[1] University of North Carolina at Chapel Hill and University of New Mexico. This research was supported in part by the Office of Naval Research under Contract N00014-67-A-0321-0002.

[2] University of North Carolina at Chapel Hill. This research was supported in part by the U.S. Air Force Office of Scientific Research under Contract AFOSR-68-1415.

[3] University of North Carolina at Chapel Hill and Kyushu University (Japan). This research was supported in part by the National Science Foundation under Grant GU-2059.

A NOTE ON A 0-1 LAW FOR STATIONARY GAUSSIAN PROCESSES

by

Clifford Qualls[1], Gordon Simons[2] *and* Hisao Watanabe[3]
Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 798

January, 1972

A NOTE ON A 0-1 LAW FOR STATIONARY GAUSSIAN PROCESSES

by

Clifford Qualls[1], Gordon Simons[2] *and* Hisao Watanabe[3]

1. Summary and Introduction. Let $\{X(t), t \in T\}$ be a real stationary Gaussian process with continuous sample functions and $T = (0,\infty)$ or with $T = N^+$, the set of positive integers. Also assume that its covariance function $r(t)$ satisfies $r(0) = 1$ and that $EX(t) \equiv 0$. Let $f(t)$ be an arbitrary non-decreasing positive function on some time interval $[a,\infty)$. For $T = N^+$ we define the event A by

$$A = [X(n) > f(n) \text{ infinitely often}].$$

In Section 2, we show that the event A has 0 or 1 probability, provided only that the covariance function satisfies the mixing condition $r(n) \to 0$ as $n \to \infty$. Under a stronger mixing condition, we also give the 0-1 law of this type (a type of the law of the iterated logarithm) containing a test to decide between 0 and 1 for each function f. The proof of this

---

[1] University of North Carolina at Chapel Hill and University of New Mexico. This research was supported in part by the Office of Naval Research under Contract N00014-67-A-0321-0002.

[2] University of North Carolina at Chapel Hill. This research was supported in part by the U.S. Air Force Office of Scientific Research under Contract AFOSR-68-1415.

[3] University of North Carolina at Chapel Hill and Kyushu University (Japan). This research was supported in part by the National Science Foundation under Grant GU-2059.

discrete time version 0-1 law with test is similar to and easier than the results in Watanabe [6] and Qualls and Watanabe [3], [4] for continuous times.

In Section 3 for $T = (0,\infty)$, we note that the event $B \equiv [X(t) > f(t)$ on some sequence of t_n's $\to \infty]$ also has 0 or 1 probability provided only that $r(t) \to 0$ as $t \to \infty$. An interesting difference between the discrete and continuous time cases is discussed in §3.

We note here that some of the results stated in the following sections for stationary processes can be extended easily to certain non-stationary processes. For a change of time $\phi: t \to \phi(t)$ where ϕ is a continuous function which is strictly increasing to ∞ as $t \to \infty$, the non-stationary process $Y(t) = X(\phi(t))$ also satisfies the 0-1 law:

$$P\{Y(t) \geq f(t) \text{ on some sequence of } t_n\text{'s} \to \infty\} = 0 \text{ or } 1.$$

2. Discrete Parameter Case.

THEOREM 1. *If* $r(n) \to 0$ *as* $n \to \infty$ *then* $P(A) = 0$ *or* 1.

PROOF. Since A is an event in the σ-field generated by $X(1), X(2), \ldots$ there exists a sequence of events $\{A_k\}$, the k-th belonging to the σ-field generated by $X(1), \ldots, X(k)$, with

$$(1) \qquad \mathcal{E}_k \equiv P(A \,\Delta\, A_k) \to 0 \quad \text{as} \quad k \to \infty$$

(cf. Halmos [2], Theorem D, page 56). Of course, the event A_k can be expressed as $[(X(1),\ldots,X(k)) \in F_k]$ for some Borel set F_k of the k-dimensional Euclidean space.

For $m \geq 0$, let

$$\begin{aligned} A^{(m)} &\equiv [X(m+n) > f(n) \text{ for infinitely many } n] \\ &= [X(n) > f(n-m) \text{ for infinitely many } n \geq m] \end{aligned}$$

and

$$A_k^{(m)} \equiv [(X(m+1),\ldots,X(m+k)) \in F_k].$$

Because of stationarity, we obtain for each m and k,

(2) $$P(A^{(m)}) = P(A),\ P(A_k^{(m)}) = P(A_k) \text{ and } \mathcal{E}_k = P(A^{(m)} \,\Delta\, A_k^{(m)}).$$

Since f is monotone, $A \subset A^{(m)}$ and, hence,

(3) $$P(A \,\Delta\, A^{(m)}) = 0 \text{ for all } m.$$

Set $D_{mk} \equiv |P(A_k A_k^{(m)}) - P(A_k)P(A_k^{(m)})|$. From (1), (2) and (3), we obtain

$$\begin{aligned}
|P(A) - P^2(A_k)| &= |P(A) - P(A_k)P(A_k^{(m)})| \\
&\le D_{mk} + |P(A) - P(A_k A_k^{(m)})| \\
&\le D_{mk} + P(A \,\Delta\, A_k) + P(A \,\Delta\, A_k^{(m)}) \\
&\le D_{mk} + \mathcal{E}_k + P(A \,\Delta\, A^{(m)}) + P(A^{(m)} \,\Delta\, A_k^{(m)}) \\
&\le D_{mk} + 2\mathcal{E}_k.
\end{aligned}$$

But the assumption $\lim_{n\to\infty} r(n) = 0$ implies $\lim_{m\to\infty} D_{mk} = 0$ (as we shall indicate below) and, consequently,

(4) $$|P(A) - P^2(A_k)| \le 2\mathcal{E}_k \quad \text{for each } k.$$

This can be seen by writing $P(A_k A_k^{(m)})$ and $P(A_k)P(A_k^{(m)})$ (when $m \ge k$) as 2k-fold integrals (over $F_k \times F_k$) and observing that the integrand of the first integral, a multivariate normal density, converges, as $m \to \infty$, to the integrand of the second integral, also a multivariate normal density (which does *not* depend on m). The convergence of the first integral to the second then follows from a version of Scheffe's theorem (cf., [1], page 224).

Finally, using (1) and letting $k \to \infty$ in (4), we obtain $P(A) - P^2(A) = 0$ and, hence, $P(A) = 0$ or 1 as claimed.

Remark. An ergodic theoretic interpretation of this proof is possible. One considers the probability space induced by the random vector $(X(1), X(2),\ldots)$ and lets T be the "shift transformation" which maps the "point" $(X(1), X(2),\ldots)$ into the point $(X(2), X(3),\ldots)$. The stationary assumption makes T a "measure preserving transformation". The events A, A_k, $A^{(m)}$ and $A_k^{(m)}$ can be readily interpreted in this new framework and one finds that $A^{(m)} = T^{-m}A$, $A_k^{(m)} = T^{-m}A_k$. With essentially the same argument as that used to show $\lim_{m\to\infty} D_{mk} = 0$, one can verify that T is "mixing". From this, it follows that T is "ergodic". (The reader is referred to Rényi [5], page 144, for a proof of this fact. The nearby text defines the concepts used in this remark.) Finally, since A is almost surely equivalent to an "invariant set" (i.e., $P(A\Delta T^{-1}A) = 0$) it follows that $P(A) = 0$ or 1.

THEOREM 2.

(i) *If* $I(f) \equiv \sum_{k=k_0}^{\infty} (f(k))^{-1} \exp(-f^2(k)/2) < \infty$, *then* $P(A) = 0$.

(ii) *If* $I(f) = \infty$ *and* $r(n) = O(n^{-\gamma})$ *as* $n \to \infty$, *for some* $\gamma > 0$, *then* $P(A) = 1$.

PROOF. Denote $X_k \equiv X(k)$ and $f_k \equiv f(k)$.

Part (i). Since $P(X_k > x) \le (2\pi)^{-\frac{1}{2}} x^{-1} \exp(-x^2/2)$ for any $x > 0$, the assumption $I(f) < \infty$ makes $\sum_{k=1}^{\infty} P(X_k > f_k) < \infty$. Then the Borel-Cantelli lemma yields $P(A) = 0$.

Before beginning the proof of part (ii), we introduce the following lemmas.

LEMMA 1. *If (ii) is true with the additional restriction* $f_k^2 \ge 2 \log k - 2 \log\log k$ *for all* $k \ge$ some k_0, *then it is true without the restriction.*

PROOF. Set $\hat{f}_k \equiv \max(f_k,u_k)$ where $u_k = (2 \log k - 2 \log\log k)^{\frac{1}{2}}$ $(k>1)$. If the additional restriction does *not* hold, then $\hat{f}_k = u_k$ infinitely often. Since the event $[X_k > \hat{f}_k$ infinitely often$] \subset A$, it suffices to show $I(\hat{f}) = \infty$. But letting $k \to \infty$ along a subsequence for which $\hat{f}_k = u_k$, we find that

$$I(\hat{f}) = \sum_{j=k_0}^{\infty} \hat{f}_j^{-1} \exp(-\hat{f}_j^{\,2}/2) \geq \sum_{j=k_0}^{k} \hat{f}_j^{-1} \exp(-\hat{f}_j^{\,2}/2)$$

$$\geq (k-k_0)\, u_k^{-1} \exp(-u_k^{\,2}/2)$$

$$= \frac{(k-k_0) \log k}{(2 \log k - 2 \log\log k)^{\frac{1}{2}} \cdot k} \to \infty.$$

That is, $I(\hat{f}) = \infty$.

LEMMA 2. *Let* $\varepsilon > 0$ *be given. If (ii) is true with the additional restriction* $|r(n)| < \varepsilon$ *for* $n \geq 1$, *then it is true without the restriction.*

PROOF. Suppose the restriction does not hold. Since $r(n) = O(n^{-\gamma})$ we have for some m, $|r(n)| < \varepsilon$ whenever $n \geq m$. Now $\hat{X}_n \equiv X_{mn}$ $(n \geq 1)$ is a stationary Gaussian sequence with covariance function $\hat{r}(n) \equiv r(mn)$ satisfying $\hat{r}(n) = O(n^{-\gamma})$ and $|\hat{r}(n)| < \varepsilon$. Let $\hat{f}_n \equiv f_{mn}$. Since $[\hat{X}_n > \hat{f}_n$ infinitely often$] \subset A$, it suffices to show that $I(f) = \infty$ entails $I(\hat{f}) = \infty$. But this follows from the monotonicity of f_n.

LEMMA 3. *Defining* $E_k = [X_k \leq f_k]$, *we have (for* $1 \leq m < n$*)*

$$\left| P\left(\bigcap_m^n E_k\right) - \prod_m^n P(E_k) \right| \leq \sum_{m \leq i < j \leq n} |r(j-i)| \int_0^1 \phi(f_i, f_j;\, \lambda r(j-i))\, d\lambda$$

where $\phi(x,y;\,\lambda r)$ *denotes the standard normal bivariate density with correlation coefficient* λr.

PROOF. This type of lemma appears in many proofs of asymptotic independence for crossing problems. Lemma 3 is a special case of Lemma 1.5 in Qualls and Watanabe [3].

Proof of Part (ii). The assumption $I(f) = \infty$ implies $\sum_1^\infty P(E_k{}^c) = \infty$ and, hence, $\Pi_1^\infty P(E_k) = 0$. In turn, we have

$$(5) \qquad 1 - P(A) = \lim_{m\to\infty} P\left(\bigcap_m^\infty E_k\right)$$
$$= \lim_{m\to\infty} \prod_m^\infty P(E_k) + \lim_{m\to\infty}\left\{P\left(\bigcap_m^\infty E_k\right) - \prod_m^\infty P(E_k)\right\}$$
$$= \lim_{m\to\infty}\left\{P\left(\bigcap_m^\infty E_k\right) - \prod_m^\infty P(E_k)\right\}.$$

In view of Lemma 3, we may conclude $P(A) = 1$ providing

$$D \equiv \sum_{1\le i<j<\infty}\sum |r(j-i)| \int_0^1 \phi(f_i, f_j;\ \lambda r(j-i))\, d\lambda < \infty.$$

Now assume the restrictions in Lemmas 1 and 2 hold with $\varepsilon = \min(1/4, \gamma/5)$. For $|r| < \varepsilon$,

$$\phi(f_i, f_j;\ \lambda r) \le (2\pi)^{-1}(1-\varepsilon^2)^{-\frac{1}{2}} \exp\left\{-\frac{f_i{}^2 - 2|r|f_i f_j + f_j{}^2}{2}\right\}$$
$$\le \text{constant} \cdot \exp\{-(1-|r|)(f_i{}^2+f_j{}^2)/2\}$$
$$\le \text{constant} \cdot \exp\{-(1-\varepsilon)(f_i{}^2+f_j{}^2)/2\}$$
$$\le \text{constant} \cdot \left[\left(\frac{\log i}{i}\right)\left(\frac{\log j}{j}\right)\right]^{1-\varepsilon}.$$

Then

$$D \le \text{constant} \cdot \sum_{1\le i<j<\infty}\sum (j-i)^{-\gamma}\left[\left(\frac{\log i}{i}\right)\left(\frac{\log j}{j}\right)\right]^{1-\varepsilon}$$
$$\le \text{constant} \cdot \sum_{i=1}^\infty\sum_{k=1}^\infty k^{-5\varepsilon}\left[\left(\frac{\log i}{i}\right)\left(\frac{\log(i+k)}{i+k}\right)\right]^{1-\varepsilon}$$
$$\le \text{constant} \cdot \sum_{i=1}^\infty\sum_{k=1}^\infty \frac{[\log(i+k)]^{2-2\varepsilon}}{(i+k)^\varepsilon}\left(\frac{1}{i}\right)^{1+\varepsilon}\left(\frac{1}{k}\right)^{1+\varepsilon} < \infty.$$

3. Continuous Parameter Case. The continuous time version of Theorem 1 depends on the sample path continuity of $X(t)$. Specifically, the values of

$X(t)$ on $(0,\infty)$ are determined by the values on the diadic rationals $\{k/2^n,\ k \text{ and } n \in N^+\}$ and hence the event B is a member of the σ-field $\mathcal{F}$ generated by $\{X(t),\ t = k/2^n,\ k \text{ and } n \in N^+\}$. Let $\mathcal{F}_n$ be the σ-field generated by $\{X(t),\ t = k/2^n,\ k = 1,\ldots,n2^n\}$ $(n \geq 1)$. The sequence $\mathcal{F}_n$ is non-decreasing and $\mathcal{F}$ is the smallest σ-field containing the field $\bigcup_1^\infty \mathcal{F}_n$. Using Halmos' result as we did in proving Theorem 1, we may obtain events $B_k \in \mathcal{F}_k$ for which $\mathcal{E}_k \equiv P(B \,\Delta\, B_k) \to 0$ as $k \to \infty$. Associated with each event B_k is a set F_k in $k2^k$-dimensional Euclidean space with $B_k = \{(X(1/2^k),\ \ldots,\ X(k2^k/2^k)) \in F_k\}$. By imitating the remaining steps in the proof of Theorem 1, we obtain:

THEOREM 3. *If* $r(t) \to 0$ *as* $t \to \infty$, *then* $P(B) = 0$ *or* 1.

Remark. However, the similarity of Theorem 3 to Theorem 1 does not extend to the 0-1 law *with test*. For example, in [3], the continuous time version of our present Theorem 2 not only assumes the mixing condition $r(t) = O(t^{-\gamma})$ as $t \to \infty$ for some $\gamma > 0$ but also a local condition:

$$r(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0, \quad \text{where} \quad 0 < \alpha \leq 2 \quad \text{and} \quad C > 0.$$

There the test is based on whether $J(f) \equiv \int_a^\infty \{f(t)\}^{2/\alpha-1} \exp(-f^2(t)/2)\, dt$ is finite or infinite. (Reference [4] gives a test for a somewhat more general local condition.) Since $x^{-1} \exp(-x^2/2)$ is a monotone function of x, the sum $I(f)$ in Theorem 2 may be replaced by the integral $\int_a^\infty \{f(t)\}^{-1} \exp(-f^2(t)/2)\, dt$, which is less than the integral $J(f)$ for the continuous time case (when a is chosen sufficiently large). In fact, there are functions $f(t)$ such that $P(B) = 1$ and $P(A) = 0$.

References

[1] Billingsley, P. *Convergence of Probability Measures.* Wiley, New York, 1968.

[2] Halmos, P.R. *Measure Theory.* D. Van Nostrand, Princeton, 1950.

[3] Qualls, C. and Watanabe, H. "An asymptotic 0-1 behavior of Gaussian processes." *Ann. Math. Statist. 42* (1971), 2029-2035.

[4] Qualls, C. and Watanabe, H. "Asymptotic properties of Gaussian processes." *University of North Carolina Institute of Statistics Mimeo Series No. 736*, Chapel Hill (1971).

[5] Rényi, A. *Foundation of Probability.* Holden-Day, San Francisco, 1970.

[6] Watanabe, H. "An asymptotic property of Gaussian processes." *Trans. AMS 148* (1970), 233-248.