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SOME ZERO-ONE LAWS FOR GAUSSIAN PROCESSES

by

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ABSTRACT

A number of interesting zero-one laws on path properties of Gaussian processes are derived by using a zero-one law for Gaussian processes which extends a result of G. Kallianpur and N. C. Jain.

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1. INTRODUCTION. A zero-one law for Gaussian processes was recently proven by G. Kallianpur [6] and extended by N. C. Jain [5]; namely for a Gaussian process of function space type satisfying certain conditions, every measurable subgroup of the function space has probability zero or one. It is pointed out in Theorem 1 of this paper that the assumptions under which this zero-one law is derived can be considerably relaxed. A number of interesting applications of this zero-one law are given in Theorem 2, where it is shown that with probability one or zero the paths of a Gaussian process have most of the important function properties.

The proof of the theorems are collected in Section 3. In Section 4 a simple proof is given for the expression of the Radon-Nikodym derivative of the translate of a Gaussian measure. This expression, which is needed in the proof of Theorem 1, is known but no straightforward proof seems to be available in the literature.

2. ZERO-ONE LAWS FOR GAUSSIAN PROCESSES. Let $\{\xi(t,\omega), t \in T\}$ be a real Gaussian process defined on the probability space $(\Omega, \mathcal{F}, P)$ with mean function zero and covariance function $R(t,s)$; T is any set and $\mathcal{F}$ is the smallest σ-algebra of subsets of Ω with respect to which the functions $\{\xi(t,\omega), t \in T\}$ are measurable. Denote by $\bar{\mathcal{F}}$ the completion of $\mathcal{F}$ with respect to P. It is the purpose of this paper to derive classes of $\bar{\mathcal{F}}$-measurable events whose P-probability is either zero or one.

Now let X be a set of real functions x on T which contains almost surely $[P]$ the paths of the Gaussian process $\xi(t,\omega)$. Denote by $\mathcal{U}(X)$ the σ-algebra of subsets of X generated by sets of the form

$\{x \in X: [x(t_1),\ldots,x(t_n)] \in B^n\}$ $(t_1,\ldots,t_n \in T$ and B^n is an n-dimensional Borel set). Then the transformation $\phi: (\Omega,F,P) \to (X,\mathcal{U}(X))$ defined by

$$\phi(\omega) = \xi(\cdot,\omega)$$

is measurable, and induces a probability measure μ on $(X,\mathcal{U}(X))$. The stochastic process $\{x(t), t \in T\}$ defined on the probability space $(X,\mathcal{U}(X),\mu)$ is clearly Gaussian with mean zero and covariance function $R(t,s)$. Denote by $\overline{\mathcal{U}}(X)$ the completion of $\mathcal{U}(X)$ with respect to μ and by $H(R)$ the reproducing kernel Hilbert space determined by the covariance R. The following conditions will be assumed:

(C1) X is a linear space of functions under the usual operation of addition of functions and multiplication by real scalars

(C2) $H(R) \subset X$

Both conditions (C1) and (C2) are easily seen to be satisfied in the following cases: (1) $X = R^T$, the set of all real functions on T; (2) T is a measurable set on the real line, $\xi(t,\omega)$ is a measurable process, and X is the set of all real measurable functions on T; (3) T is a compact metric space, the paths of ξ are a.s. [P] continuous on T, and $X = C(T)$, the set of all real continuous functions on T.

<u>THEOREM 1</u>. If conditions (C1) and (C2) are satisfied and G is a $\overline{\mathcal{U}}(X)$-measurable subgroup of X, then $\mu(G) = 0$ or 1.

This theorem generalizes a zero-one law proven in [6] for $\overline{\mathcal{U}}(X)$-measurable modules over the rationals and for $\mathcal{U}(X)$-measurable groups and extended to $\overline{\mathcal{U}}(X)$-measurable groups in [5]. In [6, 5] Theorem 1 is proven under the additional assumptions that T is a complete separable metric space and R is a continuous function on $T \times T$. This assumption on R, which is equivalent to the mean square continuity of ξ, is quite restrictive, while the assumption on T is satisfied in most applications. For $\overline{F}$-measurable events we have the following

COROLLARY 1. Let X be a set of real functions on T which contains the paths of $\xi(t,\omega)$ a.s. [P], and satisfies conditions (C1) and (C2). If $F \in \overline{F}$ is such that $F = \phi^{-1}(G)$, where G is a $\overline{U}(X)$-measurable subgroup of X, then $P(F) = 0$ or 1.

By applying Corollary 1 we obtain a number of zero-one laws for path properties of a Gaussian process.

THEOREM 2. Let $\{\xi(t,\omega), t \in T\}$ be real separable Gaussian process with zero mean on the probability space (Ω,F,P), where T is an interval on the real line. Then with probability zero or one the paths of $\xi(t,\omega)$ are

(1) bounded on T

(2) free of oscillatory discontinuities on T

(3) of bounded variation on every compact subinterval of T

(4) continuous on T

(5) uniformly continuous on T

(6) absolutely continuous on T

(7) (satisfy) a Lipshitz condition of order α, $0 < \alpha \leq 1$, on T

(8) differentiable on T.

The zero-one law on the boundedness of the paths is known [7]. An important property for which we have not been able to prove a zero-one law is the measurability of the paths. However, under the additional assumption that the process is product measurable, it is shown in [12] that its paths belong to L_p, $1 \leq p < \infty$, with probability zero or one and also necessary and sufficient conditions for the two alternatives are derived. Sharper results to the effect that with probability one the paths are either continuous or very irregular (the latter defined in a precise way) are known for mean square continuous stationary Gaussian processes [1] and for mean square continuous Gaussian processes [11].

In the course of the proof of Theorem 2 it is shown that for a separable process the sets of paths that have each of the stated properties are measurable, and also explicit expressions for these sets are given. This result is of independent interest. It should be remarked that for the measurability of the sets in (1), (2), (4) and (5) separability relative to closed intervals suffices, while for the measurability of the sets in (3) and (6) to (8) the separability relative to closed sets is needed. However, in the particular case of Gaussian processes, the measurability of the sets in (6) to (8) can be shown assuming only separability relative to closed intervals.

Additional zero-one laws can be obtained by using the techniques employed in the proof of Theorem 2. For instance, it is easily seen that with probability zero or one the paths of a separable, zero mean Gaussian process: (i) vanish at ∞ if $T = [a,\infty)$ (f vanishes at ∞ if for every $\epsilon > 0$ there exists $0 < N(\epsilon) < +\infty$ such that $|f(t)| < \epsilon$ whenever $t > N$); (ii) have uniform right (left) limits on T; (iii) are uniformly right (left) continuous on T; (iv) are uniformly right (left) differentiable on T. It should be noted that we have not been able to prove the more interesting zero-one laws that are obtained by deleting the word "uniformly" in (ii) to (iv).

The problem of finding necessary and sufficient conditions for the two alternatives in the zero-one laws of Theorem 2 is wide open at present and seemingly a difficult one. Sufficient conditions for a number of path properties to hold are known, especially for stationary Gaussian processes, and some are very close to being also necessary.

3. PROOFS. The notation and assumptions introduced in Section 2 are used here.

PROOF OF THEOREM 1. This theorem is proven in [6, 5] with the additional assumptions that T is a complete separable metric space and that $R(t,s)$ is a continuous function on $T \times T$. These assumptions are used in two places in the proof given in [6, 5].

First they are used in concluding the validity of (i) and (ii) as stated in Section 4; i.e., that $\mu_m \sim \mu_o$ $(= \mu)$ if and only if $m \in H(R)$ and that the Radon-Nikodym derivative of μ_m with respect to μ_o is given by (1). However, as explained in Section 4, (i) and (ii) are valid in general with no restrictive assumptions on either T or R.

Secondly, Lemma 6 as stated and proved in [6] employes the separability of $H(R)$, which is implied by the above mentioned assumptions on T and R. If the statement of this Lemma 6 of [6] is modified to read as follows: "If g is a $\mathcal{U}(X)$-measurable real function such that for every $x \in X$ and $m \in H(R)$, $g(x + m) = g(x)$ then $g(x) =$ constant a. e. $[\mu]$", then it is proven as Lemma 6 of [6], the separability of $H(R)$ is not needed, and this modified lemma is precisely what is needed in the course of the proof of this theorem.

By separability of the stochastic process $\{\xi(t,\omega), t \in T\}$, we mean separability relative to closed sets of the extended real line [2, p. 52], unless otherwise specified. This type of separability is equivalent to the following condition: "There exists a set $\Omega_o \in F$

with $P(\Omega_o) = 0$ and a countable dense subset S of T such that for every $\omega \in \Omega - \Omega_o$ and every $t \in T - S$ there exists a sequence $\{s_n\}_{n=1}^{\infty}$ in S converging to t such that $\lim_n \xi(s_n,\omega) = \xi(t,\omega)$" [2, p. 59]. This equivalent condition for separability will play an important role in some of the proofs. However, it will be clear from the following proofs that the weaker kind of separability relative to closed intervals of the extended real line is sufficient for some of the zero-one laws.

PROOF OF THEOREM 2. If F is the set of $\omega \in \Omega$ for which each property is satisfied we will show that $F \in \overline{F}$ and $P(F) = 0$ or 1.

Proof of (1). Let $F = \{\omega \in \Omega: \sup_{t\in T}|\xi(t,\omega)| < +\infty\}$, and

$$F' = \{\omega \in \Omega: \sup_{t\in S}|\xi(t,\omega)| < +\infty\}$$

$$= \bigcup_{N=1}^{\infty} \bigcap_{t\in S} \{\omega \in \Omega: |\xi(t,\omega)| \leq N\}.$$

Then $F' \in F$, and because of the separability of ξ, $F \in \overline{F}$ and $P(F) = P(F')$. Also if we take $X = R^T$ and define

$$G' = \{x \in X: \sup_{t\in S} |x(t)| < +\infty\}$$

$$= \bigcup_{N=1}^{\infty} \bigcap_{t\in S} \{x \in X: |x(t)| \leq N\}$$

then G' is clearly a $\mathcal{U}(X)$-measurable subgroup of X and $F' = \phi^{-1}(G')$. Corollary 1 implies that $P(F') = 0$ or 1. Note that separability relative to closed intervals only is used here.

Proof of (2). Let $F = \{\omega \in \Omega: \xi(t,\omega)$ is free of oscillatory discontinuities on $T\}$. It is shown in [9, p. 61] that $F \in F$. Similarly, if $X = R^T$ and $G = \{x \in X: x(t)$ is free of oscillatory discontinuities on $T\}$, then $G \in \mathcal{U}(X)$. Since G is a subgroup of X and $F = \phi^{-1}(G)$, Corollary 1 implies $P(F) = 0$ or 1.

Proof of (3). Let $\{T_k = [a_k,b_k]\}_{k=1}^{\infty}$ be a sequence of compact intervals such that $T_k \subset T_{k+1} \subset T$ and $\bigcup_{k=1}^{\infty} T_k = T$. If $F = \{\omega \in \Omega:$ $\xi(t,\omega)$ is of bounded variation on every compact subinterval of $T\}$, then

$$F = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{\Pi_n(t_i;T_k)} \{\omega \in \Omega: \sum_{i=1}^{n} |\xi(t_i,\omega) - \xi(t_{i-1},\omega)| \leq N\} ,$$

where $\Pi_n(t_i;T_k)$ is the class of all sets of points $\{t_i\}_{i=0}^{n}$ in T_k such that $a_k = t_o < t_1 < \cdots < t_n = b_k$. Let F' be given by the same expression as F with T_k replaced by $T_k \cap S$ (here the separating set S is augmented so as to include all end points $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$). Then $F' \in \mathcal{F}$ and $F' = \{\omega \in \Omega: \xi(t,\omega)$ is of bounded variation on every bounded subset of $S\}$. By the separability of ξ it is shown as in the Proof of (4) that $F \in \overline{\mathcal{F}}$ and $P(F) = P(F')$. If we take $X = R^T$ we have that $G' = \{x \in X: x(t)$ is of bounded variation on every bounded subset of $S\} \in \mathcal{U}(X)$ and since G' is a group and $F' = \phi^{-1}(G')$, Corollary 1 implies $P(F') = 0$ or 1.

Proof of (4). Let $\{T_k\}_{k=1}^{\infty}$ be a sequence of compact intervals such that $T_k \subset T_{k+1} \subset T$ and $\bigcup_{k=1}^{\infty} T_k = T$. Then, noting that a function is continuous on T if and only if it is uniformly continuous on every T_k, we have

$$F = \{\omega \in \Omega: \xi(t,\omega) \text{ is continuous on } T\} = \bigcap_{k=1}^{\infty} F_k$$

where

$$\begin{aligned} F_k &= \{\omega \in \Omega: \xi(t,\omega) \text{ is continuous on } T_k\} \\ &= \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{\omega \in \Omega: \sup_{\substack{t,t' \in T_k \\ |t-t'| \leq 1/m}} |\xi(t,\omega) - \xi(t',\omega)| \leq \tfrac{1}{n}\} \\ &= \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{t \in T_k} \{\omega \in \Omega: \sup_{\substack{t' \in T_k \\ |t-t'| \leq 1/m}} |\xi(t,\omega) - \xi(t',\omega)| \leq \tfrac{1}{n}\} \\ &= \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{t \in T_k} E^t_{n,m}(T_k), \end{aligned}$$

Let $F'_k = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{s \in S_k} E^s_{n,m}(S_k)$, where $S_k = T_k \cap S$, and let $F' = \bigcap_{k=1}^{\infty} F'_k = \{\omega \in \Omega: \xi(t,\omega)$ is continuous on $S\}$. Then clearly $F'_k \in \mathcal{F}$ for all k, and $F' \in \mathcal{F}$. If we take $X = R^T$ we obtain in a similar way that the set $G' = \{x \in X: x(t)$ is continuous on $S\}$ is $\mathcal{U}(X)$-measurable. Since G' is clearly a group and $F' = \phi^{-1}(G')$, Corollary 1 implies $P(F') = 0$ or 1. Clearly $F_k \subset F'_k$. It will be shown that separability implies $F'_k - F_k \subset \Omega_o$. Then $F_k \in \overline{\mathcal{F}}$ and $P(F_k) = P(F'_k)$ and thus $F \in \overline{\mathcal{F}}$ and $P(F) = P(F') = 0$ or 1.

In order to prove $F'_k - F_k \subset \Omega_o$, it suffices to show that $\omega \in F'_k - \Omega_o$ implies $\omega \in F_k$. Fix any $\omega \in F'_k - \Omega_o$. For every fixed n, since $\omega \in F'_k$, there exists m (depending on n) such that for every $s \in S_k$, $\omega \in E^s_{2n,m}(S_k)$; i.e. such that whenever

$$s,s' \in S_k \quad , \quad |s - s'| \le \frac{1}{m} \text{ then } |\xi(s,\omega) - \xi(s',\omega)| \le \frac{1}{2n} .$$

Let $t \in T_k - S_k$ be arbitrary but fixed. It follows by separability and the fact that $\omega \notin \Omega_o$ that there exists a sequence $\{s_i\}_i$ in S_k such that

$$\lim_i s_i = t \quad \text{and} \quad \lim_i \xi(s_i,\omega) = \xi(t,\omega)$$

Then there exists integer I such that whenever $i \ge I$ then

$$|t - s_i| \le \frac{1}{2m} \quad \text{and} \quad |\xi(t,\omega) - \xi(s_i,\omega)| \le \frac{1}{2n} .$$

If $s \in S_k$ with $|t - s| \le \frac{1}{2m}$, then, for $i \ge I$,

$$|s - s_i| \le |s - t| + |t - s_i| \le \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m} \quad \text{and}$$

$$|\xi(t,\omega) - \xi(s,\omega)| \le |\xi(t,\omega) - \xi(s_i,\omega)| + |\xi(s_i,\omega) - \xi(s,\omega)| \le \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} .$$

Thus whenever $s \in S_k$ with $|t - s| \le \frac{1}{2m}$ then $|\xi(t,\omega) - \xi(s,\omega)| \le \frac{1}{n}$, i.e., $\omega \in E^t_{n,2m}(S_k)$. Clearly $E^t_{n,2m}(T_k) \subset E^t_{n,2m}(S_k)$. Since for fixed $t \in T$ the process $\{\xi(t',\omega) - \xi(t,\omega), t' \in T\}$ is separable, it follows that $E^t_{n,2m}(S_k) - E^t_{n,2m}(T_k) \subset \Omega_o$. Hence $\omega \in E^t_{n,2m}(S_k)$, $\omega \notin \Omega_o$ imply $\omega \in E^t_{n,2m}(T_k)$. It is thus seen that for every n there exists m (depending on n) such that for all $t \in T$, $\omega \in E^t_{n,2m}(T_k)$. Hence $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{t\in T} E^t_{n,m}(T_k) = F_k$ and the proof is complete.

This proof is presented in some detail because it is typical of the use of the separability of the process in proving measurability of path sets and because a similar argument is used again in this paper. It should be remarked however that this proof uses separability relative to closed sets. As pointed out in the Proof of (5), this zero-one law on path continuity can be proven under the weaker kind of separability relative to closed intervals.

Proof of (5). Let $F = \{\omega \in \Omega\colon \xi(t,\omega)$ is uniformly continuous on $T\}$. It follows from the proof of (4), that $F \in \overline{F}$ and $P(F) = 0$ or 1. This proof uses separability of ξ relative to closed sets. We now give a proof employing the weaker kind of separability relative to closed intervals. It should be clear then, that under the same weaker separability assumptions, the sets F_k appearing in the Proof of (4) can be similarly shown to belong to $\overline{F}$, which proves the zero-one law (4). Let

$$F' = \{\omega \in \Omega\colon \xi(t,\omega) \text{ is uniformly continuous on } S\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{\substack{s,s'\in S \\ |s-s'|\le 1/m}} \{\omega \in \Omega\colon |\xi(s,\omega) - \xi(s',\omega)| \le \tfrac{1}{n}\}$$

Then $F \in \overline{F}$ and $F \subset F'$. Fix any $\omega \in F' - \Omega_o$. Then $\xi(\cdot,\omega)$ is uniformly continuous on S, and since S is dense in T, it can be uniquely extended to a uniformly continuous function $\hat{\xi}(\cdot,\omega)$ on T. Now let $t \in T - S$ be arbitrary but fixed. Then

$$\hat{\xi}(t,\omega) = \lim_{\substack{s\in S \\ s\to t}} \hat{\xi}(s,\omega) = \lim_{\substack{s\in S \\ s\to t}} \xi(s,\omega)$$

and the separability of ξ relative to closed intervals implies that $\lim_{\substack{s\in S \\ s\to t}} \xi(s,\omega) = \xi(t,\omega)$ [8, p. 505]. Hence $\xi(t,\omega) = \hat{\xi}(t,\omega)$, i.e., $\xi(t,\omega)$ is uniformly continuous and $\omega \in F$. It follows that $F' - \Omega_0 \subset F$ which implies $F \in \bar{F}$ and $P(F) = P(F')$. If we take $X = R^T$ we obtain in a similar way that the set $G' = \{x \in X: x(t)$ is uniformly continuous on $S\}$ is $\mathcal{U}(X)$-measurable. Since G' is clearly a group and $F' = \phi^{-1}(G')$, Corollary 1 implies $P(F') = 0$ or 1 which completes the proof.

<u>Proof of (6)</u>. Let

$$F = \{\omega \in \Omega: \xi(t,\omega) \text{ absolutely continuous on } T\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{\Pi_k^m(t_i,t_i';T)} \{\omega \in \Omega: \sum_{i=1}^{k} |\xi(t_i,\omega) - \xi(t_i',\omega)| \le \frac{1}{n}\}$$

where $\Pi_k^m(t_i,t_i';T)$ is the class of all sets of points $\{t_i,t_i'\}_{i=1}^k$ in T such that the intervals $\{(t_i,t_i')\}_{i=1}^k$ are disjoint and with total length $\le \frac{1}{m}$. Let also F' be the subset of Ω given by the same expression as F with T replaced by S. Then $F' \in F$ and clearly $F' = \{\omega \in \Omega: \xi(t,\omega)$ is absolutely continuous on $S\}$. By the separability of ξ, it is shown as in the Proof of (4) that $F \in \bar{F}$ and $P(F) = P(F')$.

This proves that $F \in \bar{F}$ for every process ξ separable relative to closed sets. For a Gaussian process ξ, by using the already established zero-one law (4), it can be shown that $F \in \bar{F}$ under the weaker condition of separability relative to closed intervals. (This applies also to (7) and (8)). Indeed if $P\{\omega \in \Omega: \xi(t,\omega)$ is continuous on $T\} = 0$ then F belongs to the completed σ-field $\bar{F}$, and $P(F) = 0$. On the other hand if $P\{\omega \in \Omega: \xi(t,\omega)$ is continuous on $T\} = 1$, then clearly ξ is separable relative to closed sets. Thus, since the zero-one law on the continuity of paths is proven under the hypothesis of separability relative to closed interval, the zero-one law of (6) is proven under the same hypothesis.

If we take $X = R^T$ we have that $G' = \{x \in X: x(t)$ is absolutely continuous on $S\} \in \mathcal{U}(X)$ and since G' is a group and $F' = \phi^{-1}(G')$, Corollary 1 implies $P(F') = 0$ or 1.

Proof of (7). This zero-one law is shown as (4) and (6) by noting that if $F = \{\omega \in \Omega: \xi(t,\omega)$ satisfies a Lipshitz condition of order α on $T\}$, for fixed α, $0 < \alpha \leq 1$, then

$$F = \{\omega \in \Omega: \sup_{\substack{t,s \in T \\ t \neq s}} \frac{|\xi(t,\omega) - \xi(s,\omega)|}{|t - s|^\alpha} < +\infty\}$$

$$= \bigcup_{N=1}^{\infty} \bigcap_{t \in T} \{\omega \in \Omega: \sup_{t \neq s \in T} \frac{|\xi(t,\omega) - \xi(s,\omega)|}{|t - s|^\alpha} \leq N\}.$$

Proof of (8). This zero-one law is shown as (4) by noting that

$$F = \{\omega \in \Omega: \xi(t,\omega) \text{ is differentiable on } T\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{\substack{s,t,\tau \in T \\ s<t<\tau \\ |s-\tau| \leq 1/m}} \{\omega \in \Omega: \max\{\Delta_1,\Delta_2,\Delta_3\} \leq \frac{1}{n}\}$$

where $\Delta_1 = \left|\frac{\xi(s,\omega) - \xi(t,\omega)}{s - t} - \frac{\xi(s,\omega) - \xi(\tau,\omega)}{s - \tau}\right|$,

$\Delta_2 = \left|\frac{\xi(\tau,\omega) - \xi(t,\omega)}{\tau - t} - \frac{\xi(\tau,\omega) - \xi(s,\omega)}{\tau - s}\right|$, $\Delta_3 = \left|\frac{\xi(t,\omega) - \xi(s,\omega)}{t - s} - \frac{\xi(t,\omega) - \xi(\tau,\omega)}{t - \tau}\right|$.

4. THE RADON-NIKODYM DERIVATIVE OF THE TRANSLATE OF A GAUSSIAN MEASURE.

In this section we discuss the properties of the translates of Gaussian measures used in the proof of Theorem 1.

The notation and assumptions of Section 2 prior to Theorem 1 are adopted here without further explanation. Let us put $P_0 = P$ and

$\mu_o = \mu$ and denote by L_2 the subspace of $L_2(\mu_o) = L_2(X,\mathcal{U}(X),\mu_o)$ spanned by the random variables $\{x(t), t \in T\}$. It is well known [10] that there is an inner product preserving isomorphism between L_2 and $H(R)$, denoted by $\leftrightarrow$, such that $x(t) \leftrightarrow R(\cdot,t)$ for all $t \in T$, and $\eta \leftrightarrow m$ if and only if

$$m(t) = \int_X x(t)\eta(x)d\mu_o(x) \qquad \text{for all } t \in T.$$

For $m \in X$ the transformation $\tau_m: (X,\mathcal{U}(X),\mu_o) \to (X,\mathcal{U}(X))$ defined by $\tau_m(x) = x + m$ is clearly measurable and induces a probability measure μ_m on $(X,\mathcal{U}(X))$. The stochastic process $\{x(t), t \in T\}$ on the probability space $(X,\mathcal{U}(X),\mu_m)$ is easily seen to be Gaussian with mean function $m(t)$ and covariance function $R(t,s)$. It is also clear that μ_m is the probability measure induced on $(X,\mathcal{U}(X))$ by the stochastic process $\{\xi(t,\omega) + m(t), t \in T\}$ by means of the measurable transformation $\phi_m: (\Omega,F,P_o) \to (X,\mathcal{U}(X))$ defined by $\phi_m(\omega) = \xi(\cdot,\omega) + m(\cdot)$. It is known that

(i) μ_m and μ_o are mutually absolutely continuous if and only if $m \in H(R)$, and

(ii) if $m \in H(R)$, the Radon-Nikodym derivative p_m of μ_m with respect to μ_o is given by

(1) $$p_m(x) = \exp[\eta(x) - \tfrac{1}{2}||\eta||^2] \qquad \text{a.e. } [\mu_o]$$

where $\eta \leftrightarrow m$ and $||\cdot||$ denotes the norm of $L_2(\mu_o)$.

The validity of (i) is established in [4]; it can also be obtained as a straightforward application of the theorem proven in [3]. The validity of (ii) is indicated in [13]. However, since the proof of (ii) is not easily reconstructed from the references cited in [13], an alternative simple proof is given here.

For any subset S of T let $L_2(S)$ be the subspace of $L_2(\mu_o)$ spanned by the set $\{x(t), t \in S\}$, let $H(R,S)$ be the reproducing kernel Hilbert space of R restricted to $S \times S$, and let $U(X,S)$ be the σ-algebra of subsets of X with respect to which the functions $\{x(t), t \in S\}$ are measurable. Clearly $L_2(T) = L_2$, $H(R,T) = H(R)$ and $U(X,T) = U(X)$. It is well known [10] that there is an inner product preserving isomorphism between $L_2(S)$ and $H(R,S)$, denoted by $\leftrightarrow$, such that $x(t) \leftrightarrow R(\cdot,t)$, $t \in S$, and $\zeta \leftrightarrow f$ if and only if $f(t) = \int_X x(t)\zeta(x)d\mu_o(x)$ for all $t \in S$.

Assume now $m \in H(R)$. Then $p_m(x)$ is $U(X,T)$-measurable and thus there exists a countable subset S_1 of T such that $p_m(x)$ is $U(X,S_1)$-measurable [2, p. 604]. Let $\eta \in L_2$ be such that $\eta \leftrightarrow m$. Then η is the $L_2(\mu_o)$-limit of a sequence $\{\eta_k\}_{k=1}^{\infty}$ of elements of $L_2(\mu_o)$ of the form $\eta_k(x) = \sum_{n=1}^{N_k} c_{k,n} x(t_{k,n})$, where the $c_{k,n}$'s are real numbers and $t_{k,n} \in T$. Hence there exists a countable subset $S_2 (= \bigcup_{1\le k} \bigcup_{1\le n\le N_k} \{t_{k,n}\})$ of T such that $\eta \in L_2(S_2)$. If $S' = S_1 \cup S_2$ then clearly $\eta \in L_2(S')$ and $p_m(x)$ is $U(X,S')$-measurable. Let $S' = \{t_k\}_{k=1}^{\infty}$, $t_k \in T$, and let S be the set of those points t_k of S' which are such that the elements $x(t_1),\dots,x(t_k)$ of $L_2(P_o)$ are linearly independent. Then S is a countable subset of T, $L_2(S) = L_2(S')$ and $\overline{U}(X,S) = \overline{U}(X,S')$, where $\overline{U}(X,S)$ is the completion of $U(X,S)$ with respect to μ_o. It follows that $\eta \in L_2(S)$ and that $p_m(x)$ is $\overline{U}(X,S)$-measurable, which implies that $p_m(x)$ is also the Radon-Nikodym derivative of μ_m restricted to $\overline{U}(X,S)$ with respect to μ_o restricted to $\overline{U}(X,S)$. Now an application of the martingale convergence theorem [2] to the martingale $\{p_m^n(x), \overline{U}(X,S_n), 1 \le n \le \infty\}$, where $S_\infty = S = \{s_k\}_{k=1}^{\infty}$, $p_m^{\infty}(x) = p_m(x)$, and for $1 \leq n < \infty$, $S_n = \{s_k\}_{k=1}^{n}$ and $p_m^n(x)$ is the Radon-Nikodym derivative of μ_m restricted to $\overline{\mathcal{U}}(X,S_n)$ with respect to μ_o restricted to $\overline{\mathcal{U}}(X,S_n)$ (or equivalently an application of Theorem 9.A of [10] to the stochastic process $\{x(t), t \in S\}$) gives

(2) $$p_m(x) = \exp[\zeta(x) - \tfrac{1}{2}||\zeta||^2] \quad \text{a.e. } [\mu_o]$$

where $\zeta \in L_2(S)$ and $\zeta \leftrightarrow m_S$; m_S being the restriction of m to S. $\zeta \leftrightarrow m_S$ implies

$$m(t) = \int_X x(t)\zeta(x)d\mu_o(x) \qquad \text{for all} \quad t \in S.$$

On the other hand, it follows from $L_2 \ni \eta \leftrightarrow m \in H(R)$ that

$$m(t) = \int_X x(t)\eta(x)d\mu_o(x) \qquad \text{for all} \quad t \in T$$

It follows now from η, $\zeta \in L_2(S)$ that $\zeta = \eta$ in $L_2(\mu_o)$, and hence (2) implies (1).

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