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Confidence Set for the Ratio of Means of Two Normal Distributions when the Ratio of Variances is Unknown*

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1. Introduction

Let $X_{11}, X_{12}, \ldots, X_{1n_1}; X_{21}, \ldots, X_{2n_2}$, be (n_1+n_2) independent normal random variables. We assume,

$$E(X_{iu}) = \mu_i, \quad Var(X_{iu}) = \sigma_i^2 \tag{1.1}$$

for $u = 1,2,\ldots,n_i$, $i = 1,2$.

We assume $\mu_2 \neq 0$ and set $\lambda = \mu_1/\mu_2$. The problem is to derive a confidence interval for λ, without any assumption about the ratio of the two variances σ_1^2 and σ_2^2.

Let

$$\bar{X}_i = \frac{1}{n_i} \sum_{u=1}^{n_i} X_{iu}, \quad S_i^2 = \sum_{u=1}^{n_i} (X_{iu}-\bar{X}_i)^2, \quad s_i^2 = S_i^2/(n_i-1), \quad i = 1,2. \tag{1.2}$$

We give a confidence set for λ with a confidence coefficient which is guaranteed to exceed or equal $(1-\alpha)$, when α is a preassigned positive number less than 1.

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2. CONFIDENCE SET FOR THE RATIO λ

Let $t_\alpha(m)$ be a constant chosen to satisfy

$$\text{Prob}\left\{|t(m)| \le t_\alpha(m)\right\} = 1 - \alpha \tag{2.1}$$

where $t(m)$ follows the Student's t distribution with m degrees of freedom and α is a preassigned positive number less than 1. For simplicity, in this article, we denote $t_\alpha(n_i-1)$ by t_i, $i = 1,2$.

We note that the random variable

$$Y = \frac{(\bar{X}_1-\lambda\bar{X}_2)^2}{\sigma_1^2/n_1 + \lambda^2\sigma_2^2/n_2} \tag{2.2}$$

follows chi-square distribution with 1 degree of freedom and the probability density of this distribution is

$$f(y) = (2\pi)^{-\frac{1}{2}} \exp(-y/2)\, y^{-\frac{1}{2}}, \quad 0 \le y \le \infty. \tag{2.3}$$

Consider the integral

$$g(v) = \int_0^v (2\pi)^{-\frac{1}{2}} \exp(-y/2)\, y^{-\frac{1}{2}}\, dy. \tag{2.4}$$

Noting that $g'(v) > 0$ and $g''(v) < 0$ it follows that

$$g(pv_1+(1-p)v_2) \ge pg(v_1) + (1-p)g(v_2) \tag{2.5}$$

for $0 < p < 1$ and $0 < v_1$, $v_2 < \infty$. That is, $g(v)$ is a concave function.

Let

$$V = \left(\frac{t_1^2\, s_1^2}{n_1(n_1-1)} + \lambda^2 \frac{t_2^2\, s_2^2}{n_2(n_2-1)}\right) \Bigg/ \left(\frac{\sigma_1^2}{n_1} + \lambda^2 \frac{\sigma_2^2}{n_2}\right)$$

$$= w_1 \frac{t_1^2\, s_1^2}{(n_1-1)\sigma_1^2} + w_2 \frac{t_2^2\, s_2^2}{(n_2-1)\sigma_2^2}, \tag{2.6}$$

where

$$w_1 = \left(\frac{\sigma_1^2}{n_1}\right) \Big/ \left(\frac{\sigma_1^2}{n_1} + \lambda^2 \frac{\sigma_2^2}{n_2}\right), \qquad w_2 = \lambda^2 \left(\frac{\sigma_2^2}{n_2}\right) \Big/ \left(\frac{\sigma_1^2}{n_1} + \lambda^2 \frac{\sigma_2^2}{n_2}\right).$$

Note that

$$w_1 + w_2 = 1 \text{ and } 0 \le w_i \le 1, \quad i = 1, 2. \tag{2.7}$$

Let $Z_i = S_i^2/\sigma_i^2$, $i = 1,2$, then Z_i has a chi-square distribution with (n_i-1) degrees of freedom and $T_i^2 = (n_i-1)\, Y/Z_i$ is distributed as the square of the Student's t with (n_i-1) degrees of freedom (or equivalently as an F with 1 and (n_i-1) degrees of freedom), $i = 1,2$.

Now V can be rewritten as

$$V = w_1 \frac{t_1^2 Z_1}{(n_1-1)} + w_2 \frac{t_2^2 Z_2}{(n_2-1)} . \tag{2.8}$$

Let

$$P = \text{Prob}\left\{(\bar{X}_1-\lambda\bar{X}_2)^2 \le \frac{t_1^2 S_1^2}{(n_1-1)n_1} + \lambda^2 \frac{t_2^2 S_2^2}{(n_2-1)n_2}\right\}$$

$$= \text{Prob}\left\{Y \le V\right\}$$

$$= \text{Prob}\left\{Y \le w_1 \frac{t_1^2 Z_1}{(n_1-1)} + w_2 \frac{t_2^2 Z_2}{(n_2-1)}\right\}. \tag{2.9}$$

We then prove the following

THEOREM $P \ge 1-\alpha$.

PROOF: Y, Z_1 and Z_2 are independently distributed as chi-square's with 1, (n_1-1) and (n_2-1) degrees of freedom respectively. Let $h(z_i;\, n_i-1)$ denote the density of the distribution of Z_i (chi-square distribution with $n_i-1)$ degrees of freedom) $i = 1,2$. Then the probability P can be written as the triple integral

$$P = \int_0^\infty \int_0^\infty \left[\int_0^{w_1 v_1 + w_2 v_2} (2\pi)^{-\frac{1}{2}} \, exp(-y/2) y^{-\frac{1}{2}} dy \right] h(z_1;n_1-1) h(z_2;n_1-1) \, dz_1 \, dz_2$$

$$= \int_0^\infty \int_0^\infty g(w_1 v_1 + w_2 v_2) \; h(z_1;n_1-1) \; h(z_2;n_2-1) \; dz_1 \; dz_2 \tag{2.10}$$

where $v_1 = (t_1^2 \; z_1/n_1-1)$ and $v_2 = (t_2^2 \; z_2/n_2-1)$. But from (2.5)

$$g(w_1 v_1 + w_2 v_2) \;\; \geq \;\; w_1 \; g(v_1) + w_2 \; g(v_2). \tag{2.11}$$

Further

$$\int_0^\infty \int_0^\infty g(v_i) \; h(z_1;n_1-1) \; h(z_2;n_2-1) \; dz_1 \; dz_2$$

$$= \text{Prob} \; \{T_i^2 \leq t_i^2\} \; = \; 1 - \alpha, \qquad i = 1,2. \tag{2.12}$$

Thus, from (2.11) and (2.12) we have

$$P \;\; \geq \;\; w_1(1-\alpha) + w_2(1-\alpha) \;\; = \;\; 1 - \alpha.$$

A confidence set for λ is thus, given by

$$(\bar{X}_1 - \lambda \bar{X}_2)^2 \;\; \leq \;\; \frac{t_1^2 \; s_1^2}{n_1} + \lambda^2 \; \frac{t_2^2 \; s_2^2}{n_1} \tag{2.13}$$

or

$$\lambda^2 \left(\bar{X}_2^2 - \frac{t_2^2 \; s_2^2}{n_2} \right) - 2 \; \lambda \; \bar{X}_1 \bar{X}_2 + \left(\bar{X}_1^2 - \frac{t_1^2 \; s_1^2}{n_1} \right) \;\; \leq \;\; 0 \tag{2.14}$$

with a confidence coefficient greater than or equal to $(1-\alpha)$.

Consider the quadratic equation

$$\lambda^2 \left(\bar{X}_2^2 - \frac{t_2^2 \; s_2^2}{n_2} \right) - 2 \; \lambda \; \bar{X}_1 \bar{X}_2 + \left(\bar{X}_1^2 - \frac{t_1^2 \; s_1^2}{n_1} \right) \;\; = \;\; 0. \tag{2.15}$$

Writing

$$a \; = \; \bar{X}_2^2 - \frac{t_2^2 \; s_2^2}{n_2}, \qquad b \; = \; \bar{X}_1 \bar{X}_2, \qquad c \; = \; \bar{X}_1^2 - \frac{t_1^2 \; s_1^2}{n_1} \; , \tag{2.16}$$

(2.15) becomes

$$a\lambda^2 - 2\lambda b + c = 0. \tag{2.17}$$

The two roots of (2.17) are given by

$$\lambda \pm = \frac{b \pm (b^2 - ac)^{\frac{1}{2}}}{a} . \tag{2.18}$$

Let $\beta = b^2 - ac$.

3. Confidence Interval. Limitations.

If $a > 0$ and $\beta > 0$, the confidence set (2.14) becomes a confidence interval for λ, namely,

$$\lambda - \leq \lambda \leq \lambda + . \tag{3.1}$$

If $(a < 0, \ \beta > 0)$, (2.14) implies that λ is outside of the interval $(\lambda-, \ \lambda+)$.

It is easy to show that the case $(a > 0, \ \beta < 0)$ is impossible. The cases $a = 0$ and $\beta = 0$ are ignored, because the probabilities are zero for these events.

The case $(a < 0, \ \beta < 0)$ may, however, occur with positive probability. In this case (2.14) will imply that $-\infty \leq \lambda \leq \infty$. The confidence set for λ, therefore, as constructed here suffers from the same limitation as the one constructed by *Fieller* (1940) for a bivariate sample from a bivariate normal distribution.

Recently, *Scheffé* (1970) has constructed a confidence set for Fieller's problem, which is free from this limitation. Using Scheffé's ideas, it is possible to derive a modified confidence set for out problem, which will not be improper in Scheffé's sense. This will be reported in a subsequent communication.

4. ILLUSTRATION

Using the formula (2.14) of this paper, confidence sets were calculated for the ratio of mean caries increment in two groups, experimental and control, from data compiled by different investigators and reported in Table II of a paper by *Marthaler* (1970). In every case, the confidence set resulted in a confidence interval of the type $\lambda- \leq \lambda \leq \lambda+$. Here we give one example based on data by *Marthaler* (1970).

n_c = number of subjects in the control group = 32

n_e = number of subjects in the experimental group = 42

$\bar{x}_c$ = mean caries increment in the control group = 15.25

$\bar{x}_e$ = mean caries increment in the experimental group = 11.33

s_c = standard deviation of the individual caries count in the control group = 8.55

s_e = standard deviation of the individual caries count in the experimental group = 7.59

$t_c^2 = t_{0.5}^2(31) = 4.17$

$t_e^2 = t_{0.5}^2(41) = 4.08.$

Let

$$\lambda = \frac{\text{mean caries increment in the experimental group}}{\text{mean caries increment in the control group}} .$$

The confidence set for λ then, with a confidence coefficient greater or equal to 0.95, is

$$0.5518 \leq \lambda \leq 0.9976,$$

and

$$\hat{\lambda} = \frac{\bar{x}_e}{\bar{x}_c} = 0.743.$$

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Summary

In this article, a confidence set for the *ratio of the means* of two normal distributions, has been derived *without making any assumption on the ratio of the two variances*. The construction of the confidence set uses the sample means and sample variances of two independent random samples from the two distributions. The solution is exact and does not use any large sample approximation. The confidence coefficient, however, is guaranteed to be greater than or equal to $(1-\alpha)$ rather than be exactly equal to $(1-\alpha)$, where α is a preassigned positive number less than 1.

Zusammenfassung

Ein Konfidenzbereich für das *Verhältnis der Mittelwerte von zwei Normalverteilungen* wird abgeleitet, *ohne Annanmen über das Verhältnis der beiden Varianzen* zu treffen. Zur Berechnung des Konfidenzbereiches werden die Mittelwerte und die Varianzen von zwei unabhängigen Zufallsstichproben aus den beiden Verteilungen benützt. Die Lösung ist genau und stellt keine Annäherung mittels grösser Stichproben dar. Vom Konfidenzkoeffizienten kann jedoch nur behauptet werden, dass er grösser oder gleich, jedoch nicht genau gleich $(1-\alpha)$ ist. Hierbei ist α eine vorgegebene positive Zahl kleiner als 1.