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On Generalized Inverses in a Linear Associative Algebra and Their Applications in the Analysis of a Class of Designs*

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1. Introduction

1.1. The association matrices $B_0, B_1, \ldots, B_m$ of an m-class association scheme on v objects are defined [3] by

$$(1.1) \qquad B_0 = I_{v\times v}, \quad B_i = ((b_{\alpha\beta}^{\ i})), \quad i = 1, 2, \ldots, m,$$

where

$$b_{\alpha\beta}^{\ i} = 1 \text{ if objects } \alpha \text{ and } \beta \text{ are i-th associates}, \quad = 0 \text{ otherwise}.$$

$$(1.2) \qquad B_0 + B_1 + \ldots + B_m = J_{v\times v},$$

where $J_{v\times v}$ is the matrix with 1 as element everywhere. Clearly, B_i is a symmetric matrix with all row and column sums equal to n_i. Further,

$$(1.3) \qquad B_jB_k = p_{jk}^{\ 0}B_0 + p_{jk}^{\ 1}B_1 + \ldots + p_{jk}^{\ m}B_m.$$

The commutative and associative laws hold for the multiplication of these matrices. It has been shown [3] that the linear forms $c_0B_0+c_1B_1+\ldots+c_mB_m$ form a linear associative and commutative algebra with a unit element, if the coefficients $c_0, c_1, \ldots, c_m$ belong to a field. In this article, c_i's will be considered as reals. We note that $B_0, B_1, \ldots, B_m$ form a basis of this algebra.

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It is also known [3] that the $(m+1)\times(m+1)$ matrices $P_0 = I_{(m+1)\times(m+1)}$, $P_1,\ldots,P_m$ defined by

(1.4) $$P_k = ((p_{ik}^{\ j})), \quad i,j = 0,1,\ldots,m, \quad k = 0,1,\ldots,m$$

provide a regular representation of the algebra defined by the matrices $B_0,B_1,\ldots,B_m$. The matrices $P_0,P_1,\ldots,P_m$ are linearly independent and provide a basis for the vector space generated by the linear forms $c_0P_0+c_1P_1+\ldots+c_mP_m$ where $c_0,c_1,\ldots,c_m$ belong to a field. P_i's are not, necessarily, symmetric.

1.2. We state here, without proof, a result which we shall need later. For proof, see [3].

Lemma 1.1. The two matrices $B = c_0B_0+\ldots+c_mB_m$ and $P = c_0P_0+\ldots+c_mP_m$ have the same minimal polynomial and hence the same distinct characteristic roots.

Let z_{ui}, $u = 0,1,\ldots,m$ denote the $m+1$ characteristic roots of P_i, $i = 0,1,\ldots,m$. It is known that z_{ui}, $u = 0,1,\ldots,m$, $i = 0,1,\ldots,m$ are all real and that the $(m+1)\times(m+1)$ matrix

(1.5) $$Z = ((z_{ui})), \quad u = 0, 1, \ldots, m, \quad i = 0, 1, \ldots, m$$

is non-singular [3]. Further, for a suitable ordering of z_{ui} for each i, the characteristic roots of the matrix $P = \sum_{i=0}^m c_iP_i$ are given by

(1.6) $$\theta_u = \sum_{i=0}^{m} c_i z_{ui}, \quad u = 0,1,\ldots,m.$$

The distinct characteristic roots of $B = \sum_{i=0}^m c_iB_i$ are, therefore, to be found among θ_u given by (1.6).

Lemma 1.2. For fixed $u = 0,\ldots,m$, the roots z_{ui} satisfy the relations

(1.7) $$z_{uj}z_{uk} = \sum_{i=0}^{m} p_{jk}^{\ i}\, z_{ui}.$$ For proof, see [3].

This lemma helps us to determine the ordering of the roots z_{ui} for a given i.

For a two-class association scheme (m=2), the matrix Z of ordered characteristic roots of association matrices is given by

$$(1.8) \qquad Z = \begin{bmatrix} 1 & n_1 & n_2 \\ 1 & \frac{\gamma-1+\sqrt{\Delta}}{2} & \frac{-\gamma-1-\sqrt{\Delta}}{2} \\ 1 & \frac{\gamma-1-\sqrt{\Delta}}{2} & \frac{-\gamma-1+\sqrt{\Delta}}{2} \end{bmatrix}$$

where

$$\gamma = p_{12}^{\ 2} - p_{12}^{\ 1}, \quad \beta = p_{12}^{\ 2} + p_{12}^{\ 1}, \quad \Delta = \gamma^2 + 2\beta + 1.$$

The multiplicities $\alpha_0, \alpha_1, \alpha_2$, are given by

$$(1.9) \qquad \alpha_0 = 1, \quad \alpha_1 = \frac{n_1+n_2}{2} - \frac{(n_1-n_2)+\gamma(n_1+n_2)}{2\sqrt{\Delta}},$$

$$\alpha_2 = \frac{n_1+n_2}{2} + \frac{(n_1-n_2)+\gamma(n_1+n_2)}{2\sqrt{\Delta}}.$$

2. Generalized Inverse of a Matrix in a Linear Associative Algebra

2.1. Consider a matrix $B = c_0B_0+c_1B_1+c_2B_2$ in the linear associative algebra generated by B_0, B_1 and B_2. Then $P = c_0P_0+c_1P_1+c_2P_2$ is the image of B in the regular representation by the algebra generated by P_0, P_1 and P_2 which are 3×3 matrices. B and P have the same minimal polynomial h(x) of degree at most three. Suppose that the distinct characteristic roots of B (and hence of P) are 0, θ_1 and θ_2. The minimal polynomial h(x) is then [5],

$$(2.1) \qquad h(x) = x(x-\theta_1)(x-\theta_2).$$

Since every matrix satisfies its minimal polynomial, we have

(2.2) $$B(B-\theta_1 B_0)(B-\theta_2 B_0) = 0.$$

We define

(2.3) $$S = -\frac{1}{\theta_1\theta_2}[B-(\theta_1+\theta_2)B_0].$$

Then

(2.4) $$BSB = B.$$

This implies that $S = B^-$ is a generalized inverse of B. In the same way we find that

(2.5) $$P^- = -\frac{1}{\theta_1\theta_2}[P-(\theta_1+\theta_2)P_0]$$

is a generalized inverse of P. It is easy to verify that the characteristic roots of P^- are $(1/\theta_1+1/\theta_2)$, $1/\theta_1$ and $1/\theta_2$ corresponding to the roots 0, θ_1 and θ_2 of P. The same is true of the distinct characteristic roots of B^-.

If B has the characteristic roots $0, \theta_1$ and θ_2, B^2 has the characteristic roots $0, \theta_1{}^2$ and $\theta_2{}^2$ and the minimal polynomial of B^2 is

(2.6) $$g(x) = x(x-\theta_1{}^2)(x-\theta_2{}^2).$$

Let

(2.7) $$M = \frac{1}{\theta_1{}^2\theta_2{}^2}[-(\theta_1{}^2+\theta_2{}^2)B_0+B^2]$$

and

(2.8) $$G = MB = \frac{1}{\theta_1{}^2\theta_2{}^2}[-(\theta_1{}^2+\theta_2{}^2)B+B^3].$$

Then we have

(2.9) $$BGB = B, \quad GBG = G, \quad (BG)' = BG, \quad (GB)' = GB.$$

Hence $G = B^+$ is the Moore-Penrose inverse of B ([7],[8]). Similarly,

$$(2.10) \qquad T = P^+ = -\frac{1}{\theta_1^2\theta_2^2}[-(\theta_1^2+\theta_2^2)P+P^3]$$

is the Moore-Penrose inverse of P. We note that the distinct characteristic roots of B^+ are 0, $1/\theta_1$ and $1/\theta_2$. The same is true of P^+. Further, B^+ and P^+ belong to the subalgebras generated by B and P respectively.

3. Generalized Inverse of a Partitioned Matrix

We consider a matrix

$$(3.1) \qquad F = \left[\begin{array}{c|c} NN' & K \\ \hline K' & 0 \end{array}\right] = \left[\begin{array}{c|c} B & K \\ \hline K' & 0 \end{array}\right]$$

where N, K, K' are real matrices, and K' is the transpose of K. We assume that the column-vectors of K belong to the vectorspace generated by the column-vectors of NN'. That is, $K = BL$. Note that F is not necessarily *non-negative*.

Let B^- be a generalized inverse of B. Define $Q = -K'B^-K$ and let Q^- be a generalized inverse of Q. Now $Q = -K'B^-K = -L'BB^-BL = -L'BL$. Consider the matrix

$$(3.2) \qquad H = \left[\begin{array}{c|c} B^- + B^-KQ^-K'B^- & -B^-KQ^- \\ \hline -Q^-K'B^- & Q^- \end{array}\right].$$

Then it is easy to verify that

$$(3.3) \qquad FHF = F,$$

and hence $H = F^-$ is a generalized inverse of F. An alternative expression for F^- is

$$F^{-} = \left[\begin{array}{c|c} B^{-} - LR^{-}L' & LR^{-} \\ \hline R^{-}L' & -R^{-} \end{array}\right] \quad (3.4)$$

where $R = L'BL$ and $K = BL$. The expression for a generalized inverse of F when it is *non-negative* Hermitian is given in [10]. For F non-singular, its inverse is given in [6].

4. Linear Model and Solution of Normal Equations. Applications.

4.1. Consider the linear model

$$E(\underline{y}) = A'\underline{\mu}\ , \quad (4.1)$$

where $\underline{y}$ is a column-vector of n random variables $y_1, y_2, \ldots, y_n$ which are uncorrelated and have the same variance σ^2. It is known ([1],[9]) that the minimum variance unbiased linear estimator of an estimable linear function $\underline{1}'\underline{\mu}$ is given by $\underline{1}'\hat{\underline{\mu}}$ where $\hat{\underline{\mu}}$ is a solution of the normal equations

$$AA'\underline{\mu} = A\underline{y}. \quad (4.2)$$

If $(AA')^{-}$ is a generalized inverse of AA', then $\hat{\underline{\mu}} = (AA')^{-}A\underline{y}$ is a solution of (4.2).

4.2. Consider a (connected) partially balanced design based on a two-class association scheme. Then the coefficient matrix C in the normal equations for estimating the treatment effects after adjusting for the block effects [2] is

$$C = r(1 - \frac{1}{k})B_0 - \frac{\lambda_1}{k} B_1 - \frac{\lambda_2}{k} B_2. \quad (4.3)$$

The representation of C is

(4.4) $$P = r(1-\frac{1}{k})P_0 - \frac{\lambda_1}{k}P_1 - \frac{\lambda_2}{k}P_2.$$

The characteristic roots of P are given by

(4.5) $Z\underline{u}$, where Z is as defined by (1.10) and $\underline{u}' = (u_0, u_1, u_2)$ is given by

(4.6) $$u_0 = r(1-\frac{1}{k}), \quad u_1 = -\frac{\lambda_1}{k}, \quad u_2 = -\frac{\lambda_2}{k}.$$

Using the relation

(4.7) $$r(k-1) = n_1\lambda_1 + n_2\lambda_2,$$

for a partially balanced design, it is easy to verify that 0 is a root of P and the other two roots θ_1 and θ_2 must be distinct if $\lambda_1 \neq \lambda_2$. Thus 0, θ_1, θ_2 will also be the distinct roots of C, 0 being a simple root and θ_1 and θ_2 will occur with multiplicities α_1 and α_2 given by (1.9). Now, it is easy to show that for solving the linear equations

(4.8) $$C\underline{\mu} = \underline{Q},$$

it is enough if one solves

(4.9) $$P\underline{\beta} = \underline{R},$$

where $\underline{\beta}' = (\beta_0, \beta_1, \beta_2)$ and $\underline{R}' = (R_0, R_1, R_2)$ are defined as follows:

(4.10) $\beta_0 = \mu_1$ (say), β_1 = the first element of $B_1\underline{\mu}$ = sum of the parameters for those treatments which are first associates of the first treatment, β_2 = the first element of $B_2\underline{\mu}$ = sum of the parameters for those treatments which are second associates of the first treatment.

R_0, R_1 and R_2 are similarly defined in terms of $\underline{Q}$. Using results (2.3) through (2.10), we can find a generalized inverse for C and P and use it for solving the equations (4.8) and (4.9) respectively.

4.3. A partially balanced weighing design based on two-association classes has been defined [11] as an arrangement of v objects in b blocks such that each object occurs in r blocks and each block is of size 2p and every block can be divided into two subblocks of size p such that

(i) any two objects which are first associates occur together in the same half block λ_{11} times and in different half blocks of the same block λ_{21} times,

(ii) any two objects which are second associates occur together in the same half block λ_{12} times and in different half blocks of the same block λ_{22} times.

Let $\underline{\mu}$ be the vector of parameters for the v objects and the model

$$E(\underline{y}) = N'\underline{\mu}, \tag{4.11}$$

where $\underline{y}$ is the vector of n random variables and $N' = ((n_{ij}))$ the design matrix.

$$(4.12) \quad \begin{aligned} n_{ij} &= 1 \text{ if the i-th object is in the first half of the j-th block,} \\ &= -1 \text{ if the i-th object is in the second half of the j-th block,} \\ &= 0 \text{ if the i-th object does not occur in the j-th block.} \end{aligned}$$

Then it is easy to verify that

$$NN' = rB_0 - \lambda_1 B_1 - \lambda_2 B_2 = B \text{ (say)} \tag{4.13}$$

where $\lambda_1 = \lambda_{21} - \lambda_{11}$, $\lambda_2 = \lambda_{22} - \lambda_{12}$ and $r = n_1\lambda_1 + n_2\lambda_2$. The normal equations are then,

$$B\underline{\mu} = N\underline{y}. \tag{4.14}$$

It is easy to verify that B will have three distinct characteristic roots 0, θ_1 and θ_2 if $\lambda_1 \neq \lambda_2$ and λ_1 and λ_2 are such that θ_1 and θ_2 are not zero. Thus using results of Section 2, we can find a generalized inverse B^- given by (2.3) and a Moore-Penrose inverse B^+ given by (2.8). We can use any one of B^- and B^+ to get a solution of the equations (4.14).

4.4. For the partially balanced weighing design of 4.3., sometimes one has restrictions on the parameters. The model is

$$(4.15) \qquad E(\underline{y}) = N'\underline{\mu} \quad \text{with the restrictions} \quad K'\underline{\mu} = \underline{m}.$$

In this situation, the equations to be solved [9] are

$$(4.16) \qquad \left[\begin{array}{c|c} NN' & K \\ \hline K' & 0 \end{array}\right] \left[\begin{array}{c} \underline{\mu} \\ \hline \underline{\Phi} \end{array}\right] = \left[\begin{array}{c} \underline{T} \\ \hline \underline{m} \end{array}\right].$$

Let F denote the matrix of coefficients in (4.16). Then if K belongs to the vectorspace generated by the column vectors of $NN' = B$, a generalized inverse of F is given by H as defined in (3.2) and (3.4).

If K is such that F is non-singular, its regular inverse is given [6] by

$$(4.17) \qquad F^{-1} = \left[\begin{array}{c|c} B^+ & H(K'H)^{-1} \\ \hline (H'K)^{-1}H' & 0 \end{array}\right]$$

where B^+ is the Moore-Penrose inverse of B and H is such that $H'B = 0$ and $H'K$ is non-singular. Such an H always exists iff F is non-singular [6].

Sommaire

On considère l'algèbre linéaire associative engendrée par les matrices d'association (v,v) $B_0, B_1, \ldots, B_m$ d'un schéma d'association à m classes. Cette algèbre est isomorphe à l'algèbre engendrée par les matrices (m+1, m+1) $P_0, P_1, \ldots, P_m$. Dans cet isomorphisme une matrice $B = \sum_{i=0}^{m} c_i B_i$ et son image $P = \sum_{i=0}^{m} c_i P_i$ ont le même polynôme minimal. A l'aide de celui-ci on obtient une inverse généralisée et l'inverse de Moore-Penrose de P et de B. On donne une formule pour le calcul d'une inverse généralisée d'une matrice symétrique partitionnée, non nécessairement non-négative. Enfin ces inverses généralisées sont utilisées pour résoudre les équations normales et pour faire l'estimation dans les plans d'expériences construits sur des schémas d'association à 2 classes.

Summary

In the regular representation of the linear association algebra generated by the $v \times v$ association matrices $B_0, B_1, \ldots, B_m$ of an m-class association scheme, in terms of the $(m+1) \times (m+1)$ matrices $P_0, P_1, \ldots, P_m$, a matrix $B = \sum_{i=0}^{m} c_i B_i$ and its image $P = \sum_{i=0}^{m} c_i P_i$ have the same minimal polynomial. A generalized inverse and the Moore-Penrose inverse of P and B have been derived using their minimal polynomial. An expression for a generalized inverse of a symmetric partitioned matrix, *not necessarily non-negative*, is given. Applications of these generalized inverses for solving normal equations and estimation in designs based on two-class association schemes are discussed.

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