MAXIMUM LIKELIHOOD ESTIMATION FOR A SPECIAL TYPE OF GROUPED DATA WITH AN APPLICATION TO A DOSE-RESPONSE PROBLEM

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SUMMARY

A short-cut method is given for calculating grouped maximum likelihood estimates when the data is relatively coarsely grouped in some directions, but more finely grouped in others. The algebraic details are then worked out for a dose-response problem that generates this kind of data. The situation envisaged is a variation on the usual quantal response problem in that dosage levels are taken to be random but grouped. Finally, the method is applied both to real and simulated response data conforming to this pattern and shown to work well in practice.

1. Introduction

When multivariate data that is basically continuous is only available in grouped form, the problem of finding the maximum likelihood (or equivalent B.A.N.) estimates of parameters presents computing problems that are usually far from trivial. An alternative to the fully grouped maximum likelihood

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solution is to carry out the corresponding analysis for continuous data after placing the observations at the mid-points of their respective groups, and then apply corrections to this approximate mid-point (a.m.p.) solution to bring it back into line. Since continuous data is much more easily handled than grouped data in general, this corrected a.m.p. solution, when available, can save considerable effort. A detailed discussion of the problem is given in Fryer and Pethybridge (1972). One of the assumptions made there was that the underlying density function could be adequately represented in a group by the first few terms of its Taylor expansion about the mid-point of that group. It was also pointed out there that examples existed for which this could only be justified for a proper subset of the variables involved, and that this type of problem needed special attention. The main purpose of this paper is to give some consideration to this kind of situation.

We shall suppose for the sake of simplicity that we are dealing with a two-variable problem. It is assumed that the variable y is too coarsely grouped to allow the approximation of the density function in a group by the first few terms of a Taylor series, but that for the other variable, z, this representation is justifiable. Firstly, we show how formulae can be derived for the appropriate 'corrections' to the a.m.p. solution when z is taken to be concentrated at its group mid-points but y remains grouped, and this is the object of the paper in general. The form of these correction terms naturally will depend to some extent on the shape of the underlying density. By making certain assumptions about the underlying distribution, we then produce simple computable answers for a particular case. It will be seen that the specific assumptions we make here amount to a description of a special kind of dose-response problem with random but grouped dosages. Finally, the a.m.p., adjusted a.m.p., and fully grouped

maximum likelihood solutions are compared for this problem using both real and simulated data.

2. Corrections to the a.m.p. solution in the general case

We first introduce some notation, and then derive some intermediate results that will be needed if we are to produce the correction formulae. Suppose that the range of y is divided into adjacent intervals, and that the i^{th} interval or group has upper and lower limits of y_i and y_{i-1} respectively. In a similar way, suppose that the range of z is also divided into groups, the j^{th} group having upper and lower end-points of z_j and z_{j-1} and write $(z_j - z_{j-1})$ as m_j. Denote the conditional density of y given z by $g(y|z; \underset{\sim}{\theta}_1)$ where $\underset{\sim}{\theta}_1$ is an unknown parameter, and similarly let the marginal density of z be written $f(z; \underset{\sim}{\theta}_2)$. The probability of an observation falling in the i^{th} y-group and j^{th} z-group is then

$$p_{ij} = \int_{z_{j-1}}^{z_j} Q_i(z; \underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2)dz \qquad (1)$$

where $Q_i(z; \underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2) = G_i(z; \underset{\sim}{\theta}_1)f(z; \underset{\sim}{\theta}_2)$ and where

$$G_i(z; \underset{\sim}{\theta}_1) = \int_{y_{i-1}}^{y_i} g(y|z; \underset{\sim}{\theta}_1)dy.$$

We can approximate to p_{ij} so avoiding the full integration problem for z in the following way. First expand Q_i in a Taylor series about the mid-point of the j^{th} z-group, and then integrate the result with respect to z which is now of course a very simple operation. This gives us a very convenient form

for p_{ij} and using a logarithmic expansion on it we find that

$$\log p_{ij} = \log m_j + \log Q_{ij} + \frac{m_j^2}{24}\frac{Q_{ij}^{(2)}}{Q_{ij}} + \frac{m_j^4}{384}\left\{\frac{Q_{ij}^{(4)}}{5Q_{ij}} - \frac{2}{3}\left[\frac{Q_{ij}^{(2)}}{Q_{ij}}\right]^2\right\} + O(m_j^6) \qquad (2)$$

An extra subscript j has been added to Q_i to indicate that the function is to be evaluated at the mid-point of the j^{th} z-group. We use $Q_i^{(t)}$ to denote the t^{th} derivative of Q_i with respect to z. Although we have included the term in m_j^4 in (2), it will not be possible in most cases to use it in the type of problem we are discussing here and simultaneously avoid non-trivial integration problems.

We are now in a position to show how the correction terms to the a.m.p. solution can be derived. Let n_{ij} be the number of observations falling in the $(i,j)^{th}$ group with $\sum_{i,j} n_{ij} = n$. Then the log-likelihood function of the grouped sample can be written as $\sum_{ij} n_{ij} \log p_{ij}$, and the maximum likelihood estimating equations for the elements of $\underset{\sim}{\phi} = (\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2)$ are

$$\sum_{i,j} n_{ij} \frac{\partial}{\partial \phi_a} \log p_{ij} = 0 \qquad \text{for } \phi_a \in \underset{\sim}{\phi} \qquad (3)$$

where a=1,2,...,k, say. Using an appropriate starting value for the estimate of $\underset{\sim}{\phi}$, $\underset{\sim}{\phi}^{(0)}$ say, we can then apply Newton's iterative scheme to (3) to find the grouped maximum likelihood estimate of $\underset{\sim}{\phi}$. But we will find the same first correction term to order m^4 if we approximate $\frac{\partial}{\partial \phi_a} \log p_{ij}$ in (3) by the derivative of the form for $\log p_{ij}$ at (2). The value of $\underset{\sim}{\phi}^{(0)}$ that we use is the

maximum likelihood estimate of $\underset{\sim}{\phi}$ found from placing the z-data at mid-points of the groups. For the technique to be of much use, the a.m.p. solution, $\underset{\sim}{\phi}^{(0)}$, must be relatively simple to calculate either directly or indirectly using a graphical method. If we use (2) to order m^2 only, then to find the adjustments to the a.m.p. solution, denoted $\underset{\sim}{\Delta}_{\phi}$, we must solve the linear equations

$$\left\{ \sum_{ij} n_{ij} \frac{m_j^2}{24} \frac{\partial}{\partial \phi_a} \left[\frac{Q_{ij}^{(2)}}{Q_{ij}} \right] \right\}_{\underset{\sim}{\phi}=\underset{\sim}{\phi}^{(0)}} =$$

$$- \left\{ \Delta_{\phi_1} \sum_{ij} n_{ij} \frac{\partial^2 \log Q_{ij}}{\partial \phi_1 \partial \phi_a} + \ldots \Delta_{\phi_k} \sum_{ij} n_{ij} \frac{\partial^2 \log Q_{ij}}{\partial \phi_k \partial \phi_a} \right\}_{\phi=\phi^{(0)}} \qquad (4)$$

for a=1,2,...,k. The corrected a.m.p. solution is then $(\underset{\sim}{\phi}^{(0)} + \underset{\sim}{\Delta}_{\phi})$. This system is formally identical to that given by Lindley (1950) who considered the single variable multiparameter case. The only difference is that Q_i is an integral in (4) but represents a simple density function in Lindley's result. In many cases, it will be found that when solving (4) to find $\underset{\sim}{\Delta}_{\phi}$ we are still faced with integration problems for y. The dose-response problem that we consider later is an exception and gives a very simple form for $\underset{\sim}{\Delta}_{\phi}$ immediately.

Of course, in such a problem we would want not only parameter estimates but also some idea of their reliability. For large samples (which we are prepared to assume here), a typical element of the information matrix in the fully grouped case is given by

$$I_{ab} = n \sum_{ij} p_{ij} \frac{\partial^2 \log p_{ij}}{\partial \phi_a \partial \phi_b} .$$

Remembering that our primary intention is to avoid heavy computations, the question we have to ask ourselves is whether we can approximate to I_{ab} without too much effort. One possibility is to substitute the form for $\log p_{ij}$ at (2) to order m^2 and the corresponding approximation for p_{ij} into I_{ab}, and then collect appropriate terms. This gives

$$I_{ab}^{(1)} = n \sum_{ij} m_j \left\{ Q_{ij} \frac{\partial^2 \log Q_{ij}}{\partial\phi_a \partial\phi_b} + \frac{m_j^2}{24}\left[Q_{ij}^{(2)} \frac{\partial^2 \log Q_{ij}}{\partial\phi_a \partial\phi_b} + Q_{ij} \frac{\partial}{\partial\phi_a \partial\phi_b}\left[\frac{Q_{ij}^{(2)}}{Q_{ij}}\right]\right]\right\}. \quad (5)$$

The problem, however, is that $I_{ab}^{(1)}$ still involves non-trivial computations. Alternatively, we could use the cruder but simpler approximation

$$I_{ab}^{(2)} = n \sum_{ij} m_j Q_{ij} \frac{\partial^2 \log Q_{ij}}{\partial\phi_a \partial\phi_b} . \quad (6)$$

Even this is not easily computed in general, but in the specific problem that we consider later on the computations turn out to be almost trivial. Whether or not $I_{ab}^{(2)}$ is an adequate approximation for I_{ab} will depend of course on the context, and in particular on the number and width of the z-groups. In our dose-response problem, it does seem in most cases that the grouping has to become quite coarse before the discrepancy between the obvious estimates of I_{ab} and $I_{ab}^{(2)}$ is serious.

3. Application to a dose-response problem

Let ψ denote the tolerance level of an organism in a quantal response problem, and suppose that it is normally distributed in the population of organisms. Denote the continuous dosage variable by z, and suppose for a given

level of z that the probability of response is of the form

$$G(z;\ \alpha,\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha+\beta z} e^{-\frac{\psi^2}{2}}\, d\psi,$$

where α and β are unknown parameters. Dosage levels are presumed to be outside of the control of the experimenter, but to be characterised by a density function $f(z;\ \underset{\sim}{\theta}_2)$. The values of z in practice are not known precisely, but it is known in which of several possible z-groups any particular dosage falls. Writing $y = (\alpha+\beta z)-\psi$, we see that this problem falls within the general category that we are considering. In this case, $g(y|z;\ \underset{\sim}{\theta}_1)$ is $N(\alpha+\beta z,\ 1)$ and there are two y-groups defined by $y_0 = -\infty$, $y_1 = 0$ and $y_2 = +\infty$. We can therefore set $G_2 = G$ and $G_1 = (1-G)$. To simplify matters, we take all z-intervals to have the same length m, and for the present leave $f(z;\ \underset{\sim}{\theta}_2)$ in arbitrary form. We are mainly interested in the estimation of α and β and the response function G. The parameters appearing in f may not be of direct interest in themselves, but we cannot condition on the values of z since they are unknown. In order to find the fully grouped maximum likelihood estimates of α and β, we have to estimate $\underset{\sim}{\theta}_2$ simultaneously. Whether the density f depends on parameters or not, we still have to use f for the estimation of α and β. This variation on the standard dose-response problem seems only to have been considered previously by Tocher (1949) who treated the simplest possible case of uniformly distributed z.

It is a fairly straightforward problem now to determine the elements of the equations at (4). To abbreviate the notation a little we denote the mid-point of the j^{th} z-group by Z_j, and the function $\frac{f(\partial G/\partial\alpha)^2}{G(1-G)}$ evaluated at Z_j by w_j. In a similar way, f_j denotes f evaluated at Z_j, and f_j' is the first derivative of f with respect to z at that point. Then dropping the zero superscript on initial

parameter estimates, we find after smoothing the equations a little that the solution for Δ_α and Δ_β to order m^2 is

$$\Delta_\alpha = \frac{m^2}{24}\left\{\beta^2\alpha - \frac{2\beta}{A}\left[\left(\sum_j w_j \frac{f_j'}{f_j}\right)\left(\sum_j w_j Z_j^2\right) - \left(\sum_j w_j Z_j\right)\left(\sum_j w_j Z_j \frac{f_j'}{f_j}\right)\right]\right\}$$

$$\Delta_\beta = \frac{m^2}{24}\left\{\beta^3 + \frac{2\beta}{A}\left[\left(\sum_j w_j Z_j\right)\left(\sum_j w_j \frac{f_j'}{f_j}\right) - \left(\sum_j w_j\right)\left(\sum_j w_j Z_j \frac{f_j'}{f_j}\right)\right]\right\} \qquad (7)$$

where

$$A = \left[\left(\sum_j w_j\right)\left(\sum_j w_j Z_j^2\right) - \left(\sum_j w_j Z_j\right)^2\right],$$

and this is clearly independent of the solution for $\underset{\sim}{\Delta}_{\theta_2}$. Even so, we may still want to calculate $\underset{\sim}{\Delta}_{\theta_2}$ because the substitution of 'corrected' versions of parameter estimates for $\underset{\sim}{\theta}_2$ in (7) could well lead to better estimates of α and β.

In order to get computable answers for Δ_α and Δ_β we now have to specify the form of f. Tocher (1949) supposed it to be uniform and found that

$$\Delta_\alpha = \frac{m^2\beta^2\alpha}{24}, \qquad \Delta_\beta = \frac{m^2\beta^3}{24} \qquad (8)$$

If, on the other hand, we supposed z to be exponential with $f = \theta e^{-\theta z}$ $(z \geq 0)$, then solving for Δ_θ as well we find that

$$\Delta_\alpha = \frac{m^2}{24}(\beta^2\alpha + 2\beta\theta) \ ;$$

$$\Delta_\beta = \frac{m^2\beta^3}{24} \ ;$$

$$\Delta_\theta = \frac{m^2\theta^3}{12} \ . \qquad (9)$$

Of course, these expressions for Δ_α and Δ_β remain unchanged if we truncate the basic exponential distribution.

The main case we want to consider, however, is the only other one for which f_j'/f_j is of the form $cz + d$ and so leads to a simple solution for Δ_α and Δ_β; that is to say, the case when z has a truncated normal distribution. This is what we shall be confining our attention to from now on. Denoting the mean of the basic normal density by μ and its standard deviation by σ, f is then given by

$$f = (\sigma\sqrt{2\pi}\Phi)^{-1} \exp\left\{-\tfrac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right\} \qquad \text{for} \qquad v_1 \le z \le v_2 \qquad (10)$$

where

$$\Phi = \int_{v_1}^{v_2} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\tfrac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right\}.$$

The solution for Δ_α and Δ_β is easily found to be

$$\Delta_\alpha = \frac{m^2}{24}\left(\beta^2\alpha - \frac{2\beta\mu}{\sigma^2}\right), \qquad \Delta_\beta = \frac{m^2}{24}\left(\beta^3 + \frac{2\beta}{\sigma^2}\right) \ . \qquad (11)$$

The corrections for μ and σ, however, are more complicated. Again to ease the notation we shall denote

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\tfrac{1}{2}\left[\frac{v_1-\mu}{\sigma}\right]^2 \right\}$$

by N_1 with a similar meaning for N_2. A further simplification comes from denoting $\Phi^{-1}\{(v_1-\mu)^c N_1 - (v_2-\mu)^c N_2\}$ by A_c for c=0 to 3. With this notation, the equations for Δ_μ and Δ_σ can be written as

$$\Delta_\mu\{1+A_1-A_0^2\sigma^2\} + \Delta_\sigma\{A_2\sigma^{-1}+2(\bar{Z}-\mu)\sigma^{-1}-A_0A_1\sigma-A_0\sigma\} = \frac{-m^2(\bar{Z}-\mu)}{12\sigma^2}$$

$$\Delta_\mu\{A_2+2(\bar{Z}-\mu)-A_0A_1\sigma^2-A_0\sigma^2\} + \Delta_\sigma\{3[S^2+(\bar{Z}-\mu)^2]\sigma^{-1}-\sigma-2A_1\sigma-A_1^2\sigma+A_3\sigma^{-1}\}$$

$$= \frac{-m^2}{12\sigma^2}\{2S^2+2(\bar{Z}-\mu)^2-\sigma^2\} \qquad (12)$$

where $\bar{Z} = \sum_j \frac{n_j Z_j}{n}$, $S^2 = \frac{1}{n}\sum_j n_j(Z_j-\bar{Z})^2$ and $n = \sum_j n_j$. If we allow $v_1 \to -\infty$ and $v_2 \to +\infty$, having smoothed some of the terms in (12) a little further, then for the full normal distribution we find that $\Delta_\mu=0$ and $\Delta_\sigma = \frac{-m^2}{24\sigma}$.

Of course, before we can apply the relatively simple correction forms at (11) and (12), we need to calculate $\underset{\sim}{\phi}^{(0)}$ also. Fortunately, this will not usually present any severe computational problems either. The calculation of the a.m.p. estimates of α and β is straightforward since if we place the values of z at the mid-points of the groups, we have then effectively reduced the problem to the standard dose-response situation with known dosage levels. In most cases the corresponding a.m.p. estimates of μ and σ are even simpler to find, since they can be read with an acceptable level of accuracy straight from Cohen's (1957) graph. Sometimes, however, the graph is difficult to read. This happens when the range of z is very narrow and is taken from a tail of the basic normal distribution, for example. Some very severe cases that we have checked have

even led us to question the accuracy of the graph itself in certain regions. When faced with such a problem, we would clearly have to adopt an alternative method for calculating the a.m.p. estimates of μ and σ. We have tried using existing iterative procedures for finding the estimates but they seem to be extremely slow to converge in particularly difficult areas. The only methods we have found to be effective here are optimisation routines based on a direct search of the likelihood function. These, of course, involve the kind of computing that we are trying to avoid, but even so call for substantially less time and effort than would be needed to find the fully-grouped maximum likelihood estimate itself.

Finally, in this section we record the form of $I_{ab}^{(2)}$, the crude information term defined at (6). Some relatively simple algebra shows that if we take $\phi_1=\alpha$ and $\phi_2=\beta$ then the corresponding elements reduce here to

$$I_{ab}^{(2)} = - nm \sum_j \frac{Z_j^{a+b-2} f_j \left(\frac{\partial G_j}{\partial \alpha}\right)^2}{G_j(1-G_j)} \qquad (13)$$

for a,b=1,2 where G_j is simply

$$\int_{-\infty}^{\alpha+\beta Z_j} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, dt.$$

To avoid calculating f_j in (13), we could replace $n\,m\,f_j$ by n_j, the number of observations in the j^{th} group, and then use the form

$$I_{ab}^{(3)} = - \sum_j \frac{n_j Z_j^{a+b-2} \left(\frac{\partial G_j}{\partial \alpha}\right)^2}{G_j(1-G_j)} \qquad (14)$$

where a,b=1,2. The important thing to note is that both (13) and (14) can be calculated quite easily on a simple desk machine using standard statistical tables. Since $I_{rs}^{(2)} = 0$ if $\phi_r \in \underset{\sim}{\theta}_1$ and $\phi_s \in \underset{\sim}{\theta}_2$, we need only invert a 2×2 matrix to find the variances of $\hat{\alpha}$ and $\hat{\beta}$ and the covariance between them.

4. Some numerical illustrations

As our first example, we consider the estimation of the perinatal mortality-birthweight relationship for a population of human infants. Taking perinatal death as the quantal response and birthweight as the level of stimulus, our aim will be to estimate the dose-response relationship for the 220 births weighing between 978g. (2lb. 2½oz.) and 1,772g. (3lb. 14½oz.) which were born in the English counties of Devon and Somerset in 1965. We use this odd-looking weight range to illustrate the method for two reasons. Firstly, because the full birthweight distribution itself corresponds to a mixture of two normal distributions, and so does not conform to the model. The two components are sufficiently well-separated, however, to justify the assumption of a truncated single normal distribution over the range 978-1,772g. Secondly, because the form of the response function in the model only fits the data for birthweights below 2,000g. approximately and above 950g. or so. The weights were originally recorded to the nearest ounce (28.35g.) and we use three different birthweight groupings. Grouping the data is relevant because most mortality-birthweight data is reported in this form.

The various estimates of α and β and the elements of their asymptotic covariance matrix are summarised in Table 1. As regards the parameter estimates, the first entry in each cell is the fully-grouped maximum likelihood solution, the second is the crude a.m.p. solution and the third is the 'corrected' solution using the a.m.p. parameter estimates in the Δ's. These 'corrected' estimates are

then used as parameter estimates in the Δ's and the 'recorrected' a.m.p. solution is the fourth cell entry. The a.m.p. estimates of μ and σ were read directly from Cohen's graph, incidentally. Turning now to the estimates of the elements of the covariance matrix, the first entry is the obvious estimate for the fully-grouped case. The others all use the approximation $I_{ab}^{(3)}$ and are found from substituting the a.m.p. solution (second entry) the 'corrected' a.m.p. solution (third entry) and finally the 'recorrected' a.m.p. solution (fourth entry). We have also calculated the corresponding values of $I_{ab}^{(2)}$ and although they are very similar to the calculations based on $I_{ab}^{(3)}$, on average they do seem to be slightly less satisfactory. The method evidently works well for 7 groups but fails for 4 when the group length is 0.35σ or so. In fact, in this case the crude a.m.p. solution itself is easily preferable.

To supplement these calculations, we have also included the corresponding results for two ranges of z based on some simulated data. Using the parameter values $\alpha=2$, $\beta=-10^{-3}$, $\mu=2{,}000$ and $\sigma=500$, we generated 2,000 observations effectively covering the complete range for z. The values of α and β were changed from those in the mortality-birthweight data so that we could use our response function for any value of z. We then ran out the calculations for different z-ranges, and the results in Tables 2 and 3 are fairly typical of reasonably wide central and off-central intervals. Table 2 is based on 1,892 observations and Table 3 on 1,247; the format used is precisely the same as that adopted for Table 1. The range of z in Table 2 is $\mu \pm 2\sigma$ and in Table 3, $\left(\mu - \frac{\sigma}{2}, \mu + \frac{3\sigma}{2}\right)$.

As regards parameter estimates, the method obviously works very well here for almost all of the groupings. Even when the groups are relatively wide, the 'corrected' and 'recorrected' a.m.p. estimates are still acceptable. The results for the estimated covariance matrix are naturally not quite so good. Using any

of the three approximations based on $I_{ab}^{(3)}$ certainly gives results of the right magnitude in most cases. However, their adequacy is questionable when the grouping is relatively coarse. By the time the group width reaches 0.8σ in Table 2, for example, the difference between the approximations and the first cell entry is of the order of 10-15% or so. The fact that the approximations are underestimates scarcely adds to their attraction.

REFERENCES

Cohen, A. C. (1957). On the solution of estimating equations for truncated and censored samples from normal populations. Biometrika, 44, 225-236.

Fryer, J. G. and Pethybridge, R. J. (1972). Maximum likelihood estimation of a linear regression function when the data is grouped. Appl. Statist. (To appear).

Lindley, D. V. (1950). Grouping corrections and maximum likelihood equations. Proc. Camb. Phil. Soc., 46, 106-110.

Tocher, K. D. (1949). A note on the analysis of grouped probit data. Biometrika, 36, 9-17.

TABLE 1

Parameter estimates and their estimated covariance matrix

for the perinatal mortality-birthweight data

Number of z-groups	Width of each z-group (in units of σ)	Parameter estimates and their estimated covariance matrix				
		$\hat{\alpha}$	$\hat{\beta}.10^3$	$\hat{v}(\hat{\alpha})$	$\hat{v}(\hat{\beta}).10^6$	$\hat{c}(\hat{\alpha},\hat{\beta}).10^3$
28	0.05 (approx.)	2.300	-1.602	0.364	0.169	-0.245
		2.300	-1.601	0.363	0.168	-0.245
		2.300	-1.602	0.364	0.169	-0.246
		2.300	-1.602	0.364	0.169	-0.246
7	0.20 (approx.)	2.434	-1.692	0.379	0.175	-0.255
		2.421	-1.684	0.373	0.173	-0.251
		2.436 }2.434	-1.692 }-1.691	0.374	0.173	-0.252
		2.433	-1.691	0.374	0.173	-0.252
4	0.35 (approx.)	2.428	-1.698	0.381	0.178	-0.257
		2.417	-1.692	0.366	0.171	-0.248
		2.468	-1.723	0.368	0.172	-0.249
		2.453	-1.712	0.367	0.172	-0.249

Note: In this and other tables the average of the last two entries to three decimal places is recorded after a curly bracket } when appropriate.

TABLE 2

Parameter estimates and their estimated covariance matrix

for the simulated data using the range $1000 \leq z \leq 3000$

Number of z-groups	Width of each z-group (in units of σ)	Parameter estimates and their estimated covariance matrix				
		$\hat{\alpha}$	$\hat{\beta}.10^3$	$\hat{v}(\hat{\alpha}).10^2$	$\hat{v}(\hat{\beta}).10^9$	$\hat{c}(\hat{\alpha},\hat{\beta}).10^6$
40	0.1	2.037	-1.028	2.163	5.214	-10.400
		2.035	-1.027	2.160	5.206	-10.334
		2.037	-1.028	2.161	5.207	-10.386
		2.037	-1.028	2.161	5.207	-10.386
20	0.2	2.045	-1.032	2.175	5.244	-10.460
		2.036	-1.028	2.158	5.201	-10.373
		2.045	-1.032	2.160	5.208	-10.385 } -10.385
		2.045	-1.032	2.160	5.208	-10.386
10	0.4	2.044	-1.031	2.211	5.329	-10.633
		2.012	-1.015	2.133	5.139	-10.250
		2.043 } 2.043	-1.031	2.144	5.164 } 5.165	-10.301
		2.044	-1.031	2.144	5.165	-10.303
5	0.8	1.974	-0.997	2.335	5.649	-11.264
		1.856	-0.938	2.026	4.875	-9.718
		1.963	-0.991 } -0.995	2.059	4.957 } 4.963	-9.881 -9.893
		1.977	-0.998	2.063	4.968	-9.904

TABLE 3

Parameter estimates and their estimated covariance matrix

for the simulated data using the range $1750 \leq z \leq 2750$

Number of z-groups	Width of each z-group (in units of σ)	Parameter estimates and their estimated covariance matrix				
		$\hat{\alpha}$	$\hat{\beta}.10^3$	$\hat{v}(\hat{\alpha}).10^2$	$\hat{v}(\hat{\beta}).10^9$	$\hat{c}(\hat{\alpha},\hat{\beta}).10^6$
20	0.1	1.891	-0.957	9.257	19.540	-42.229
		1.889	-0.956	9.253	19.533	-42.213
		1.891	-0.957	9.252	19.534	-42.212
		1.891	-0.957	9.252	19.532	-42.208
10	0.2	1.861	-0.943	9.345	19.713	-42.620
		1.854	-0.939	9.283	19.571	-42.324
		1.861	-0.943	9.285	19.580 } 19.581	-42.339 } -42.342
		1.861	-0.943	9.287	19.583	-43.344
5	0.4	1.870	-0.947	9.616	20.313	-43.896
		1.844	-0.934	9.338	19.665	-42.553
		1.873	-0.949	9.355	19.709 } 19.707	-42.639
		1.869	-0.947	9.353	19.706	-42.631