOREM A5. If $r^{IV}(0) = \lambda_4 < \infty$, then $\mu_u^d \Rightarrow \mu_u$ on $\{D, \mathcal{D}\}$ as $d \rightarrow 0$.

REFERENCES

- [1] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.
- [2] H. Cramér and M.R. Leadbetter, *Stationary and Related Stochastic Processes*, Wiley, New York, 1967.
- [3] M. Kac and D. Slepian, "Large excursions of Gaussian processes," *Ann. Math. Statist.* 30 (1959), pp. 1215-1228.
- [4] G. Lindgren, "Some properties of a normal process near a local maximum," *Ann. Math. Statist.* 41 (1970), pp. 1870-1883.
- [5] G. Lindgren, "Extreme values of stationary normal processes," *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, 17 (1971), pp. 39-47.
- [6] G. Lindgren, "Wave-length and amplitude in Gaussian noise," To appear in *Advances in Appl. Probability* (1972).
- [7] D. E. Longuet-Higgins, "The distribution of intervals between zeros of a stationary random function," *Philos. Trans. Roy. Soc. London, Ser. A* 254 (1962), pp. 557-599.
- [8] C. R. Rao, *Linear Statistical Inference and its Applications*, Wiley, New York, 1965.
- [9] D. Slepian, "The one-sided barrier problem for Gaussian noise," *Bell System Tech. J.* 41 (1962), pp. 463-501.
- [10] D. Slepian, "On the zeros of Gaussian noise," *Time Series Analysis*, M. Rosenblatt ed., Wiley, New York, 1962, pp. 104-115.
- [11] L. J. Tich and P. Shaman, "Sampling rates and appearance of stationary Gaussian processes," *Technometrics* 8 (1966), pp. 91-106.
- [12] E. Wong, "Some results concerning zero-crossings in Gaussian noise," *SIAM J. Appl. Math.* 14 (1966), pp. 1246-1254.
- [13] N. D. Ylvisaker, "A note on the absence of tangencies in Gaussian sample paths," *Ann. Math. Statist.* 39 (1968), pp. 261-262.