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ON THE RELATIVE EFFICIENCY OF RANDOMIZED RESPONSE MODELS

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1. INTRODUCTION

Warner introduced in [5] a technique for estimating the proportion π of a human population having an unobservable sensitive or stigmatizing characteristic A. The method, which he called "randomized response", is designed to eliminate untruthful responses in sample surveys which would result in a biased estimate of π. Each subject is asked to observe the outcome, concealed to the interviewer, of a randomization device producing one of two outcomes A, $\bar{A}$ with known probabilities p, $1-p$, respectively, and then to respond either "yes" or "no" according as he does or does not belong to the group indicated by the outcome of the randomization device. (A refers to the group with the sensitive characteristic; $\bar{A}$ refers to the complementary group.) The randomization probability p must be chosen sufficiently distant from 0 or 1 to encourage truthful responses, as extreme values will tend to arouse suspicion in the respondent.

A second technique, called the "alternate question randomized response model", is discussed by Greenberg, *et.al.* in [1]. The method involves an unobservable innocuous characteristic Y, possessed by a proportion μ of the population, which respondents would presumably have no reason to conceal, and a randomization device producing one of two outcomes A, Y with respective known probabilities p, $1-p$. Each subject is asked to respond "yes" or "no" according as he does or does not belong to the group indicated by the outcome

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of the randomization device. Both the cases when μ is known and when μ is unknown are discussed in [1]. In the latter case, the total sample must be split into two samples with different randomization probabilities p_1, p_2 in order to estimate π. As with Warner's method, randomization probabilities near 1 must be excluded to encourage truthful responses, but in the alternate question models these need not be bounded away from 0.

The purpose of this note is to prove, as is suggested by numerical evidence in [1], that the variance of the alternate question estimator of π is less than that of Warner's estimator, uniformly in π and μ, provided that p (or $\max(p_1,p_2)$ in the two-sample case) is greater than one-third (roughly).

Section 2[2] deals with the one-sample case, while in Section 3 we correct and extend a proof of this result due to Moors [3] for the two-sample estimator, optimized with respect to sample size allocation and the smaller randomization probability.

Further work on randomized response models may be found in [2,6].

2. THE ONE-SAMPLE CASE

Let π be the unknown proportion of people in the population having the sensitive characteristic A, and μ be the proportion with the innocuous characteristic Y. We assume in this section that μ is known, and write q in place of μ. This situation can always be achieved by incorporating μ in the randomization device (see [1], p. 532). We assume, for both the Warner model and the one-sample alternate question model, a fixed sample size n,

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2 The two models studied in this section were discovered independently by the authors, before learning of [1,3].

and a fixed randomization probability p of selecting the sensitive question: "Do you belong to Group A?". We further assume that p is within the allowable range required to obtain completely truthful responses, and denote by X the number of "yes" responses obtained in the sample.

In the Warner model [5], X is a binomial random variable with parameters n and $\lambda_w = p\pi + (1-p)(1-\pi)$. The latter is independent of π when $p = \frac{1}{2}$, so X carries no information about π in this case. When $p \neq \frac{1}{2}$, the maximum likelihood estimator of π is

$$\hat{\pi}_w = \frac{X - n(1-p)}{n(2p-1)} ,$$

which is unbiased, with variance

$$V(\hat{\pi}_w) = \frac{\lambda_w(1-\lambda_w)}{n(2p-1)^2} = \frac{\pi(1-\pi)}{n} + \frac{p(1-p)}{n(2p-1)^2} . \qquad (2.1)$$

Note that the term $\pi(1-\pi)/n$ in (2.1) is the variance of the direct, nonrandomized estimator of π if truthful responses were obtained. The term $p(1-p)/n(2p-1)^2$ represents the additional variance introduced by the randomization procedure, necessary to obtain truthful responses. Clearly $V(\hat{\pi}_w)$ is symmetric about $p = \frac{1}{2}$ and is minimized by taking p as far from $\frac{1}{2}$ as practicable.

For the one-sample alternate question model [1] X is a binomial random variable with parameters n and $\lambda = p\pi + (1-p)q$. The maximum likelihood estimator of π is

$$\hat{\pi}_1 = \frac{X - n(1-p)q}{np} ,$$

which is unbiased, with variance

$$V(\hat{\pi}_1) = \frac{\lambda(1-\lambda)}{np^2} = \frac{[p\pi + (1-p)q][1 - p\pi - (1-p)q]}{np^2} . \qquad (2.2)$$

Theorem 1. Let $p \in (p_0,1)$, where $p_0 = .339333...$ is the unique solution in $[0,\frac{1}{2}]$ of

$$\frac{1}{1+p^2} = 4p(1-p). \tag{2.3}$$

Then

$$V(\hat{\pi}_1) < V(\hat{\pi}_w)$$

for all $(q,\pi) \in [0,1]^2$.

Proof. We first note that $1/(1+p^2)$ is strictly decreasing from 1 to 4/5 in $[0,\frac{1}{2}]$, and that $4p(1-p)$ is strictly increasing from 0 to 1 in $[0,\frac{1}{2}]$. Hence (2.3) has a unique solution $p_0 \in [0,\frac{1}{2}]$, the approximate value of which is given.

By expanding the numerator of (2.2), then adding and subtracting the term $p^2\pi$, we can write (2.2) in the form

$$V(\hat{\pi}_1) = \frac{\pi(1-\pi)}{n} + \frac{1-p}{np^2} K_p(q,\pi), \tag{2.4}$$

where (see Figure 1)

$$K_p(q,\pi) = p(1-2q)\pi + q[1-(1-p)q]. \tag{2.5}$$

The term $(1-p)K_p(q,\pi)/np^2$ in (2.4) is the additional variance of the alternate question estimator over the direct estimator of π. Comparing (2.1) and (2.4), we see that $V(\hat{\pi}_1) < V(\hat{\pi}_w)$ if and only if

$$K_p(q,\pi) < p^3/(2p-1)^2. \tag{2.6}$$

We must show that (2.6) holds for all $(q,\pi) \in [0,1]^2$. Now K_p is continuous in (q,π) over the compact set $[0,1]^2$, so that a maximum exists and may be obtained by maximizing first with respect to π, then with respect to q.

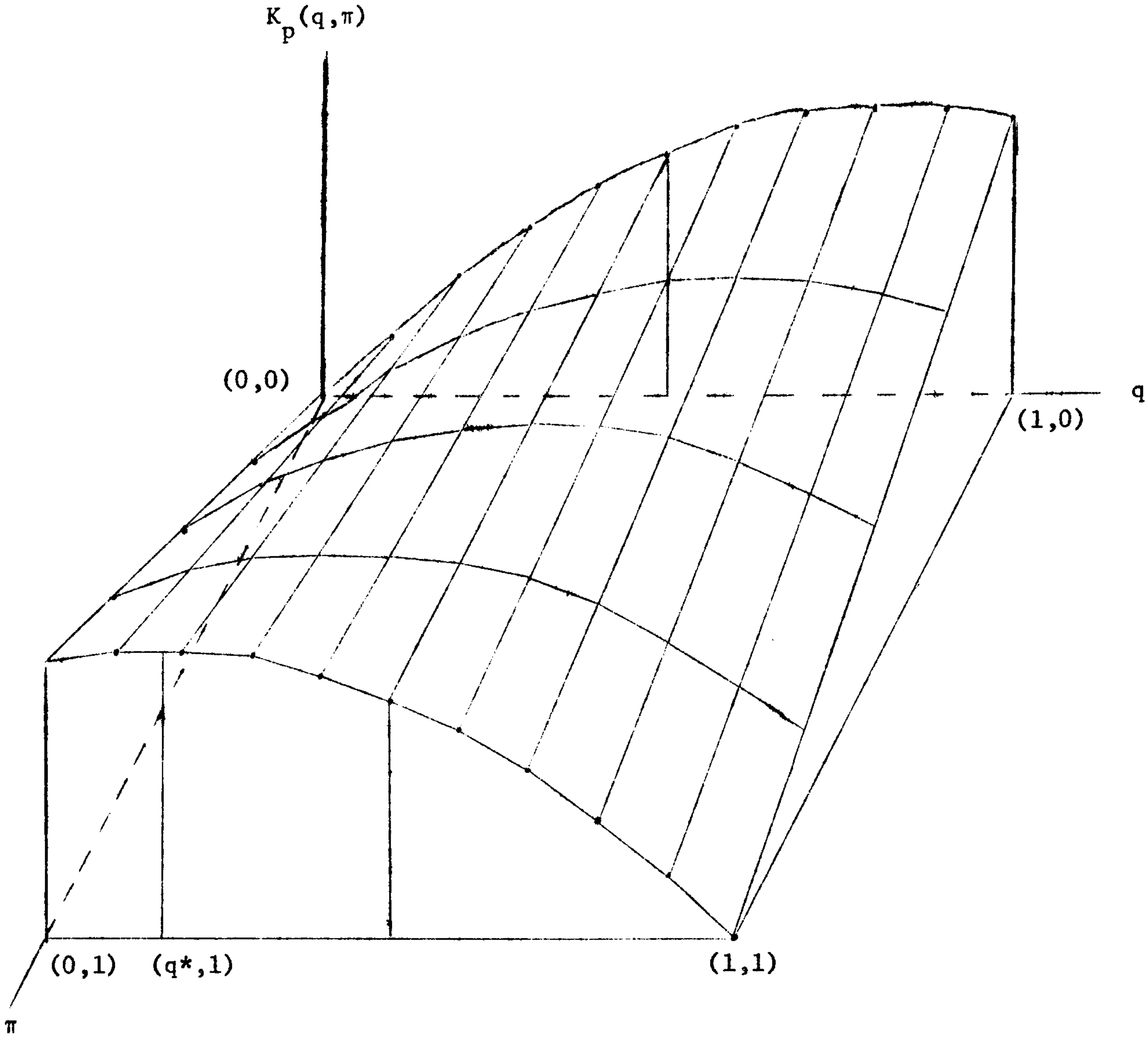

FIGURE 1

The Surface $K_p(q,\pi)$

Further, since $K_p(q,\pi) = K_p(1-q, 1-\pi)$, we may restrict attention to $(q,\pi) \in [0,\frac{1}{2}] \times [0,1]$. Fixing $q \in [0,\frac{1}{2}]$, it is clear that

$$\begin{aligned}\max_{\pi \in [0,1]} K_p(q,\pi) &= K_p(q,1) \\ &= -(1-p)q^2 + (1-2p)q + p,\end{aligned}$$

a concave quadratic in q with maximum at $q^* = (1-2p)/2(1-p)$. We may assume $p \neq \frac{1}{2}$ since $\hat{\pi}_w$ is not defined at $p = \frac{1}{2}$. If $p \in (0,\frac{1}{2})$, then $q^* \in (0,\frac{1}{2})$ also, while if $p \in (\frac{1}{2},1)$, $q^* < 0$. Hence

$$\begin{aligned}\max_{(q,\pi) \in [0,1]^2} K_p(q,\pi) &= \max_{q \in [0,\frac{1}{2}]} K_p(q,1) \\ &= \begin{cases} K_p(q^*,1) = 1/4(1-p), & p \in (0,\frac{1}{2}), \\ K_p(0,1) = p, & p \in (\frac{1}{2},1). \end{cases}\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6), we have $V(\hat{\pi}_1) < V(\hat{\pi}_w)$ for all $(q,\pi) \in [0,1]^2$ if either

(a) $p \in (0,\frac{1}{2})$ and $1/4(1-p) < p^3/(2p-1)^2$,

or

(b) $p \in (\frac{1}{2},1)$ and $p < p^3/2p-1)^2$.

Now the inequality in (b) holds for all $p \in (\frac{1}{2},1)$, while the inequality in (a) is equivalent to $1/(1+p^2) < 4p(1-p)$, hence holds if $p \in (p_0,\frac{1}{2})$. □

The general shape of the surface $K_p(q,\pi)$ is shown in Figure 1, for a value of $p \in (0,\frac{1}{2})$. The q-sections of $K_p(q,\pi)$ are linear in π, while the π-sections are quadratic in q. If $p \in (\frac{1}{2},1)$, the π-sections take their maximum value at values $q^* \notin [0,1]$ for extreme values of π, hence are monotone in $[0,1]$.

It is clear from Figure 1 (and easily proved) that for p fixed, $V(\hat{\pi}_1)$ is minimized over $q \in [0,1]$ by taking $q = 0$ if $\pi < \frac{1}{2}$ or $q = 1$ if

$\pi > \frac{1}{2}$. More realistically, if q is constrained to an interval $[q_1,q_2]$, then $V(\hat{\pi}_1)$ is minimized at $q = q_1$ if $\pi < \frac{1}{2}$ or $q = q_2$ if $\pi > \frac{1}{2}$, provided $q_2 = 1-q_1$. If q_1, q_2 are not symmetric about $\frac{1}{2}$, then one may attain smaller variance at q_2 than at q_1, even when $\pi < \frac{1}{2}$.

3. THE TWO-SAMPLE CASE

Assume now that μ is unknown. Then the total sample of n is split into two samples of sizes n_1 and n_2 with respective randomization probabilities p_1 and p_2. Without loss of generality, we may assume $p_1 \geq p_2$, so that $p = \max(p_1,p_2) = p_1$. Then if X_i denotes the number of "yes" responses in the i^{th} sample $(i = 1,2)$, X_1 and X_2 are independent random variables and X_i is binomially distributed with parameters n_i and $\lambda_i = p_i\pi + (1-p_i)\mu$. The maximum likelihood estimator [1] of π is

$$\hat{\pi}_2 = \frac{n_2(1-p_2)X_1 - n_1(1-p_1)X_2}{n_1 n_2(p_1-p_2)},$$

which exists for all $p_1 > p_2$, is unbiased, and has variance

$$V(\hat{\pi}_2) = \frac{1}{(p_1-p_2)^2}\left[\frac{(1-p_2)^2\lambda_1(1-\lambda_1)}{n_1^2} + \frac{(1-p_1)^2\lambda_2(1-\lambda_2)}{n_2^2}\right] \tag{3.1}$$

It is shown in [1] that (3.1) is minimized over n_1, n_2, subject to $n = n_1 + n_2$, by taking n_1, n_2 in the ratio

$$\frac{n_1}{n_2} = \sqrt{\frac{\lambda_1(1-\lambda_1)(1-p_2)^2}{\lambda_2(1-\lambda_2)(1-p_1)^2}} \tag{3.2}$$

in which case, (3.1) becomes

$$V(\hat{\pi}_2) = \frac{1}{n(p_1-p_2)^2}\left[(1-p_1)\sqrt{\lambda_2(1-\lambda_2)} + (1-p_2)\sqrt{\lambda_1(1-\lambda_1)}\right]^2. \tag{3.3}$$

Of course the optimal allocation ratio (3.2) depends on the unknown parameters μ, π so is unavailable to the statistician. We shall assume, however, as in [3], that n_1, n_2 satisfy (3.2) in order to compare the variance with Warner's variance independently of total sample size n. Differentiating (3.3) with respect to p_i, we observe (see [2]) that $\partial V(\hat{\pi}_2)/\partial p_1$ is negative and that $\partial V(\hat{\pi}_2)/\partial p_2$ is positive, so that $V(\hat{\pi}_2)$ in (3.3) is minimized over p_1, p_2 by taking $p = p_1$ as large as practicable, and taking p_2 as small as possible, i.e., $p_2 = 0$. In other words, the second sample is a direct survey, involving no randomization, used solely to estimate μ.

The estimator with optimal sample allocation (3.2) and with $p_2 = 0$ is called the "optimized" estimator in [3]. In this case, setting $p = p_1$, $\lambda = \lambda_1 = p\pi + (1-p)\mu$, (3.3) becomes

$$V(\hat{\pi}_2) = \frac{1}{np^2}\left[(1-p)\sqrt{\mu(1-\mu)} + \sqrt{\lambda(1-\lambda)}\right]. \tag{3.4}$$

<u>Theorem 2</u>. Let $p \in (p_{00},1)$ where $p_{00} = (3-\sqrt{5})/2 = .381966...$ Then

$$V(\hat{\pi}_2) < V(\hat{\pi}_w)$$

for all $(\mu,\pi) \in [0,1]^2$.

<u>Proof</u>. We write

$$p\sqrt{nV(\hat{\pi}_2)} = L_p(\mu,\pi) = (1-p)\sqrt{\mu(1-\mu)} + \sqrt{\lambda(1-\lambda)}.$$

L_p is continuous in (μ,π) over the compact set $[0,1]^2$, so as in Section 2, we may iterate maxima again to obtain the maximum of L_p over $[0,1]^2$, which will maximize $V(\hat{\pi}_2)$. We first maximize over $\mu \in [0,1]$. L_p is a positive linear combination of square roots of unimodal functions. Since the square root of a unimodal function is unimodal, $L_p(\cdot,\pi)$ will be maximized at a point μ^* between the values $\mu_1 = \frac{1}{2}$ maximizing $\mu(1-\mu)$ and $\mu_2 = (1-2p\pi)/2(1-p)$ maximizing $\lambda(1-\lambda)$. Computing $\partial L_p/\partial\mu$ and setting it equal to zero, we obtain

$$(1-2\mu)\sqrt{\mu(1-\mu)} = [2(1-p)\mu - (1-2p\pi)]\sqrt{-(1-p)^2\mu^2+(1-p)(1-2p\pi)\mu+p\pi(1-p\pi)}. \quad (3.5)$$

Each side of (3.5) will have the same sign if and only if μ is between μ_1 and μ_2. In particular, this holds for $\mu = \mu^*$, thus we may square each side, and simplify, obtaining the quadratic in μ,

$$(2-p)\mu^2 - [1+2\pi(1-p)]\mu + \pi(1-p\pi) = 0. \quad (3.6)$$

One root of (3.6) is $(1-p\pi)/(2-p)$, which is in [0,1] for all p, π; the second is π. If $\pi < \frac{1}{2}$, then $\mu_2 > \mu_1$, and if $\pi > \frac{1}{2}$, then $\mu_2 < \mu_1$, hence π is not the maximizing root. Consequently, $\mu^* = (1-p\pi)/(2-p)$. (Observe that for $\mu = \mu^*$, $\lambda(1-\lambda) = \mu(1-\mu)$.)

Thus

$$\max_{\mu \in [0,1]} L_p(\mu,\pi) = L_p(\mu^*,\pi).$$

and at this value,

$$V(\hat{\pi}_2) = \frac{\pi(1-\pi)}{n} + \frac{1-p}{np^2}. \quad (3.7)$$

Comparing (3.7) with (2.1), we see that $V(\hat{\pi}_2) < V(\hat{\pi}_w)$ if and only if

$$\frac{1-p}{p^2} < \frac{1-p}{(2p-1)^2},$$

or

$$p^3 - 4p^2 + 4p - 1 > 0. \quad (3.8)$$

The roots of the cubic in (3.8) are 1, $(3-\sqrt{5})/2 = p_{00}$, and $(3+\sqrt{5})/2 > 1$. Hence (3.8) holds for $p \in (0,1)$ if $p \in (p_{00},1)$. □

4. REMARKS

As is expected, if we compare the "optimized" $V(\hat{\pi}_2)$ (with respect to n_1, n_2 and p_2) and $V(\hat{\pi}_1)$, we find that $V(\hat{\pi}_1) \leq V(\hat{\pi}_2)$ for all p, $q = \mu$ and π. This can be seen if the variances are written as follows

$$V(\hat{\pi}_1) = \frac{\pi(1-\pi)}{n} + \frac{(1-p)}{np^2}\left\{[1-(1-p)q]q + p\pi(1-2q)\right\}$$

and

$$V(\hat{\pi}_2) = \frac{\pi(1-\pi)}{n} + \frac{(1-p)}{np^2}\left\{[2-p-2(1-p)q]q + p\pi(1-2q) + 2\sqrt{q(1-q)\lambda(1-\lambda)}\right\} .$$

A sufficient condition is then

$$1 - (1-p)q \leq 2 - p - 2(1-p)q$$

which is equivalent to $(1-p)(1-q) \geq 0$.

$V(\hat{\pi}_1)$, as a function of (q,π) in Theorem 1, and L_p, as a function of (μ,π) in Theorem 2, enjoy a stronger property than that of unimodality; in fact they are concave. Hence an alternate justification for the proofs may proceed along the lines of showing this concavity and, hence, the concavity of the iterated maxima. This may be seen by evaluating the Hessians (matrices of second order partial derivatives) of $V(\hat{\pi}_1)$ and L_p which turn out to be non-positive definite; see [4, p. 27].

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