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In this note a model for shot noise generated by a semi-Markov process is developed. The moments of the shot noise process are found, and some applications of this model are briefly indicated.

SECTION 1: A General Shot Noise Model

Before developing the shot noise model, we introduce and define some notation for semi-Markov processes. In semi-Markov processes, S-MP, as in Markov processes, each jump is a regeneration point eliminating the influence of past events. However, in the S-MP the distribution of times between jumps is arbitrary, whereas in the Markov process the sojourn time in any state is exponentially distributed. Let t_n be the time of the n-th transition, $n = 0,1,2,\ldots$. Throughout this paper, we let $t_0 = 0$. Let X_n be the value of the S-MP after the n-th transition. The process X_n is a homogeneous Markov chain. The S-MP, $X(t)$, is completely defined by a set of defective probability distributions, G_{ij}; $G_{ij}(\infty) \leq 1$ and $G_{ij}(0) = 0$. $G_{ij}(t)$ is the probability that the sojourn time in state i has duration $\leq t$ and ends with a jump to state j,

$$G_{ij}(t) = \Pr(X_{n+1}=j,\ t_{n+1}-t_n \leq t \mid X_n=i).$$

In this paper, we will assume that the S-MP, $X(t)$, has a finite number of

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states i, $i = 0,1,2,\ldots,N$, that

(1) $$S_i(t) = \sum_{j=0}^{N} G_{ij}(t)$$

is not a lattice distribution function, and that

(2) $$\alpha_i = \int_0^\infty t dS_i(t) < \infty.$$

Let P denote the one-step transition matrix for the embedded Markov chain X_n, $P = [G_{ij}(\infty)]$. We assume that the matrix P is irreducible and has the stationary probability vector $\pi = (\pi_0, \pi_1, \ldots, \pi_N)$,

(3) $$\pi = \pi P.$$

The shot noise process, $Y(t)$, is then defined by

(4) $$Y(t) = \sum_{0<t_n\leq t} f(t-t'_n, X_n, W_n)$$

where $f(t,j,w)$ is a known function and $\{W_n\}$ is a sequence of independent random variables with distribution function

(5) $$H(x) = \Pr(W_n \leq x).$$

The shape of the n-th pulse which is initiated at time t_n depends on the state of the S-MP, X_n, and the random variable W_n. The model presented here is a straightforward generalization of a shot noise process generated by a renewal process investigated by Takács (1956).

We define the distribution function for the process $Y(t)$,

$$F_j(t,y) = \Pr(Y(t) \leq y | X_0 = j),$$

and let

(6) $$\Phi_j(t,\omega) = \int_{-\infty}^{+\infty} e^{i\omega x} d_x F_j(t,x)$$

denote its characteristic function. Let

(7) $$\Gamma_j(t,\omega) = \int_{-\infty}^{+\infty} e^{i\omega f(t,j,x)}\, dH(x)$$

denote the characteristic function of a single impulse.

THEOREM 1: The functions $\Phi_i(t,\omega)$ satisfy the system of linear integral equations

(8) $$\Phi_i(t,\omega) = \sum_{j=0}^{N} \int_0^t \Gamma_j(t-t',\omega)\, \Phi_j(t-t',\omega)\, dG_{ij}(t') + 1 - \sum_{j=0}^{N} G_{ij}(t).$$

PROOF: $Y(t) = 0$ if $t_1 > t$ and if $t_1 < t$ then $Y(t)$ is the sum of two random variables, $f(t-t_1,W_1)$ and $Y(t-t_1)$. If $\Phi_i(t,\omega|x,j)$ denotes the characteristic function of the conditional distribution of $Y(t)$, given that the first transition occurs at time $t_1 = x$ and is a transition to the state j, then

$$\Phi_i(t,\omega|x,j) = \begin{cases} \Phi_j(t-x,\omega)\Gamma_j(t-x,\omega) & x \le t \\ 1 & x > t. \end{cases}$$

From this expression, we obtain the unconditional characteristic function given in equation (8).

The general solution for this system of integral equations is not known; however, one can obtain from (8) equations for the moments of the $Y(t)$ process which can be solved. Let

(9) $$M_n(t;j) = E[Y^n(t)|X_0 = j]$$

and let

(10) $$M_n = \lim_{t \to \infty} M_n(t;j)$$

denote the moments of the stationary process if it exists. Further let

(11) $$\lambda_n(t,j) = E[f^n(t,j,W)].$$

THEOREM 2: The first two moments of the $Y(t)$ process are

(12) $$M_1(t;j) = \int_0^t \sum_{i=0}^{N} \lambda_1(t-t',i)\, dm_{ji}(t')$$

and

(13) $$M_2(t;j) = \int_0^t \sum_{i=0}^{N} \left\{2M_1(t-t';i)\lambda_1(t-t',i) + \lambda_2(t-t',i)\right\} dm_{ji}(t'),$$

provided that the integrals exist, where

(14) $$m_{ij}(t) = \sum_{n=1}^{\infty} G^{(n)}{}_{ij}(t),$$

and $G^{(n)}{}_{ij}$ is the n-fold convolution

$$G^{(n)}{}_{ij}(t) = \int_0^t \sum_{k=0}^{N} G^{(n-1)}{}_{kj}(t-t')\, dG_{ik}(t').$$

If as $t \to \infty$ $t\lambda_1(t,i) \to 0$ and $t\lambda_2(t,i) \to 0$ for $i = 1,2,\ldots,N$, then the first and second moments of the stationary process are

(15) $$M_1 = \frac{1}{\alpha}\int_0^{\infty} \sum_{i=0}^{N} \pi_i\, \lambda_1(t,i)\, dt$$

and

(16) $$M_2 = \frac{1}{\alpha}\int_0^{\infty} \sum_{i=0}^{N} \pi_i\left\{2M_1(t;i)\lambda_1(t,i) + \lambda_2(t,i)\right\} dt,$$

provided that the integrals exist, where

$$\alpha = \sum_{i=0}^{N} \pi_i\alpha_i;$$

π_i and α_i are defined in (2) and (3).

PROOF: Differentiating (8) with respect to ω and setting $\omega = 0$ and multiplying by $-i$ we obtain

$$(17)\qquad M_1(t;j) = \int_0^t \sum_{k=0}^{N} (\lambda_1(t-t',k)+M_1(t-t';k))\, dG_{jk}(t'),\ j = 0,1,2,..,N.$$

This system of Volterra integral equations (Tricomi (1957) p. 40) has a solution given by (12). One obtains the equations for the second moments by differentiating (8) twice with respect to ω and solving the corresponding system of integral equations.

To show that (15) holds, we first write

$$(18)\qquad \lim_{t\to\infty} M_1(t;j) = \lim_{t\to\infty} \int_0^{t/2} \sum_{i=0}^{N} \lambda_1(t-t',i)\, dm_{ji}(t') + \lim_{t\to\infty} \int_{t/2}^{t} \sum_{i=0}^{N} \lambda_1(t-t',i)\, dm_{ji}(t').$$

A well-known result for Markov renewal processes, Pyke (1961), states that

$$(19)\qquad \lim_{t\to\infty} (m_{ji}(t+h)-m_{ji}(t)) = \frac{\pi_i h}{\alpha}.$$

Under the conditions of Theorem 2, the first integral on the right-hand side of (18) goes to 0 as $t \to \infty$, and using the result given in (19) as $t \to \infty$ the second term goes to (15). The proof for the second moment follows by the same general argument.

The first moments for the stationary process are obtained directly from equation (12). However, to obtain an explicit solution for the second moment M_2, we will consider the special case

$$(20)\qquad f(t,j,w) = wa_j \exp(-\gamma_j t).$$

If $E[W] = \mu < \infty$ and $Var[W] = \sigma^2 < \infty$, then

$$(21)\qquad \lambda_1(t,j) = \mu a_j \exp(-\gamma_j t)$$

and

(22) $\lambda_2(t,j) = (\mu^2+\sigma^2)\, a_j^{\,2} \exp(-2\gamma_j t).$

The exponential form of λ_1 and λ_2 enables us to take advantage of the relationship between the convolution operator and the Laplace-Stieltjes transforms. Let $G^*_{ij}(s)$ and $m_{ij}(s)$ denote the Laplace-Stieltjes transform of $G_{ij}(t)$ and $m_{ij}(t)$ respectively. It can be shown (Pyke (1961)) that

(23) $M^*(s) = G^*(s)[1-G^*(s)]^{-1}$

where M^* and G^* are N by N matrices whose ij-th element is m^*_{ij} and G^*_{ij} respectively. For the special case that we are considering, the Laplace transform of (12) is

(24) $M^*_1(s;j) = \mu \sum_{i=0}^{N} \frac{a_i}{\gamma_i+s}\, m^*_{ji}(s).$

Making use of the exponential form of λ_1, equation (16) becomes

(25) $M_2 = \frac{1}{\alpha} \sum_{i=0}^{N} \pi_i \left\{ 2\mu a_i M^*_1(\gamma_i, i) + \frac{a^2_i(\mu^2+\sigma^2)}{2\gamma_i} \right\}.$

Equations (25), (24) and (23) then give a straightforward, if somewhat cumbersome, method for computing the variance of this stationary process.

In the interest of brevity, we have not discussed other properties of the $Y(t)$ process that can be obtained by similar arguments. For instance, one can obtain from (8) equations for all the moments of the $Y(t)$ process. The covariance function of the $Y(t)$ process can be found directly from the results given here by defining a new process $Z(t) = Y(t) + Y(t+h)$. This process is also a shot noise process and the covariance of the $Y(t)$ process can be found from the variance of the $Z(t)$ process, since

$$\mathrm{Cov}(Y(t),Y(t+h)) = \frac{1}{2}[\mathrm{Var}(Z(t)) - \mathrm{Var}(Y(t)) - \mathrm{Var}(Y(t+h))].$$

SECTION 2: Discussion

A shot noise process generated by a semi-Markov process can arise in many situations where for various reasons one cannot consider the impulses to be generated by a renewal process. One such situation is the modeling of traffic noise generated by automobiles on a highway. This model has been described by Marcus (1971).

In neurophysiology, the analysis of neural spike trains indicates that the interspike times are often serially dependent random variables (Perkel, Gerstein and Moore, 1967). Serial dependence can be simply introduced by using S-MPs. Models for neuron firing proposed by Coleman and Gastwirth (1969) lend to spike train processes that can be described by semi-Markov processes (Smith, 1971).

Shot noise models can also be applied to queueing theory. Takács (1958) has pointed out that the infinitely many server queue is a special case of the shot noise process. In particular the S-MP/GI/∞ queue is a special case of the model developed in this note where

$$f(t,i,w) = \begin{cases} 1 & \text{if } 0 \le t \le w \\ 0 & \text{otherwise} \end{cases}$$

and the random variable W_n represents the service time of the n-th individual.

For simplicity, we have discussed only the one-sided impulse function, $f(t,i,w) = 0$ for $t < 0$. However, the two-sided impulse function can also be analyzed using the same techniques. For instance in the two-sided case equation (8) of Theorem 1 becomes

$$(26) \qquad \Phi_i(t,\omega) = \sum_{j=1}^{N} \int_0^{\infty} \Gamma_j(t-t',\omega)\ \Phi_j(t-t',\omega)\ dG_{ij}(t)$$

the integral equations for the one-sided impulse function is of course a special case of this equation. From (26) one can find integral equations for the moments of the two-sided shot noise process. For the two-sided process the results in Theorem 2 are the same except that the upper limits of integration in equations (12) and (13) are infinity instead of t, and the lower limits of equations (15) and (16) are $(-\infty)$ instead of 0. Marcus (1971) used this two-sided shot noise model to describe traffic noise near a highway. He found the mean and variance of the stationary process by using different, more heuristic methods.

In general, the usefulness of the shot noise process discussed in this note is that complex stochastic systems can be represented in a model which can be analyzed using straightforward mathematical techniques.

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