

AN ALMOST SURE INVARIANCE PRINCIPLE FOR MULTIVARIATE
KOLMOGOROV-SMIRNOV STATISTICS

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Institute of Statistics Mimeo Series No. 806

February 1972

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An almost sure invariance principle for Kolmogorov-Smirnov statistics for vector chance variables is established along the lines of Theorems 1.4 and 4.9 of Strassen [Proc. Fifth Berkeley Symp. Math Statist. Prob. (1967), 2, 315-343]. This strengthens certain asymptotic expressions on the probability of moderate deviations for Kolmogorov-Smirnov statistics, obtained earlier by Gnedenko, Karoluk and Skorokhod, and by Kiefer and Wolfowitz, among others.

AMS 1970 Subject classification

Primary 60B10, 60F15

Secondary 62E20

Key Words and Phrases:

Almost sure invariance principle, multivariate Kolmogorov-Smirnov statistics, probability of moderate deviations and reverse sub-martingales.

- 1) Work sponsored by the Aerospace Research Laboratories, Air Force Systems Command, U. S. Air Force, Contract No. F33615-71-C-1927. Reproduction in whole or in part permitted for any purpose of the U. S. Government.

1. Introduction. Let $\{X_{\underline{i}}=(X_{i1}, \dots, X_{ip})'; i \geq 1\}$ be a sequence of independent and identically distributed random vectors (idrv) defined on a probability space (Ω, \mathcal{A}, P) , where $X_{\underline{i}}$ has a continuous distribution function $F(x)$, $x \in R^p$, the $p(\geq 1)$ -dimensional Euclidean space. Define the empirical dfs by

$$(1.1) \quad F_n(x) = n^{-1} \sum_{i=1}^n c(x - X_{\underline{i}}), \quad x \in R^p, \quad n \geq 1,$$

where $c(u)=1$, if all the p components of u are ≥ 0 ; otherwise $c(u)=0$. Consider then the general p -variate Kolmogorov-Smirnov statistics

$$(1.2) \quad D_n^+ = \sup\{F_n(x) - F(x) : x \in R^p\}, \quad n \geq 1;$$

$$(1.3) \quad D_n^- = \sup\{F(x) - F_n(x) : x \in R^p\}, \quad n \geq 1;$$

$$(1.4) \quad D_n = \sup\{|F_n(x) - F(x)| : x \in R^p\} \\ = \max\{D_n^+, D_n^-\}, \quad n \geq 1;$$

these are all non-negative random variables.

In the particular case of $p=1$, Gnedenko, Koroluk and Skorokhod (1961, pp. 154-155) reported the results of Karplevskaia and of Li-Tsian that if $\{\lambda_n\}$ be a sequence of positive numbers such that $n\lambda_n^3 = o(1)$, then as $n \rightarrow \infty$,

$$(1.5) \quad P\{D_n^+ > \lambda_n\} = P\{D_n^- \geq \lambda_n\} = \frac{1}{2} P\{D_n \geq \lambda_n\} [1 + o(1)] \\ = \exp\{-2n\lambda_n^2\} [1 + o(1)].$$

Kiefer and Wolfowitz (1958) have shown that for $p \geq 1$, there exist two positive constants $c_p^{(1)}$ and $c_p^{(2)}$, such that for every $n \geq 1$, $\lambda_n > 0$,

$$(1.6) \quad P\{D_n^+ > \lambda_n\} < P\{D_n^- > \lambda_n\} \leq c_p^{(1)} \exp\{-c_p^{(2)} n \lambda_n\}.$$

That $c_p^{(2)}$ cannot be, in general, equal to 2, have been proved by Kiefer (1961),

who shows that for every $\epsilon > 0$ and $p \geq 1$, there exists a positive $c(p, \epsilon)$ such that for every $n \geq 1$, $\lambda_n > 0$,

$$(1.7) \quad P\{D_n^- \geq \lambda_n\} \leq c(p, \epsilon) \exp\{-(2-\epsilon)n\lambda_n^2\}.$$

Since D_n (or D_n^+ or D_n^-) can not be less than the corresponding statistic for the p univariate marginals, we obtain from (1.5) and (1.7) by letting $\epsilon (> 0)$ to be arbitrarily small that if $n\lambda_n^2 \rightarrow \infty$ as $n \rightarrow \infty$, then for every $p \geq 1$, $n\lambda_n^3 = o(1)$,

$$(1.8) \quad (n\lambda_n^2)^{-1} \log P\{D_n^- \geq \lambda_n\} \rightarrow -2, \text{ as } n \rightarrow \infty,$$

and the same result holds for $\{D_n^+\}$ and $\{D_n^-\}$.

For a positive function $\phi(t)$ increasing at a faster (slower) rate than $t^{1/2}$ ($t^{3/5}$) and satisfying certain regularity conditions, Theorems 1.4 and 4.9 of Strassen (1967) relate to the asymptotic (as $n \rightarrow \infty$) expression for

$$(1.9) \quad P\{S_m \geq \phi(m) \text{ for some } m \geq n\},$$

where $S_m = \sum_{i=1}^m (X_i - \mu)/\sigma$, $\mu = EX_i$ and $0 < \sigma^2 = V(X_i) < \infty$, and this relates to his (third) almost sure invariance principle for cumulative sums. The object of the present investigation is to show that this almost sure invariance principle also holds for $\{nD_n^+\}$, $\{nD_n^-\}$ and $\{nD_n\}$. This shows that under certain mild regularity conditions, in the asymptotic expression in (1.8), $[D_n^- \geq \lambda_n]$ can be replaced by $[D_m^- \geq \lambda_m \text{ for some } m \geq n]$.

The main theorems along with the basic regularity conditions are formulated in section 2, where a few remarks are also appended. The proofs of the theorems, based on a reverse sub-martingale property of $\{D_n^+\}$ and Theorem 1 of Kiefer (1961), is considered in section 3.

2. ~~The main theorems.~~ Let $\phi = \{\phi(t): 0 \leq t < \infty\}$ be a positive function, defined on

$[0, \infty)$, with a continuous derivative $\phi'(t)$, such that (i)

$$(2.1) \quad \psi(t) = t^{-\frac{1}{2}} \phi(t) \text{ is } \uparrow \text{ but } t^{-h} \phi(t) \text{ is } \downarrow \text{ in } t, \quad \frac{1}{2} < h < 3/5,$$

(ii) as $t \rightarrow \infty$, with $s/t \rightarrow 1$,

$$(2.2) \quad \phi'(s)/\phi'(t) \rightarrow 1, \quad (\Rightarrow \psi'(s)/\psi'(t) \rightarrow 1),$$

and (iii) the Kolmogorov-Petrovski-Erdős criterion holds for 2ϕ , i.e., for every $n \geq 1$ and $k \geq 2$,

$$(2.3) \quad \begin{aligned} J_n(k\phi) &= (2\pi)^{-\frac{1}{2}} (k/2) \int_n^\infty t^{-3/2} \phi(t) \exp\{-\frac{1}{2} k^2 t^{-1} \phi^2(t)\} dt \\ &= (2\pi)^{-\frac{1}{2}} \int_{\log n}^\infty (k/2) \psi(e^t) \exp\{-\frac{1}{2} k^2 \psi^2(e^t)\} dt < \infty. \end{aligned}$$

Note that by (2.1)

$$(2.4) \quad \psi'(t) = t^{-\frac{1}{2}} \phi'(t) - \frac{1}{2} t^{-3/2} \phi(t) (>0) \text{ is continuous in } t;$$

$$(2.5) \quad 0 < (2t)^{-1} \phi(t) < \phi'(t) < 3(5t)^{-1} \phi(t).$$

Also, by (2.3) and a few routine steps it follows that there exists a positive t_0 ($< \infty$), such that for all $t \geq t_0$ and $k \geq 2$,

$$(2.6) \quad v_k(t) = \frac{1}{2} k^2 \psi^2(t) - \log \psi(t) - \log \log t \text{ is } \uparrow \text{ to } \infty \text{ as } t \uparrow \infty.$$

Thus, for large t , $\psi^2(t) > \frac{1}{2} \log \log t$ when (2.3) holds. We further assume that for large n , $v_k(n)/v_2(n)$ is a continuous function of $k \in [2, 2+\delta]$, for some $\delta > 0$. Thus for every δ' ($0 < \delta' \leq \delta$), there exists an $\eta > 0$, such that for $n \geq n_0$,

$$(2.7) \quad |v_k(n)/v_2(n) - 1| < \eta \text{ whenever } |k-2| < \delta'.$$

In fact, if for some $\varepsilon > 0$, $\overline{\lim}_{t \rightarrow \infty} [(\log \log t)/\psi^2(t)] \leq 2 - \varepsilon$, then (2.1)-(2.3) imply (2.7). A counter example where (2.1)-(2.3) hold but not (2.7) is $\psi^2(t) = \frac{1}{2} \log \log t + \lambda \log \log \log t$ where $\lambda (> 0)$ is a positive number. Let us now define

$$(2.8) \quad P_n(\psi) = P\{m \frac{1}{2} D_m \geq \lambda_m \text{ for some } m \geq n\},$$

and in (2.8) on replacing D_m by D_m^+ and D_m^- , we define $P_n^+(\psi)$ and $P_n^-(\psi)$, respectively. Then, we have the following.

Theorem 1. Under (2.1), (2.2), (2.3) and (2.7)

$$(2.9) \quad \lim_{n \rightarrow \infty} \{[\log P_n(\psi)]/\psi^2(n)\} = -1,$$

and, if, in addition, $\lim_{n \rightarrow \infty} \{(\log \log n)/\psi^2(n)\} = 0$, then

$$(2.10) \quad \lim_{n \rightarrow \infty} \{[\log P_n(\psi)]/\psi^2(n)\} = -2.$$

Both (2.9) and (2.10) hold for $\{P_n^+(\psi)\}$ and $\{P_n^-(\psi)\}$.

Upon letting $\psi(n) = n^{\frac{1}{2}} \lambda_n$ i.e., $\phi(n) = n \lambda_n$, we note that in (2.1) we limit ourselves to $\lambda_n = O(n^{h-1})$ where $\frac{1}{2} < h < 3/5$. On the other hand, in (1.5), we had $\lambda_n = O(n^{-1/3})$. This leads us to inquire whether (2.9) [or (2.10)] holds for λ_n satisfying (1.5) but not (2.1). Towards this end, we may remark that whenever $\psi^2(n)$ increases with n in such a way that $(\log \log n)/\psi^2(n) \rightarrow 0$ with $n \rightarrow \infty$, we do not require (2.2), (2.3) and (2.7), and we may also extend the range of $\psi(n)$ to $O(n^{1/6})$. We have the following.

Theorem 2. If for some $C: 0 < C < \infty$, $\psi(n) < Cn^{1/6}$ and $\lim_{n \rightarrow \infty} (\log \log n)/\psi^2(n) = 0$,
then,

$$(2.11) \quad \lim_{n \rightarrow \infty} \{[\log P_n(\psi)]/\psi^2(n)\} = -2,$$

and the same result holds for $\{P_n^+(\psi)\}$ and $\{P_n^-(\psi)\}$.

We postpone the proof of the theorems to section 3. The following remarks illustrate a few applications of the theorem:

(I) If we let $\psi^2(t) = (\frac{1}{2} + \epsilon) \log \log t$, $\epsilon > 0$, it follows that (2.1), (2.2), (2.3) and (2.7) hold, where

$$(2.12) \quad v_2(t) = 2\varepsilon \log \log t - \frac{1}{2} \log \log \log t - \frac{1}{2} \log(\frac{1}{2} + \varepsilon) \quad (\rightarrow \infty \text{ as } t \rightarrow \infty).$$

Therefore, by (2.9), $P_n(\psi) \rightarrow 0$ as $n \rightarrow \infty$ i.e.,

$$(2.13) \quad P \left\{ \limsup_n (2n)^{\frac{1}{2}} (\log \log n)^{-\frac{1}{2}} D_n = 1 \right\} = 1,$$

and the same result holds for $\{D_n^+\}$ and $\{D_n^-\}$. Thus, in the general p -variate case, the law of iterated logarithm holds for Kolmogorov-Smirnov statistics; this was proved in Theorem 2 of Kiefer (1961) by a different approach.

(II) If we let $\psi(n) = n^{\frac{1}{2}} \lambda_n$, we observe that under the condition that $(\log \log n)/\psi^2(n) \rightarrow 0$ with $n \rightarrow \infty$, (2.11) extends (1.8) in the sense that $[D_n \geq \lambda_n]$ is replaced by $[D_m \geq \lambda_m \text{ for some } m \geq n]$. This extension is comparable to Theorem 1.4 of Strassen (1967) which provides similar extension of the probability of moderate deviations for sample cumulative sums, studied earlier by Cramér (1938), Linnik (1961), Rubin and Sethuraman (1965), and others.

3. ~~The proofs of Theorems 1 and 2.~~ For a positive k , we define

$$(3.1) \quad I_n(k\phi) = (2\pi)^{-\frac{1}{2}} \int_n k\phi'(t) t^{-\frac{1}{2}} \exp\{-\frac{1}{2}k^2 t^{-1} \phi^2(t)\} dt, \quad n \geq 1.$$

Note that by (2.3), (2.5) and (3.1), for every $n \geq 1$, $k \geq 2$,

$$(3.2) \quad 0 < \frac{1}{2} J_n(k\phi) < I_n(k\phi) < \frac{3}{5} J_n(k\phi).$$

We consider first the following.

Lemma 3.1. Under (2.1), (2.2), (2.3) and (2.7), as $n \rightarrow \infty$,

$$(3.3) \quad \{\log I_n(k\phi)\}/v_k(n) \rightarrow -1, \text{ for every } k \geq 2.$$

Proof. By virtue of (3.2), it suffices to show that as $n \rightarrow \infty$

$$(3.4) \quad \{\log J_n(k\phi)\}/v_k(n) \rightarrow -1, \text{ for every } k \geq 2.$$

Since $u \exp\{-\frac{1}{2}u^2\}$ is \downarrow in $u(\geq 1)$, on denoting t_0 by $\psi^2(t_0) = k^{-2}$, we obtain that for all $n \geq t_0$,

$$(3.5) \quad \sum_{m=n}^{\infty} km^{-1} \psi(m+1) \exp\{-\frac{1}{2}k^2\psi^2(m+1)\} \leq 2\sqrt{2\pi} J_n(k\phi) \\ \leq \sum_{m=n}^{\infty} km^{-1} \psi(m) \exp\{-\frac{1}{2}k^2\psi^2(m)\}; \quad k \geq 2.$$

Let us now define a set of points $\{n_s, s=0,1,\dots\}$ by $n_0 = n$, and

$$(3.6) \quad n_s = [\exp\{(\log n)^{1+\epsilon s}\}] + 1, \quad s = 0,1,\dots, \quad \epsilon > 0,$$

where $[k]$ denotes the integral part of $k(\geq 0)$, and ϵ is arbitrarily small. Then, the right hand side of (3.5) is bounded from above by

$$(3.7) \quad \sum_{s=0}^{\infty} k\psi(n_s) \exp\{-\frac{1}{2}k^2\psi^2(n_s)\} \sum_{m=n_s}^{n_{s+1}-1} m^{-1} \\ \leq [(\log n)^{\epsilon} - 1] \sum_{s=0}^{\infty} k\psi(n_s) \exp\{-\frac{1}{2}k^2\psi^2(n_s)\} \log n_s \\ \leq (\log n)^{\epsilon} k[\exp\{-v_k(n)\}] [\sum_{s=0}^{\infty} \chi_n(s)],$$

where $\chi_n(0) = 1$, and for $s \geq 1$,

$$\chi_n(s) = \exp\{-\frac{1}{2}k^2[\psi^2(n_s) - \psi^2(n)] + \log[\psi(n_s)/\psi(n)] + \log \log n_s - \log \log n\} \\ = \exp\{-[v_k(n_s) - v_k(n)]\}.$$

We prove the lemma first for a $k > 2$ i.e., $k = 2 + \delta$, $\delta > 0$. Since $v_k(n) - v_2(n) = (\frac{1}{2}k^2 - 2)\psi^2(n) > 2\delta\psi^2(n)$, by (2.6) and the remark made thereafter, for large n and every $s \geq 1$,

$$(3.9) \quad v_{2+\delta}(n_s) - v_{2+\delta}(n) > \delta(\log \log n_s - \log \log n) = \delta\epsilon s \log \log n.$$

Hence, for every $\epsilon > 0$, $\delta > 0$, for large n ,

$$(3.10) \quad \sum_{s=0}^{\infty} \chi_n(s) \leq \sum_{s=0}^{\infty} [(\log n)^{-\delta \varepsilon}]^s = \{1 - (\log n)^{-\delta \varepsilon}\}^{-1} < K_{\varepsilon, \delta} < \infty.$$

Therefore, for $k = 2 + \delta$, $\delta > 0$, we have by (3.7) and (3.8) that

$$(3.11) \quad \limsup_n \{[\log J_n(k\phi)]/v_k(n)\} \leq -1 + \varepsilon [\limsup_n \{(\log \log n)/v_k(n)\}].$$

Now, by the remark following (2.6), for $k = 2 + \delta$, $v_k(n) > v_2(n) + 2\delta\psi^2(n) > 2\delta\psi^2(n)$

$> \delta \log \log n$, so that the right hand side of (3.11) is bounded above by $-1 + \varepsilon/\delta$.

Thus, for every $\delta > 0$ and $\eta > 0$, there exists an $\varepsilon > 0$, such that $\varepsilon/\delta < \eta$ and

$$(3.12) \quad \limsup_n \{[\log J_n(k\phi)]/v_k(n)\} \leq -1 + \eta \text{ for } k = 2 + \delta, \delta > 0.$$

In a similar way, by working with the left hand side of (3.7), it follows that for every $\eta > 0$,

$$(3.13) \quad \liminf_n \{[\log J_n(k\phi)]/v_k(n)\} \geq -1 - \eta \text{ for every } k = 2 + \delta, \delta > 0.$$

Thus, for every $k > 2$,

$$(3.14) \quad \lim_{n \rightarrow \infty} \{[\log J_n(k\phi)]/v_k(n)\} = -1.$$

Now to consider the case of $k = 2$, we note that, by definition in (2.3), $J_n(k\phi)$ is \downarrow in k , and hence, for every $\delta > 0$,

$$(3.15) \quad [\log J_n(2\phi)]/v_2(n) \geq [v_{2+\delta}(n)/v_2(n)] \{[\log J_n((2+\delta)\phi)]/v_{2+\delta}(n)\}.$$

Thus, from (2.7), (3.13) and (3.15), it follows that for every $\eta > 0$,

$$(3.16) \quad \liminf_n \{[\log J_n(2\phi)]/v_2(n)\} \geq -1 - \eta.$$

Also, by (2.3) and a few simple steps, we obtain on using (2.7) that for every $\eta > 0$, there exists a $\delta > 0$, such that as $n \rightarrow \infty$,

$$\begin{aligned}
(3.17) \quad J_n(2\phi) &= (2\pi)^{-1/2} \int_{t=n}^{\infty} \exp\{-v_2(t)\} (\log t)^{-1} d(\log t) \\
&= (2\pi)^{-1/2} \int_{t=n}^{\infty} \exp\{-v_{2+\delta}(t) [v_2(t)/v_{2+\delta}(t)]\} (\log t)^{-1} d(\log t) \\
&\leq (2\pi)^{-1/2} \int_{t=n}^{\infty} \exp\{-v_{2+\delta}(t) (1-\eta)\} (\log t)^{-1} d(\log t) \\
&= (2\pi)^{-1/2} \int_{t=n}^{\infty} \exp\{-\frac{1}{2}(1-\eta) (2+\delta)^2 \psi^2(t) + (1-\eta) \log \psi(t)\} (\log t)^{-\eta} t^{-1} dt \\
&\leq (2\pi)^{-1/2} (\log n)^{-\eta} \int_n^{\infty} \exp\{-\frac{1}{2}(1-\eta) (2+\delta)^2 \psi^2(t)\} [\psi(t)]^{1-\eta} t^{-1} dt.
\end{aligned}$$

Therefore, by the same technique as in (3.7) through (3.12), we obtain on choosing $\delta(>0)$ sufficiently small that

$$(3.18) \quad \limsup_n \{[\log J_n(2\phi)]/v_2(n)\} \leq -1+\eta, \eta>0.$$

By letting η in (3.16) and (3.18) to be arbitrarily small, we conclude that

$$(3.19) \quad \lim_{n \rightarrow \infty} \{[\log J_n(2\phi)]/v_2(n)\} = -1. \quad \text{Q.E.D.}$$

Lemma 3.2. Under (2.1), (2.2), (2.3) and (2.7) for every $x \in \mathbb{R}^P$,

$$\begin{aligned}
(3.20) \quad \lim_{n \rightarrow \infty} \{[\log P\{m^{\frac{1}{2}} [F_m(\tilde{x}) - F(\tilde{x})] \geq \psi(m) \text{ for some } m \geq n\}] / v_{k(\tilde{x})}(n)\} \\
= -1, \text{ where } k(\tilde{x}) = \{2F(\tilde{x})[1-F(\tilde{x})]\}^{-1} (>2).
\end{aligned}$$

Proof. Since $F_m(\tilde{x})$ involves an average of iidrv's which assume only the values 0 and 1, the existence of the moment generating function is insured, and $mV[F_m(\tilde{x}) - F(\tilde{x})] = F(\tilde{x})[1-F(\tilde{x})] (\leq \frac{1}{4} \text{ for all } \tilde{x} \in \mathbb{R}^P)$. Thus, by theorems 1.4 and 4.9 of Strassen (1967), as $n \rightarrow \infty$,

$$(3.21) \quad P\{m^{\frac{1}{2}}[F_m(\tilde{x}) - F(\tilde{x})] \geq \psi(m) \text{ for some } m \geq n\} \\ \sim I_n(k(\tilde{x})\phi),$$

where $I_n(k\phi)$ is defined by (3.1) and $k(\tilde{x})$ by (3.20), and \sim indicates that the ratio of the two sides converges to 1 as $n \rightarrow \infty$. The rest of the proof follows from Lemma 3.1 and (3.21). Q.E.D.

Lemma 3.3. Under (2.1), (2.2), (2.3) and (2.7)

$$(3.22) \quad \liminf_n \{[\log P_n(\psi)]/\nu_2(n)\} \geq -1.$$

Proof. By (1.2), for every $n \geq 1$.

$$(3.23) \quad P_n^+(\psi) \geq \sup_{\tilde{x} \in R^p} [P\{m^{\frac{1}{2}}[F_m(\tilde{x}) - F(\tilde{x})] \geq \psi(m) \text{ for some } m \geq n\}] \\ \geq P\{m^{\frac{1}{2}}[F_m(\tilde{x}^0) - F(\tilde{x}^0)] \geq \psi(m) \text{ for some } m \geq n\},$$

where \tilde{x}^0 is a point (in R^p) for which $F(\tilde{x}^0) = \frac{1}{2}$. Then, by Lemma 3.2, we obtain that

$$(3.24) \quad \liminf_n \{[\log P_n^+(\psi)]/\nu_2(n)\} \\ \geq \lim_{n \rightarrow \infty} \{[\log P\{m^{\frac{1}{2}}[F_m(\tilde{x}^0) - F(\tilde{x}^0)] \geq \psi(m) \text{ for some } m \geq n\}]/\nu_2(n)\} \\ = -1, \text{ as } k(\tilde{x}^0) = 2.$$

In a similar way, it follows that

$$(3.25) \quad \liminf_n \{[\log P_n^-(\psi)]/\nu_2(n)\} \geq -1,$$

and hence, the lemma follows from (3.24) and (3.25). Q.E.D.

For every $n \geq 1$, let C_n be the σ -field generated by the unordered $X_{\tilde{1}}, \dots, X_{\tilde{n}}$ and by $X_{\tilde{n}+1}, X_{\tilde{n}+2}, \dots$; C_n is \downarrow in n . Then we have the following.

Lemma 3.4. $\{D_n^+, C_n; n \geq 1\}$ and $\{D_n^-, C_n; n \geq 1\}$ are both non-negative reverse submartingales for every $p > 1$.

Proof. For every $n \geq 1$, let $Z_{\sim n}$ (a random vector) be a point in R^p where $F_n(\underline{x}) - F(\underline{x})$ attains a maximum i.e.,

$$(3.26) \quad D_n^+ = \sup\{F_n(\underline{x}) - F(\underline{x}) : \underline{x} \in R^p\} = F_n(Z_{\sim n}) - F(Z_{\sim n});$$

$Z_{\sim n}$ need not be unique. Also, by (1.2), $D_n^+ \geq 0$ for $n \geq 1$. Therefore, using the fact that

$$(3.27) \quad \begin{aligned} D_n^+ &= \sup\{F_n(\underline{x}) - F(\underline{x}) : \underline{x} \in R^p\} \\ &\geq F_n(Z_{\sim{n+1}}) - F(Z_{\sim{n+1}}), \text{ for every } n \geq 1, \end{aligned}$$

we obtain that

$$(3.28) \quad \begin{aligned} E(D_n^+ | C_{n+1}) &\geq E\{F_n(Z_{\sim{n+1}}) - F(Z_{\sim{n+1}}) | C_{n+1}\} \\ &= n^{-1} \sum_{i=1}^n E\{[c(Z_{\sim{n+1}} - X_i) - F(Z_{\sim{n+1}})] | C_{n+1}\} \\ &= E\{[c(Z_{\sim{n+1}} - X_1) - F(Z_{\sim{n+1}})] | C_{n+1}\} \\ &= (n+1)^{-1} \sum_{j=1}^{n+1} [c(Z_{\sim{n+1}} - X_j) - F(Z_{\sim{n+1}})] \\ &= F_{n+1}(Z_{\sim{n+1}}) - F(Z_{\sim{n+1}}) \\ &= D_{n+1}^+ (> 0), \text{ for every } n \geq 1. \end{aligned}$$

In a similar way, it follows that for every $n \geq 1$, $D_n^- \geq 0$, and

$$(3.29) \quad E(D_n^- | C_{n+1}) \geq D_{n+1}^-, \text{ for every } n \geq 1.$$

Hence the lemma follows.

Lemma 3.5. For each $p (> 1)$ and every $\epsilon > 0$, there exists a positive $c(p, \epsilon) (< \infty)$,

such that for every $k > 0$,

$$(3.30) \quad E[(n^{\frac{1}{2}} D_n^+)^k] \leq c(p, \varepsilon) (2-\varepsilon)^{-k/2} \overline{1/2k+1}, \text{ for all } F,$$

and the same result holds for $\{D_n^-\}$ and $\{D_n\}$.

Proof. It follows from Theorem 1 of Kiefer (1961) that for each $p (> 1)$ and every $\varepsilon > 0$, there exists a positive $c(p, \varepsilon) (< \infty)$, such that for every $n \geq 1$, $r > 0$ and F ,

$$(3.31) \quad P\{n^{\frac{1}{2}} D_n^+ > r\} \leq P\{n^{\frac{1}{2}} D_n^+ \geq r\} \leq c(p, \varepsilon) \exp\{-(2-\varepsilon)r^2\}.$$

Therefore, by routine steps,

$$(3.32) \quad \begin{aligned} E[(n^{\frac{1}{2}} D_n^+)^k] &= \int_0^\infty x^k dP\{n^{\frac{1}{2}} D_n^+ \leq x\} \\ &= k \int_0^\infty x^{k-1} P\{n^{\frac{1}{2}} D_n^+ > x\} dx \\ &\leq k c(p, \varepsilon) \int_0^\infty x^{k-1} \exp\{-(2-\varepsilon)x^2\} dx \\ &= c(p, \varepsilon) \overline{1/2k+1} (2-\varepsilon)^{-k/2}. \end{aligned}$$

The other two cases follow similarly. Q.E.D.

Lemma 3.6. If $\{c_m\}$ be non-decreasing, then for every $N > n > 1$, $t > 0$,

$$(3.33) \quad \begin{aligned} P\{\max_{n \leq m \leq N} c_m D_m^+ \geq t\} &\leq t^{-k} c(p, \varepsilon) \overline{1/2k+1} (2-\varepsilon)^{-k/2} \\ &\quad \left\{ n^{-k/2} c_n^k + \sum_{s=n+1}^N (c_s^k - c_{s-1}^k) s^{-k/2} \right\} \end{aligned}$$

The proof is a direct consequence of Lemmas 3.4 and (3.5) and of Theorem 1 of Chow (1960) which extends the Hájek-Rényi inequality for sub-martingales.

Lemma 3.7. Under (2.3) and (2.7), for all non-decreasing $\{\psi(t)\}$ such that $\psi(t) < ct^{1/3}$, $c < \infty$,

$$(3.34) \quad \limsup_n \{[\log P_n^+(\psi)] / v_2(n)\} = -1,$$

and the same result holds for $\{P_n^-(\psi)\}$ and $\{P_n(\psi)\}$.

Proof. We only consider the proof for $\{P_n^+(\psi)\}$; the case of $\{P_n^-(\psi)\}$ follows on identical lines. Also, noting that $P_n^+(\psi) \leq P_n(\psi) \leq P_n^+(\psi) + P_n^-(\psi)$, the case of $\{P_n(\psi)\}$ follows trivially. We consider the events

$$(3.35) \quad A_n(\lambda\psi) = \{m^{\frac{1}{2}D_m^+} \geq \lambda\psi(m) \text{ for some } m \geq n\}, \quad n \geq 1, \lambda > 0,$$

$$(3.36) \quad A_n^o(\lambda\psi) = \{\bigcup_{s=0}^{\infty} [m^{\frac{1}{2}D_m^+} \geq \lambda\psi(n_s) \text{ for some } n_s \leq m < n_{s+1}]\}, \quad n \geq 1,$$

where $n_0 = n$ and $\{n_s\}$ is an increasing sequence of positive integers to be chosen later on. Since $\psi(t)$ is \uparrow in t , we have

$$(3.37) \quad P(A_n(\lambda\psi)) = P_n^+(\lambda\psi) \leq P(A_n^o(\lambda\psi)) \text{ for all } \lambda > 0, n \geq 1.$$

Then, by Lemma 3.6 and a few simple steps, we obtain that

$$(3.38) \quad \begin{aligned} P(A_n^o(\lambda\psi)) &\leq \sum_{s=0}^{\infty} P\{n_s \leq \max_{m < n_{s+1}} m^{\frac{1}{2}D_m^+} > \psi(n_s)\} \\ &\leq e(p, \epsilon) \sum_{s=0}^{\infty} [\lambda\psi(n_s)]^{-k_{n,s}(2-\epsilon)} e^{-k_{n,s}/2} \sqrt{\frac{1}{\frac{1}{2}k_{n,s}+1}} \\ &\quad \{1 + \sum_{j=n_s}^{n_{s+1}-1} [1 - (1-j)^{-1}]^{\frac{1}{2}k_{n,s}}\}, \quad \epsilon > 0, \end{aligned}$$

where we let

$$(3.39) \quad k_{n,s} = 2\lambda^2(2-\epsilon)\psi^2(n_s), \quad s=0,1,\dots$$

By Sterling's approximations, for large n , for every $s=0,1,\dots$,

$$(3.40) \quad \begin{aligned} \log \sqrt{\frac{1}{\frac{1}{2}k_{n,s}+1}} &= -\frac{1}{2}\log(2\pi) + \frac{1}{2}(k_{n,s}+1)\log\left(\frac{1}{\frac{1}{2}k_{n,s}}\right) - \frac{1}{2}k_{n,s} + 0(1) \\ &= -\frac{1}{2}\log(2\pi) + [(2-\epsilon)\lambda^2\psi^2(n_s) + \frac{1}{2}]\log[(2-\epsilon)\lambda^2\psi^2(n_s)] - (2-\epsilon)\lambda^2\psi^2(n_s) + 0(1). \end{aligned}$$

Also, by (3.39) and (2.1),

$$(3.41) \quad 1 - (1-j)^{-1}]^{\frac{1}{2}k_{n,s}} = \frac{1}{2}k_{n,s}j^{-1} + 0([k_{n,s}/j]^2),$$

so that for every $s \geq 0$,

$$(3.42) \quad \sum_{j=n_s}^{n_{s+1}-1} [1-(1-j^{-1})^{\frac{1}{2}k_{n,s}}] = \frac{1}{2}k_{n,s} \left(\frac{1}{n_s+1} + \dots + \frac{1}{n_{s+1}-1} \right) + O(k_{n,s}^2 n_s^{-1})$$

$$\leq \frac{1}{2}k_{n,s} (\log n_{s+1} - \log n_s) + O(1), \text{ by (2.1) and (3.39).}$$

Let us now set $\lambda = 1 + \frac{1}{2}\varepsilon$, so that $\frac{1}{2}k_{n,s} = (2 + \varepsilon')\psi^2(n_s)$ where $\varepsilon' = \varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{4}\varepsilon^4 > 0$ for every $0 < \varepsilon < \varepsilon_0 (> \frac{1}{2})$. We consider first the case of $\psi(t)$ satisfying (2.1), (2.2), (2.3) and (2.7), such that $\psi^2(t) < C \log t$, $0 < C < \infty$.

We set then

$$(3.43) \quad \log n_s \approx (\log n)^{1 + \varepsilon s/3}, \quad s=0,1,\dots, 0,$$

where \approx indicates that n_s is the least positive integer for which the left hand side is \geq the right hand side. Then, for every $s \geq 0$,

$$(3.44) \quad \log n_{s+1} - \log n_s = (\log n_s)((\log n)^{\varepsilon/3} - 1) \leq (\log n)^{\varepsilon/3} (\log n_s),$$

so that by (3.38) through (3.44), we have for large n ,

$$(3.45) \quad P(A_n^O((1 + \varepsilon)\psi)) \leq K_\varepsilon \left[\sum_{s=0}^{\infty} \psi^2(n_s) \exp\{-\nu_2(n_s) - \varepsilon' \psi^2(n_s) + (\varepsilon/3) \log \log n_s\} \right]$$

$$= K_\varepsilon \psi^2(n) \exp\{-\nu_2(n) - \varepsilon' \psi^2(n) + (\varepsilon/3) \log \log n\} \sum_{s=0}^{\infty} \chi_n(s),$$

where $K_\varepsilon (< \infty)$ depends only on $\varepsilon (> 0)$, $\nu_2(n)$ is defined by (2.6), and

$$(3.46) \quad \chi_n(s) = \exp\{-[\nu_2(n_s) - \nu_2(n)] - \varepsilon' [\psi^2(n_s) - \psi^2(n)] + (\varepsilon/3) [\log \log n_s - \log \log n] + 2 \log[\psi(n_s)/\psi(n)]\}, \quad s \geq 0.$$

By the remark made after (2.6) and (3.44), it follows that for large n ,

$$(3.47) \quad \chi_n(s) \leq \exp\{-[\frac{1}{2}\varepsilon \varepsilon' s \log \log n - (\varepsilon^2 s/9) \log \log n]$$

$$[-(\varepsilon'/12) (\psi^2(n_s) - \psi^2(n)) - \log(\psi^2(n_s)/\psi^2(n))]\}$$

$$\leq \exp\{-(\varepsilon^2 s/18) \log \log n\} = (\log n)^{-(\varepsilon^2 s/18)}, \text{ for } 0 < \varepsilon < \varepsilon_0 (< \frac{1}{2}),$$

as $\varepsilon' = \varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{4}\varepsilon^2 > 23\varepsilon/32 > (2/3)\varepsilon$ for all $0 < \varepsilon < \varepsilon_0 (< \frac{1}{2})$, and for large n , $(\varepsilon'/12)[\psi^2(n_s) - \psi^2(n)]$ can be made larger than $\log[\psi^2(n_s)/\psi^2(n)]$. Therefore, for n sufficiently large,

$$(3.48) \quad 1 \leq \sum_{s=0}^{\infty} \chi_n(s) \leq \{1 - (\log n)^{-\varepsilon^2/18}\} + 1 \text{ as } n \rightarrow \infty.$$

In a similar way, it follows that for every $\varepsilon > 0$, as $n \rightarrow \infty$,

$$(3.49) \quad \varepsilon' \psi^2(n) \geq (\varepsilon'/3) \log \log n + 2 \log \psi(n), \text{ when (2.3) holds.}$$

Therefore from (3.45) through (3.49), it follows that for every $\varepsilon (0 < \varepsilon < \varepsilon_0 < \frac{1}{2})$,

$$(3.50) \quad \limsup_n \{[\log P(A_n^0((1+\frac{1}{2}\varepsilon)\psi))]/v_2(n)\} \leq -1,$$

and (3.34) follows from (2.7), (3.37) and (3.50) by letting $\varepsilon (> 0)$ to be arbitrarily small.

We next consider the case when $\psi(t)$ is \uparrow in t such that $(\log t)/\psi^2(t) \rightarrow 0$ and $t^{-1/3}\psi^2(t) \leq c < \infty$ as $t \rightarrow \infty$. In this case, we let

$$(3.51) \quad n_s = [n(1+\varepsilon)^s], \quad s=0,1,\dots; \quad \varepsilon > 0,$$

and virtually repeat the steps (3.44)-(3.50), with some further simplifications; for brevity, the details are omitted. Q.E.D.

Returning now to the proof of theorem 1, we note that (2.9) follows directly from Lemmas 3.3 and 3.7, and (2.10) follows from (2.6), (2.9) and the fact that $\psi^2(n)/\log \log n \rightarrow \infty$ with $n \rightarrow \infty$ implies that $v_2(n)/\psi^2(n) \rightarrow -2$ as $n \rightarrow \infty$.

For the proof of Theorem 2, we define by \tilde{D}_n^+ , \tilde{D}_n^- and \tilde{D}_n the Kolmogorov-Smirnov statistics, defined by (1.2), (1.3) and (1.4), but based on the observations X_{11}, \dots, X_{n1} ; all these statistics are based on univariate observations and thereby are strictly distribution-free when the first marginal df of F is

continuous. Then,

$$(3.52) \quad D_n^+ \geq \tilde{D}_n^+, \quad D_n^- \geq \tilde{D}_n^- \quad \text{and} \quad D_n \geq \tilde{D}_n \quad \text{for all } n \geq 1.$$

Also, as in Gnedenko, Koroluk and Skorokhod (1961), for every $0 < x < cn^{1/6}$,

$$(3.53) \quad \begin{aligned} P\{n^{1/2} \tilde{D}_n^+ \geq x\} &= P\{n^{1/2} \tilde{D}_n^- \geq x\} \\ &= \exp\{-2x^2\} \{1 + 2x/3n^{1/2} + o(1/n)\}; \end{aligned}$$

$$(3.54) \quad P\{n^{1/2} \tilde{D}_n^- > x\} = 2\exp\{-2x^2\} \{1 + o(n^{-1/3})\}.$$

Therefore, by (1.7), (3.52) and (3.54) we have

$$(3.55) \quad \begin{aligned} 2\exp\{-2\psi^2(n)\} \{1 + o(n^{-1/3})\} &\leq P\{D_n^- \geq n^{-1/2}\psi(n)\} \\ &\leq c(p, \varepsilon) \exp\{-(2-\varepsilon)\psi^2(n)\}, \end{aligned}$$

for all $0 \leq \psi^2(n) < cn^{1/3}$, $0 < c < \infty$. Further, by (2.8),

$$(3.56) \quad P_n(\psi) \geq P\{D_n^- \geq n^{-1/2}\psi(n)\} \quad \text{for every } n \geq 1,$$

and hence, by (3.55)

$$(3.57) \quad \liminf_n \{[\log P_n(\psi)]/\psi^2(n)\} \geq -2.$$

Since $v_2(n)/\psi^2(n) \rightarrow 2$ as $n \rightarrow \infty$ (under the hypothesis of Theorem 2), we obtain from Lemma 3.7 that

$$(3.58) \quad \limsup_n \{[\log P_n(\psi)]/\psi^2(n)\} \leq -2,$$

which completes the proof for $\{P_n(\psi)\}$; the cases of $\{P_n^+(\psi)\}$ and $\{P_n^-(\psi)\}$ follow similarly.

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