AN ASYMPTOTICALLY EFFICIENT TEST FOR THE BUNDLE STRENGTH OF FILAMENTS

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Based on a Wiener process approximation, a sequential test for the bundle strength of filaments is proposed and studied here. Asymptotic expressions for the OC and ASN functions are derived, and it is shown that asymptotically the test is more efficient than the usual fixed sample size procedure based on the asymptotic normality of the standardized form of the bundle strength of filaments, studied earlier by Daniels (1945), and Sen, Bhattacharyya and Suh (1972).

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1. INTRODUCTION

Consider a sequence $\{X_i, i\geq 1\}$ of independent and identically distributed (iid) non-negative random variables (rv) with an absolutely continuous distribution function (df) $F(x)$, defined on $(0,\infty)$. For every $n\geq 1$, the ordered variables corresponding to $X_1,\ldots,X_n$ are denoted by $X_{n,1}\leq\cdots\leq X_{n,n}$. Let then

$$D_n = \max_{1\leq i\leq n}[(n-i+1)X_{n,i}], \quad Z_n = n^{-1}D_n. \tag{1.1}$$

When the X_i represent the breaking stresses of filaments, D_n is the maximum stress which a bundle of n parallel filaments of equal length can stand, and is termed the bundle strength [cf. Daniels (1945)]. We assume that F has a finite second moment, so that

$$0 < \lambda^2 = \int_0^\infty x^2 dF(x) < \infty, \tag{1.2}$$

and $x[1-F(x)]$ has a unique maximum at x_0 $(0<x_0<\infty)$ i.e.,

$$\theta = \sup_x x[1-F(x)] = x_0[1-F(x_0)] > 0. \tag{1.3}$$

Note that

$$\theta\varepsilon(0,\lambda) \quad \text{and} \quad \pi_0 = F(x_0): \quad 0<\pi_0<1. \tag{1.4}$$

Further, the first derivative of $x[1-F(x)]$ vanishes at $x=x_0$, so that $f(x_0) = F'(x_0) > 0$. Our second assumption is that for $\delta(>0)$ sufficiently small, for all $x\varepsilon[x_0-\delta, x_0+\delta]$,

$$x[1-F(x)] \leq \theta - C|x-x_0|^k, \ k\geq 1, \ C<\infty. \tag{1.5}$$

In fact, if $x[1-F(x)]$ is twice differentiable in some neighborhood of x_0 with a continuous and non-null second derivative, then (1.5) holds for k=2. If follows from Sen, Bhattacharyya and Suh (1972), Bhattacharyya, Suh and Grandage (1970), and Sen (1972) that Z_n almost surely (a.s.) converges to θ as $n\to\infty$. We term θ as the <u>mean (per unit) bundle strength of filaments</u>.

Our parameter of interest is θ, and the unknown df F is treated as a nuisance parameter (in the family of all dfs satisfying the aforesaid conditions). We want to test

$$H_0: \theta=\theta_0 \quad \text{vs} \quad H_1: \theta=\theta_1=\theta_0+\Delta, \ \Delta>0, \tag{1.6}$$

where θ_0 and Δ are known. It follows from Daniels (1945), and Sen, Bhattacharyya and Suh (1972) that $n^{\frac{1}{2}}(Z_n-\theta)$ has asymptotically (as $n\to\infty$) a normal distribution with mean 0 and variance

$$\nu^2 = \theta^2\pi_0(1-\pi_0)^{-1} = x_0^2\pi_0(1-\pi_0), \tag{1.7}$$

and writing (1.1) equivalently as

$$D_n = (n-r_n+1)\ X_{n,r_n} \quad (\text{where } 1\le r_n\le n), \tag{1.8}$$

that the integer valued random variable r_n is unique with probability 1. Then, from Sen (1972), it follows that

$$p_n = n^{-1}r_n \to \pi_0 \text{ a.s., as } n\to\infty. \tag{1.9}$$

Thus, as $n\to\infty$

$$\mathcal{L}(n^{\frac{1}{2}}(p_n^{-1}-1)^{\frac{1}{2}}(1-Z_n^{-1}\theta)) \to \mathcal{N}(0,1). \tag{1.10}$$

The weak convergence in (1.10) enables one to construct a large sample test for (1.6) which achieves asymptotically a specified level of significance. However, like the usual fixed sample size procedures, this fails to have any predetermined power.

Now, by the results of Sen (1972) and Strassen (1967), (1.10) can be strengtened to a Wiener process approximation of a continuous sample path version of $\{D_n - n\theta; n \geq 1\}$. Thus, in the same way as a sequential probability ratio test (SPRT) based on a Wiener process approximation is an improvement over the fixed sample likelihood ratio test based merely on the asymptotic normality, it should be possible to obtain, at least asymptotically, a better test for (1.6) based on the above Wiener process approximation. With this objective, a sequential test for (1.6) is proposed in section 2 and its termination probability is studied. The allied OC and ASN functions are then studied in section 3. The test is compared with the fixed sample size test in section 4. Results of sections 3 and 4 are of asymptotic nature, where we let $\Delta \to 0$. In practice, these are valid whenever Δ is small.

2. THE PROPOSED SEQUENTIAL TEST

To motivate the proposed sequential test, we first consider (in Theorem 2.1) a Wiener process approximation of a continuous sample path version of $\{D_n - n\theta, n \geq 1\}$. Subsequently, we make use of the results of Dvoretzky, Kiefer and Wolfowitz (1953) on sequential probability ratio tests for Wiener processes.

Let us define $D_0 = 0$, and for $t \in [n, n+1]$, let

$$D_t = D_n + (t-n)[D_{n+1} - D_n], \quad n=0,1,\ldots, \tag{2.1}$$

so that $\{D_t, t \geq 0\}$ has continuous sample paths. Then, along the lines of Strassen (1967), we frame the following almost sure invariance principle for $\{D_t - t\theta; t \geq 0\}$.

Theorem 2.1. *Under* (1.2) - (1.5),

$$D_t - t\theta = \nu W(t) + o(t^{\frac{1}{2}}) \text{ a.s.,} \quad \textit{as } t \to \infty, \tag{2.2}$$

where $W = \{W(t), t\geq 0\}$ is a standard Brownian motion on $[0,\infty)$.

Proof. Let $c(u)$ be equal to 0 or 1 according as u is $<$ or ≥ 0, and let

$$U_i = x_0[F(x_0) - c(x_0 - X_i)],\ i\geq 1, \tag{2.3}$$

$$T_n = \sum_{i=1}^{n} U_i,\ n\geq 1,\ \text{and}\ T_0 = 0; \tag{2.4}$$

$$T_t = T_n + (t-n)U_{n+1} \quad \text{for} \quad t\varepsilon[n,n+1],\ n\geq 0. \tag{2.5}$$

Then, it follows from Theorem 3.2 of Sen (1972) that under (1.2) - (1.5), as $n\to\infty$,

$$n^{-\frac{1}{2}}|D_n - n\theta - T_n| \to 0 \quad \text{a.s.} \tag{2.6}$$

Also, by (2.1), (2.5) and (2.6),

$$\sup_{t\geq n}\{t^{-\frac{1}{2}}|D_t - t\theta - T_t|\} \leq (1+n^{-1})^{\frac{1}{2}} \max_{m\geq n}|m^{-\frac{1}{2}}(D_m - m\theta - T_m)|$$
$$\to 0 \text{ a.s., as } n\to\infty. \tag{2.7}$$

Consequently, it suffices to show that

$$T_t = \nu W(t) + o(t^{\frac{1}{2}}) \text{ a.s., as } t\to\infty. \tag{2.8}$$

Now, the U_i, $i\geq 1$, are bounded valued iidrv with $EU=0$, $EU^2=\nu^2$, and hence, by Theorems 1.5 and 4.4 of Strassen (1967), we have

$$T_t = \nu W(t) + O((t \log\log t)^{\frac{1}{4}}(\log t)^{\frac{1}{2}}) \text{ a.s., as } t\to\infty, \tag{2.9}$$

which implies (2.8), and completes the proof. Q.E.D.

Theorem 2.1 provides us with at least a heuristic justification for replacing the process $\{D_t - t\theta, t\geq 0\}$ by $\{\nu W_t, t\geq 0\}$; a rigorous justification under an asymptotic setup (allowing $\Delta\to 0$) will be considered in section 3. Thus, if ν

were known, transmitting the hypotheses H_0 and H_1 in (1.6) in terms of drifts (per unit of time) of the process $\{\nu W(t)+t\theta, t\geq 0\}$, and then using the results in section 3 of Dvoretzky, Kiefer and Wolfowitz (1953), one could have constructed the following sequential test.

Corresponding to the desired strength (α,β) of the test, we define two positive numbers (B,A) (where $0<\beta/(1-\alpha)\leq B<1<A\leq(1-\beta)/\alpha<\infty$), and define a stopping variable $N(\Delta)$ as the smallest positive integer for which the inequality

$$\nu^2 b < \Delta[D_m - \tfrac{m}{2}(\theta_0+\theta_1)] < \nu^2 a, \quad m\geq 1, \tag{2.10}$$

(where $b = \log B$ and $a = \log A$) is vitiated; if for $N(\Delta)=n$, $\Delta[D_n - \tfrac{1}{2}(\theta_0+\theta_1)] \leq \nu^2 b$ (or $\geq \nu^2 a$), then H_0 (or H_1) is accepted.

Now ν^2 is unknown, but as has been noted earlier that

$$\hat{\nu}_n^2 = Z_n^2 p_n (1-p_n)^{-1} \to \nu^2 \text{ a.s., as } n\to\infty. \tag{2.11}$$

Thus, if we start with an initial sample of size $n_0 (=n_0(\Delta)$, moderately large for small Δ), and replace for $m\geq n_0$, ν by $\hat{\nu}_m$, we may, by analogy to (2.10), consider the following proposed sequential test for (1.6):

Starting with an initial sample of size $n_0(=n_0(\Delta))$, continue drawing observations, one by one, so long as

$$\hat{\nu}_m^2 b < \Delta[D_m - \tfrac{m}{2}(\theta_0+\theta_1)] < \hat{\nu}_m^2 a, \quad m\geq n_0(\Delta); \tag{2.12}$$

if $N^*(\Delta)=n$ is the smallest positive integer $(\geq n_0(\Delta))$ for which (2.12) is vitiated, accept H_0 or H_1 according as $\Delta[D_n - \tfrac{n}{2}(\theta_0+\theta_1)]$ is $\leq \hat{\nu}_n^2 b$ or $\geq \hat{\nu}_n^2 a$.

We first show that for every pair (θ_0,Δ) of positive constants, the proposed test like the Wald (1947) SPRT terminates with probability 1.

Theorem 2.2. Under (1.2) and (1.5), for every $\theta_0>0$ and $\Delta>0$,

$$\lim_{n\to\infty} P_\theta\{N^*(\Delta)>n\} = 0, \tag{2.13}$$

that is, the process in (2.12) terminates with probability one.

Proof. By (2.12), for every $n\geq n_0(\Delta)$,

$$P_\theta\{N^*(\Delta)>n\} \leq P_\theta\{\hat{\nu}_n^2 b < \Delta[D_n - \tfrac{n}{2}(\theta_0+\theta_1)] < \hat{\nu}_n^2 a\}. \tag{2.14}$$

Consider now the two possible cases (a) $\theta = (\theta_0+\theta_1)/2$ and (b) $\theta \neq (\theta_0+\theta_1)/2$. Now, by (2.11), for every $\gamma>0$, as $n\to\infty$,

$$|n^{-\gamma}\hat{\nu}_n b/\Delta| \to 0 \text{ a.s.}, \quad |n^{-\gamma}\hat{\nu}_n a/\Delta| \to 0 \text{ a.s.}, \tag{2.15}$$

while, by (1.10), $n^{-\frac{1}{2}}[D_n-n\theta]/\hat{\nu}_n$ is asymptotically normally distributed with 0 mean and unit variance. Hence, in the first case,

$$P_\theta\{N^*(\Delta)>n\} \leq P_\theta\{n^{-\frac{1}{2}}\hat{\nu}_n b/\Delta \leq n^{-\frac{1}{2}}[D_n-n\theta]/\hat{\nu}_n < n^{-\frac{1}{2}}\hat{\nu}_n a/\Delta\}$$
$$\to 0 \quad \text{as} \quad n\to\infty \text{ (for every fixed } \Delta>0). \tag{2.16}$$

In the second case, we rewrite (2.14) as

$$P_\theta\{N^*(\Delta)>n\} \leq P_\theta\{n^{-1}\hat{\nu}_n^2 b/\Delta \leq n^{-1}D_n - \tfrac{1}{2}(\theta_0+\theta_1) < n^{-1}\hat{\nu}_n^2 a/\Delta\}, \tag{2.17}$$

where by the results of Sen (1972), $n^{-1}D_n = Z_n\to\theta$ a.s., as $n\to\infty$, and hence, for $\theta \neq \frac{1}{2}(\theta_0+\theta_1)$, by (2.15) and (2.17), $P_\theta\{N^*(\Delta)>n\} \to 0$ as $n\to\infty$. Q.E.D.

3. OC AND ASN OF THE PROPOSED TEST

For theoretical justifications, here we consider the asymptotic situation where we let $\Delta\to 0$; in practice, the results provide good approximation when Δ is small. In this set up, the situation is similar to the dual problem of sequential bounded length confidence intervals, treated in Chow and Robbins (1965), and others, where the results are justified in the limiting case when the width of the confidence intervals is made to converge to 0.

As in the SPRT, for small Δ, the excess over the boundaries are negligible, so that we can take

$$e^a = A = (1-\beta)/\alpha \quad \text{and} \quad e^b = B = \beta/(1-\alpha). \tag{3.1}$$

Secondly, proceeding as in section 2, it can be shown that for every $\theta(\neq\theta_0)$, the OC of the proposed test converges (as $\Delta\to 0$) to 0 or 1 according as θ is > or < θ_0. Hence, to avoid the limiting degeneracy, we let

$$\theta = \theta_0+\phi\Delta, \; \phi\varepsilon I = \{\phi: \; |\phi|<K\}, \tag{3.2}$$

where $K(>\frac{1}{2})$ is a finite number. Finally, it will be shown later on that the ASN of $N^*(\Delta)$ is proportional to Δ^{-2} as $\Delta\to 0$, and, up to a first order of approximation, both the OC and ASN are not affected by an initial large sample size $n_0(\Delta)$, provided $\Delta^2 n_0(\Delta)$ is small. On the otherhand, a reasonably large initial sample size is needed to insure the accuracy of the estimation $\hat{\nu}_n$ of ν for all $n\geq n_0(\Delta)$. Hence, we assume that

$$\lim_{\Delta\to 0} n_0(\Delta) = \infty \; \text{ but } \lim_{\Delta\to 0}\{\Delta^2 n_0(\Delta)\} = 0. \tag{3.3}$$

Let us then denote by $L_F(\phi,\Delta)$ the OC function of the proposed test when $\theta = \theta_0+\phi\Delta$ and the underlying df is F. Then, in the following theorem, we show that $L_F(\phi,\Delta)$ is ADF (asymptotically distribution-free) for all $\phi\varepsilon I$ and F satisfying (1.2) - (1.5).

Theorem 3.1. Under (1.2), (1.3), (1.5), (3.1) and (3.3), for every $\phi \in I$,

$$\lim_{\Delta \to 0} L_F(\phi,\Delta) = \begin{cases} (A^{1-2\phi}-1)/(A^{1-2\phi}-B^{1-2\phi}), & \phi \neq \frac{1}{2} \\ (\log A)/[\log A - \log B], & \phi = \frac{1}{2}. \end{cases} \tag{3.4}$$

Hence, asymptotically (as $\Delta \to 0$), the test has the prescribed strength (α,β) for all F satisfying (1.2) and (1.4), that is, the test is asymptotically consistent.

Proof. We only prove (3.4), as the later part of the theorem follows directly by putting $\phi=0$ and 1, and using (3.1).

By (2.2), for every $\varepsilon>0$ and $\eta>0$, there exists a $t_0(\varepsilon,\eta)$ such that

$$P\{\sup_{t \geq t_0(\varepsilon,\eta)} t^{-\frac{1}{2}}|D_t - t\theta - \nu W(t)| > \tfrac{1}{2}\varepsilon\} < \tfrac{1}{2}\eta, \tag{3.5}$$

and by (2.11), there exists an $n_0(\varepsilon,\eta)$, such that

$$P\{\max_{n_0(\varepsilon,\eta) \leq n < \infty} |\hat{\nu}_n^2 - \nu^2| > \tfrac{1}{2}\varepsilon\} < \tfrac{1}{2}\eta. \tag{3.6}$$

Thus, by (3.3), we can select a $\Delta_0 = \Delta_0(\varepsilon,\eta)$ such that

$$n_0(\Delta_0) \geq \max[t_0(\varepsilon,\eta),\ n_0(\varepsilon,\eta)] \quad \text{and} \quad \Delta_0^2 n_0(\Delta_0) < \varepsilon. \tag{3.7}$$

Now, for every $\delta>0$, $t_0>0$ and $\phi \in I$, let

$$P(\phi,\delta,a,b,t_0) = P\left\{\begin{array}{l} W(t) \leq \delta^{-1}b + t\delta(\frac{1}{2}-\phi) \text{ for a smaller } t \\ (\geq t_0) \text{ than any other } t(\geq t_0) \text{ for which } W(t) \geq \delta^{-1}a \\ \qquad\qquad + t\delta(\frac{1}{2}-\phi) \end{array}\right\} \tag{3.8}$$

Also, we define $a = \log A$, $b = \log B$,

$$a_\varepsilon^{(i)} = [1+(-1)^i\varepsilon]a \quad \text{and} \quad b_\varepsilon^{(i)} = [1+(-1)^i\varepsilon]b,\ i=1,2. \tag{3.9}$$

Then, from (2.12) and (3.5) through (3.9), it follows that for every $\Delta\varepsilon[0,\Delta_0]$, $\theta = \theta_0+\phi\Delta$, $\phi\varepsilon I$,

$$P(\phi,\Delta/\nu,a_\varepsilon^{(2)},b_\varepsilon^{(1)},n_0(\Delta)) - \eta \leq L_F(\phi,\Delta)$$
$$\leq P(\phi,\Delta/\nu,a_\varepsilon^{(1)},b_\varepsilon^{(2)},n_0(\Delta)) + \eta. \tag{3.10}$$

Now, $|\phi-\frac{1}{2}|$ is bounded for all $\phi\varepsilon I$, and $t\Delta^2 \leq \varepsilon$ for all $t\varepsilon[0,n_0(\Delta)]$, $\Delta\varepsilon[0,\Delta_0]$. Also, $\{W(t),t\geq 0\}$ is a process of independent increment. Hence, by the Levy inequality, the probability that $[W(t),0\leq t\leq n_0(\Delta)]$ crosses either of the two lines $(\nu/\Delta)b_\varepsilon^{(j)} + (t\Delta/\nu)(\frac{1}{2}-\phi)$ or $(\nu/\Delta)a_\varepsilon^{(i)} + (t\Delta/\nu)(\frac{1}{2}-\phi)$ (for every $i,j=1,2$) can be made smaller than $\eta'(>0)$, where $\eta'\to 0$ as $\varepsilon\to 0$. Thus, by (3.10),

$$P(\phi,\Delta/\nu,a_\varepsilon^{(2)},b_\varepsilon^{(1)},0) - \eta - \eta' \leq L_F(\phi,\Delta)$$
$$\leq P(\phi,\Delta/\nu,a_\varepsilon^{(1)},b_\varepsilon^{(2)},0) + \eta + \eta'. \tag{3.11}$$

Now, by the results of section 3 of Dvoretzky, Kiefer and Wolfowitz (1953), for every $\delta>0$, $d<0<c$,

$$P(\phi,\delta,c,d,0) = \begin{cases} (e^{c(1-2\phi)}-1)/(e^{c(1-2\phi)}-e^{d(1-2\phi)}), & \phi\neq\frac{1}{2}, \\ c/(c-d), & \phi=\frac{1}{2}. \end{cases} \tag{3.12}$$

The proof of the theorem is then completed by noting that $P(\phi,\delta,c,d,0)$ is continuous in c and d, so that by choosing ε and η sufficiently small, both the left and right hand sides of (3.11) can be made arbitrarily close to (3.4). Q.E.D.

<u>Theorem 3.2</u>. <u>Under</u> (1.2) - (1.5) <u>and</u> (3.1) - (3.3), <u>for every</u> $\phi\varepsilon I$,

$$\lim_{\Delta\to 0}\{\Delta^2 E_\phi[N^*(\Delta)]\} = \psi(\phi,\nu), \tag{3.13}$$

where E_ϕ stands for the expectation under $\theta=\theta_0+\phi\Delta$,

$$\psi(\phi,\nu) = \begin{cases} \nu^2(\phi-\tfrac{1}{2})^{-1}[bP(\phi) + a\{1-P(\phi)\}], & \phi\neq\tfrac{1}{2}, \\ -\nu^2P'(\tfrac{1}{2})(a-b), & \phi=\tfrac{1}{2}, \end{cases} \tag{3.14}$$

ν^2 is defined in (1.7), $P(\phi) = P(\phi,\Delta/\nu,a,b,0)$ in (3.12), and $P'(\tfrac{1}{2}) = (d/d\phi)P(\phi)\big|_{\phi=\frac{1}{2}}$.

Proof. We first consider the case of $\phi\neq\tfrac{1}{2}$, and let for every $\Delta>0$,

$$n_{\phi,\Delta} = [K^2\nu^2\Delta^{-2}(\phi-\tfrac{1}{2})^{-2}] + 1, \quad K<\infty, \tag{3.15}$$

where K will be chosen later on. Then, noting that for $k\geq 0$,

$$\textstyle\sum_{n\geq k} nP_\phi\{N^*(\Delta)=n\} = \sum_{n\geq k} P_\phi\{N^*(\Delta)>n\} + kP_\phi\{N^*(\Delta)>k\}, \tag{3.16}$$

we have by (3.3) that as $\Delta\to 0$,

$$\Big|\Delta^2E_\phi N^*(\Delta) - \Delta^2[\textstyle\sum_{n=n_0(\Delta)}^{n_{\phi,\Delta}} P_\phi\{N^*(\Delta)>n\} + \sum_{n>n_{\phi,\Delta}} P_\phi\{N^*(\Delta)>n\}]\Big|$$

$$= \Delta^2n_0(\Delta)\ P_\phi\{N^*(\Delta)>n_0(\Delta)\} \to 0. \tag{3.17}$$

Now, for every $\varepsilon>0$ and $n\geq n_{\phi,\Delta}$, by (1.1) and (2.12),

$$\begin{aligned} P_\phi\{N^*(\Delta)>n\} &\leq P_\phi\{\hat{\nu}_n^2 b < \Delta[D_n - \tfrac{n}{2}(\theta_0+\theta_1)] < \hat{\nu}_n^2 a\} \\ &\leq P\{\hat{\nu}_n^2>\nu^2+\varepsilon\} + P_\phi\{\nu^2b(1+\varepsilon) < \Delta[D_n - \tfrac{n}{2}(\theta_0+\theta_1)] < \nu^2a(1+\varepsilon)\} \\ &= P\{\hat{\nu}_n^2>\nu^2+\varepsilon\} + \\ &\quad P_\phi\{n^{\frac{1}{2}}\Delta(\tfrac{1}{2}-\phi) + \frac{b\nu^2(1+\varepsilon)}{\Delta n^{\frac{1}{2}}} < n^{\frac{1}{2}}(Z_n-\theta) < n^{\frac{1}{2}}\Delta(\tfrac{1}{2}-\phi) + \frac{a\nu^2(1+\varepsilon)}{\Delta n^{\frac{1}{2}}}\}. \end{aligned} \tag{3.18}$$

With the aid of Theorems 3.1 and 3.4 of Sen (1972), it follows that for every $\varepsilon>0$ and s (which we select > 1), there exist a finite positive $c(\varepsilon,s)$ and an $n_0(\varepsilon,s)$, such that for all $n\geq n_0(\varepsilon,s)$,

$$P\{\hat{\nu}_n^2>\nu^2+\varepsilon\} \leq c(\varepsilon,s)n^{-s}, \; s>1. \tag{3.19}$$

Now specifying in Theorem 2.3 of Sen (1972), a power rate $O(n^{-s})$, as in Lemma 1 of Bahadur (1966), $s>1$, and then proceeding as in Theorems 2.1 and 3.2 of Sen (1972), it follows that for all $n\geq n_0(\varepsilon,s)$,

$$P\{|n^{\frac{1}{2}}(Z_n-\theta)-n^{-\frac{1}{2}}T_n| > c^*n^{-1/4k}(\log n)\} \leq c(\varepsilon,s)n^{-s}, \; s>1, \tag{3.20}$$

where k is defined in (1.5) and T_n in (2.4). On the other hand, for $n\geq n_{\phi,\Delta}$, $n^{\frac{1}{2}}\Delta \geq (n/n_{\phi,\Delta})^{\frac{1}{2}}K\nu|\phi-\frac{1}{2}|^{-1}$, so that by proper choice of K, $\nu^2(a-b)\ (1+\varepsilon)/\Delta n^{\frac{1}{2}}$ can be made smaller than $\frac{1}{4}|\phi-\frac{1}{2}|\Delta n^{\frac{1}{2}}$. Thus, the second term on the right hand side of (3.18) is bounded above by

$$\begin{aligned}
&P_\phi\{n^{\frac{1}{2}}|Z_n-\theta| \geq \tfrac{3}{4}n^{\frac{1}{2}}\Delta|\phi-\tfrac{1}{2}|\} \\
&\leq P_\phi\{n^{\frac{1}{2}}|Z_n-\theta| \geq \tfrac{3}{4}K\nu(n/n_{\phi,\Delta})^{\frac{1}{2}}\} \\
&\leq P_\phi\{n^{\frac{1}{2}}|Z_n-\theta-n^{-1}T_n| \geq \tfrac{1}{4}K\nu(n/n_{\phi,\Delta})^{\frac{1}{2}}\} + P_\phi\{n^{-\frac{1}{2}}|T_n| \geq \tfrac{1}{2}K\nu(n/n_{\phi,\Delta})^{\frac{1}{2}}\}.
\end{aligned} \tag{3.21}$$

Now, $T_n(=\sum_1^n U_i)$ involves summation over iidrv's, where $|U_i/x_0| \leq 1$, $\forall i$. Hence, by Theorem 1 of Hoeffding (1963),

$$\begin{aligned}
&P_\phi\{n^{-\frac{1}{2}}|T_n| \geq \tfrac{1}{2}K\nu(n/n_{\phi,\Delta})^{\frac{1}{2}}\} \\
&\quad\leq 2\ \exp\{-\tfrac{1}{2}(n/n_{\phi,\Delta})\pi_0(1-\pi_0)K^2\}.
\end{aligned} \tag{3.22}$$

Thus, if we let $\rho(\phi,\Delta) = \exp\{-\pi_0(1-\pi_0)K^2/2n_{\phi,\Delta}\} \simeq \exp\{-\Delta^2(\phi-\frac{1}{2})^2/2x_0^2\}$, we have

$$\begin{aligned}
&2\Delta^2 \textstyle\sum_{n>n_{\phi,\Delta}} \exp\{-\tfrac{1}{2}(n/n_{\phi,\Delta})\pi_0(1-\pi_0)K^2\} \\
&= 2\Delta^2\ [\rho(\phi,\Delta)]^{n_{\phi,\Delta}+1}\ [1-\rho(\phi,\Delta)]^{-1} \\
&= 2[\exp\{-\tfrac{1}{2}\pi_0(1-\pi_0)K^2\}][2x_0(\phi-\tfrac{1}{2})^{-2}\{1+O(\Delta^2)\}] \\
&< \tfrac{1}{4}\varepsilon, \text{ where } \varepsilon(>0) \text{ is arbitrarily small,}
\end{aligned} \tag{3.23}$$

by proper choice of K in (3.14). Let us now select a $\Delta_0=\Delta_0(\varepsilon)$ (>0) such that $n_0(\Delta_0) \geq n_0(\varepsilon,s)$, defined in (3.19) and (3.20). Then, by (3.17) through (3.23), it follows that for every $\varepsilon>0$, there exists a $K_\varepsilon(<\infty)$, such that for $n_{\phi,\Delta}$, defined by (3.14) with $K=K_\varepsilon$, and $0<\Delta\leq\Delta_0$,

$$|\Delta^2 E_\phi N^*(\Delta) - \Delta^2 \sum_{n=n_0(\Delta)}^{n_{\phi,\Delta}} P_\phi\{N^*(\Delta)>n\}| < \tfrac{1}{2}\varepsilon. \tag{3.24}$$

Let us now define two stopping variables $N_\phi^{(1)}(\Delta)$ and $N_\phi^{(2)}(\Delta)$ as the least positive integer $(\geq n_0(\Delta))$, for which $T_n+n\Delta(\phi-\tfrac{1}{2})$ is not contained in $[\nu^2 b(1-\varepsilon)/\Delta, \nu^2 a(1-\varepsilon)/\Delta]$, and $[\nu^2 b(1+\varepsilon)/\Delta, \nu^2 a(1+\varepsilon)/\Delta]$, respectively, and the terminal decisions are to accept H_0 or H_1 according as for $N_\phi^{(i)}(\Delta) = n$, $T_n+n\Delta(\phi-\tfrac{1}{2})$ is $\leq \nu^2 b(1+(-1)^i\varepsilon)/\Delta$ or $\geq \nu^2 a(1+(-1)^i\varepsilon)/\Delta$, i=1,2. Then, by the same technique as in (3.22) and (3.23), it follows that, parallel to (3.24),

$$|\Delta^2 E_\phi N_\phi^{(i)}(\Delta) - \Delta^2 \sum_{n=n_0(\Delta)}^{n_{\phi,\Delta}} P_\phi\{N_\phi^{(i)}(\Delta)>n\}| < \tfrac{1}{2}\varepsilon,\ i=1,2. \tag{3.25}$$

By a two-sided version of (3.19) and (3.20), for every $\varepsilon>0$, there exists a $\Delta_0(\varepsilon)>0$, such that $n_0(\Delta) > n_0(\varepsilon/2,s)$ for all $0<\Delta\leq\Delta_0(\varepsilon)$. Then for every $n\geq n_0(\Delta)$, by (3.19) and (3.20),

$$\begin{aligned} P_\phi\{N^*(\Delta)>n\} &\leq P_\phi\{\nu^2 b(1+\varepsilon)/\Delta < T_m+m\Delta(\phi-\tfrac{1}{2}) < \nu^2 a(1+\varepsilon)/\Delta,\ n_0(\Delta)\leq m\leq n\} \\ &\quad + P\left\{\begin{matrix}\hat{\nu}_m^2>\nu^2+\varepsilon/2,\ m^{-\frac{1}{2}}\Delta|Z_m-\theta-m^{-1}T_m| > \varepsilon/2, \\ \text{for at least one } m:\ n_0(\Delta)\leq m\leq n\end{matrix}\right\} \\ &\leq P_\phi\{N_\phi^{(2)}(\Delta)>n\} + 2c(\varepsilon/2,s)[n_0(\Delta)]^{-s+1}. \end{aligned} \tag{3.26}$$

Similarly,

$$P_\phi\{N^*(\Delta)>n\} \geq P_\phi\{N_\phi^{(1)}(\Delta)>n\} - 2c(\varepsilon/2,s)[n_0(\Delta)]^{-s+1}. \tag{3.27}$$

Since $\Delta^2 n_{\phi,\Delta}$ is bounded, while s>1, so that by (3.3), $[n_0(\Delta)]^{-s+1} \to 0$ as $\Delta\to 0$, we obtain from (3.26) and (3.27) that as $\Delta\to 0$,

$$\Delta^2 \sum_{n=n_0(\Delta)}^{n_{\phi,\Delta}} P_\phi\{N_\phi^{(1)}(\Delta)>n\} - \eta < \Delta^2 \sum_{n=n_0(\Delta)}^{n_{\phi,\Delta}} P_\phi\{N^*(\Delta)>n\}$$
$$< \Delta^2 \sum_{n=n_0(\Delta)}^{n_{\phi,\Delta}} P_\phi\{N_\phi^{(2)}(\Delta)>n\} + \eta,\ \eta>0, \tag{3.28}$$

where η can be made arbitrarily small by choosing Δ small. Consequently, by (3.24), (3.25) and (3.28), it follows that we are only to show that for every $\eta>0$, there exist an $\varepsilon>0$ and a $\Delta_0(=\Delta_0(\eta)>0)$, such that for every $\phi(\neq\frac{1}{2})\ \varepsilon I$ and $0<\Delta\leq\Delta_0$,

$$|\Delta^2 E_\phi\{N_\phi^{(j)}(\Delta)\} - \psi(\phi,\nu)| < \eta \quad \text{for} \quad j=1,2. \tag{3.29}$$

Since $N_\phi^{(j)}(\Delta)$ is based on $\{T_m+m\Delta(\phi-\frac{1}{2}),\ m\geq n_0(\Delta)\}$, where T_m involves a summation over independent, bounded valued, random variables, the excess over the boundaries is negligible for small Δ, and hence, by the Wald (1947, p. 171) fundamental identity and Theorem 3.1, we have

$$\lim_{\Delta\to 0}|\Delta^2 E_\phi\{N_\phi^{(j)}(\Delta)\} - (1+(-1)^j\varepsilon)\psi(\phi,\nu)| = 0,\ j=1,2. \tag{3.30}$$

Thus, (3.29) follows from (3.30) by letting $\varepsilon(>0)$ to be arbitrarily small. Hence the theorem follows for $\phi\neq\frac{1}{2}$.

For $\phi=\frac{1}{2}$, we can not adopt the above proof. However, we may note that $P(\phi)$ is continuous and differentiable in some neighborhood of $\phi=\frac{1}{2}$, so that $P'(\frac{1}{2})$ exists. Hence, by considering a sequence of values of ϕ, say, $\phi_r = \frac{1}{2}\pm\varepsilon_r$, $r\geq 1$, where $\varepsilon_r\to 0$ as $r\to\infty$, and then using the above proof, we obtain by the L' Hospital's rule that

$$\lim_{\Delta\to 0}\{\Delta^2 E_{\frac{1}{2}}[N^*(\Delta)]\} = \lim_{n\to\infty} \psi(\phi_r,\nu) = -\nu^2 P'(\tfrac{1}{2})(a-b). \quad \text{Q.E.D.}$$

4. COMPARISON OF THE TWO TESTS

Since $\hat{\nu}_n$ stochastically converges to ν, for large n, the test for (1.6) based on the convergence in (1.10) has the same properties as the test based on $n^{\frac{1}{2}}(Z_n-\theta)$ assuming ν to be known. Had ν been known, the sample size needed to have a test for (1.6) with strength (α,β) is given by $n(\Delta)$, where

$$n(\Delta) \simeq \nu^2(\tau_\alpha+\tau_\beta)^2/\Delta^2, \tag{4.1}$$

and τ_ε is the upper $100\varepsilon\%$ point of the standard normal distribution. Thus,

$$\lim_{\Delta\to 0}\{\Delta^2 n(\Delta)\} = \nu^2(\tau_\alpha+\tau_\beta)^2. \tag{4.2}$$

Now, from Theorem 3.2 and (4.2), we conclude that the asymptotic (as $\Delta\to 0$) relative efficiency of the fixed sample size procedure [based on (1.10)] with respect to the proposed sequential procedure is equal to

$$\begin{aligned} e_\phi &= \lim_{\Delta\to 0}\{E_\phi[N^*(\Delta)]/n(\Delta)\} \\ &= \psi(\phi,\nu)/\nu^2(\tau_\alpha+\tau_\beta)^2 \\ &= \begin{cases} [bP(\phi) + a\{1-P(\phi)\}]/[(\phi-\tfrac{1}{2})(\tau_\alpha+\tau_\beta)^2], & \phi\neq\tfrac{1}{2} \\ -\,P'(\tfrac{1}{2})(a-b)/(\tau_\alpha+\tau_\beta)^2, & \phi=\tfrac{1}{2}. \end{cases} \end{aligned} \tag{4.3}$$

Numerical computation of (4.3) for various $\phi(\varepsilon I)$ and (α,β) shows that e_ϕ lies close to $\frac{1}{2}$ to $\frac{3}{5}$, indicating a substantial saving in the sample size for the sequential procedure. In particular, for $\phi=0$ or 1, the table on page 57 of Wald (1947), provides the numerical values of (4.3) for various (α,β).

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