ASYMPTOTIC PROPERTIES OF SOME SEQUENTIAL NONPARAMETRIC ESTIMATORS IN SOME MULTIVARIATE LINEAR MODELS

by

Pranab Kumar Sen and Malay Ghosh

Department of Biostatistics
University of North Carolina at Chapel Hill
and
Indian Statistical Institute
Calcutta, India

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Asymptotic Properties of Some Sequential Nonparametric Estimators in Some Multivariate Linear Models

PRANAB KUMAR SEN[1]
Department of Biostatistics
University of North Carolina
Chapel Hill, N. C. 27514 U.S.A.

MALAY GHOSH
Research Training School
Indian Statistical Institute
Calcutta - 35. India

1. INTRODUCTION

For a general multivariate linear model (which includes the one-sample and two-sample location models as special cases), robust sequential point as well as interval estimators based on suitable rank order statistics are proposed and studied. In a non-sequential set up, parallel procedures were considered by Sen and Puri [14]. Also, the sequential point estimation problem based on sample means (in the univariate case) has been studied earlier by Blum, Hanson and Rosenblatt [3], and later, in a more general set up, by Mogyorodi [10], among others. Finally, the sequential interval estimation procedures, based on the principles of Chow and Robbins [4], extends the univariate theory developed in Sen and Ghosh [13], and Ghosh and Sen [5, 6, 7] to the general multivariate case.

In Section 2, along with our basic model, we briefly sketch the problems. Preliminary notions and basic assumptions are then considered in Section 3. Section 4 is devoted to the study of the asymptotic properties of sequential point estimators based on robust rank order statistics. The problem of robust

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sequential interval estimation is then treated in Section 5. The last section is devoted to a comparison with the corresponding parametric procedures, and presents the allied asymptotic relative efficiency (ARE) results.

2. THE PROBLEMS

Consider a sequence $\{\underset{\sim}{X}_i = (X_{i1},\ldots,X_{ip})',\ i\geq 1\}$ of $p(\geq 1)$-variate stochastic vectors, defined on a probability space (Ω,A,P), where $\underset{\sim}{X}_i$ has an absolutely continuous cumulative distribution function (cdf) $F_i(\underset{\sim}{x})$, $\underset{\sim}{x}\in R^p$, the p-dimensional Euclidean space. It is assumed that

$$F_i(\underset{\sim}{x}) = F(\underset{\sim}{x}-\underset{\sim}{\alpha}-\underset{\sim}{\beta}c_i), \quad i\geq 1, \tag{2.1}$$

where $\underset{\sim}{\alpha}=(\alpha_1,\ldots,\alpha_p)'$ and $\underset{\sim}{\beta}=(\beta_1,\ldots,\beta_p)'$ are unknown parameters (vectors), and $\{c_i,\ i\geq 1\}$ is a sequence of known (scalar) constants.

Robust point as well as interval estimators of $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ based on suitable rank order statistics when the sample size is large but non-random were studied in detail in Sen and Puri [14]. We are primarily concerned here with the following two sequential extensions of this theory.

Let $\{N_\nu,\ \nu\geq 1\}$ be a sequence of non-negative inter-valued random variables, such that

$$\nu^{-1}N_\nu \to \lambda, \text{ in probability, as } \nu\to\infty, \tag{2.2}$$

where λ is a positive random variable having an arbitrary distribution

$$H(u) = P\{\lambda\leq u\}, \quad 0<u<\infty, \tag{2.3}$$

and defined on the same probability space (Ω,A,P). Consider then an estimator $(\hat{\underset{\sim}{\alpha}}_{N_\nu},\ \hat{\underset{\sim}{\beta}}_{N_\nu})$ of $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ based on $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_{N_\nu}$ through a general class of rank order statistics, to be precisely defined in section 3. Our first problem is to derive (along the lines of Blum, Hanson and Rosenblatt [3], and Mogyorodi [10]) the asymptotic normality of $N_\nu^{\frac{1}{2}}[(\hat{\underset{\sim}{\alpha}}_{\underset{\sim}{N}_\nu}-\underset{\sim}{\alpha}),(\hat{\underset{\sim}{\beta}}_{\underset{\sim}{N}_\nu}-\underset{\sim}{\beta})]$ (as $\nu\to\infty$). This enables us to study various asymptotic properties of $(\hat{\underset{\sim}{\alpha}}_{\underset{\sim}{N}_\nu},\hat{\underset{\sim}{\beta}}_{\underset{\sim}{N}_\nu})$.

In the second problem, our sample size N_ν remains a random variable, but so determined by a "stopping rule" that we have a simultaneous confidence interval for $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$, with the property that the confidence coefficient is asymptotically equal to a predetermined $1-\varepsilon$: $0<\varepsilon<1$, and the length of the interval for each component of $\underset{\sim}{\alpha}$ (or $\underset{\sim}{\beta}$) is bounded above by 2d (or by a known multiple of 2d), where $d>0$ is a predetermined (small) number. The theory is an extension of the corresponding univariate theory developed in Sen and Ghosh [13], and Ghosh and Sen [5, 6, 7]. It is also a sequential extension of the theory developed in Sen and Puri [14], and a nonparametric analogue of the theory developed in Gleser [8] and Albert [2].

3. PRELIMINARY NOTIONS AND BASIC ASSUMPTIONS

Let $\mathcal{F}_p$ be the class of all p-variate absolutely continuous cdf's with finite Fisher information matrix, and let $\mathcal{F}_p^o$ be the subclass of $\mathcal{F}_p$ for which the distribution is diagonally symmetric about $\underset{\sim}{0}$.

Assumption I. If we are only interested in $\underset{\sim}{\beta}$, we assume that $F\varepsilon\mathcal{F}_p$, otherwise, we assume that $F\varepsilon\mathcal{F}_p^o$, where F is defined in (2.1). For every $\nu\geq1$, let

$$\bar{c}_\nu = \nu^{-1}\sum_{i=1}^{\nu} c_i, \qquad C_\nu^2 = \sum_{i=1}^{\nu}(c_i-\bar{c}_\nu)^2. \tag{3.1}$$

We have then the following problems: (a) estimation of $\underset{\sim}{\alpha}$ assuming $\underset{\sim}{\beta}=\underset{\sim}{0}$; (b) estimation of $\underset{\sim}{\beta}$ treating $\underset{\sim}{\alpha}$ as a nuisance parameter; (c) simultaneous estimation of $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. For (a) no assumptions are needed on $\{c_i,\ i\geq1\}$; for (b) and (c) our assumptions are respectively (II, III) and (II, III', IV), where, Assumption II. As $\nu\to\infty$,

$$\max_{1\le i\le\nu} \nu(c_i-\bar{c}_\nu)^2/C_\nu^2 = O(1); \tag{3.2}$$

Assumption III.

$$\lim_{\nu\to\infty} C_\nu^2 = \infty; \tag{3.3}$$

Assumption III'.

$$\lim_{\nu\to\infty} \nu^{-1}C_\nu^2 = C^2 \quad (0<C<\infty); \tag{3.4}$$

Assumption IV.

$$\lim_{\nu\to\infty} \bar{c}_\nu = \bar{c} \text{ (finite)}. \tag{3.5}$$

It is easy to verify that all these assumptions hold true for the multivariate one sample (where $c_i=0 \ \forall_i$) and two-sample (where c_i is either 0 or 1) models.

For every $\nu\ge1$, let $R_{\nu i}^{(j)}$ (or $R_{\nu i}^{(j)+}$) be the rank of X_{ij} (or $|X_{ij}|$) among $X_{1j},\ldots,X_{\nu j}$ (or $|X_{1j}|,\ldots,|X_{\nu j}|$) for $1\le i\le\nu$, $1\le j\le p$. To estimate β, consider the following linear (regression) rank statistics

$$S_{\nu j} = \sum_{i=1}^{\nu}(c_i-\bar{c}_\nu)a_\nu^{(j)}(R_{\nu i}^{(j)}), \quad j=1,2,\ldots,p, \tag{3.6}$$

$$S_\nu = (S_{\nu 1},\ldots,S_{\nu p})', \tag{3.7}$$

where the rank scores $a_\nu^{(j)}(i)$, $1\le i\le\nu$, $j=1,\ldots,p$ are defined by

$$a_\nu^{(j)}(i) = E\phi_j(U_{\nu i}) \ [\text{or } \phi_j(i/(\nu+1))], \quad i=1,\ldots,\nu; \tag{3.8}$$

$\phi_j(u)$ is non-decreasing and absolutely continuous inside [0,1], $U_{\nu 1}\le\cdots<U_{\nu\nu}$ are the ordered random variable in a sample of size ν from the rectangular [0,1] distribution. Regarding the score functions $\phi_1,\ldots,\phi_p$ one assumes as in Ghosh and Sen [6] that for every $j(=1,\ldots,p)$,

$$|\phi_j(u)| \le K[-\log(u(1-u))], \quad |\phi_j'(u)| \le K[u(1-u)]^{-1}, \quad 0<u<1 \tag{3.9}$$

where $0<K<\infty$. This implies the existence of a $t_o(>0)$ such that

$$M_j(t) = \int_{-\infty}^{\infty} \exp(t\phi_j(u))du < \infty \text{ for all } t: \ |t| \le t_o, \tag{3.10}$$

for all $j=1,\ldots,p$.

For estimating $\underset{\sim}{\alpha}$, we need an alignment procedure and the following type of one sample rank order statistics

$$T_{\nu j} = \sum_{i=1}^{\nu} c(X_{ij})a_{\nu}^{(j)*}(R_{\nu i}^{(j)+}), \quad j=1,\ldots,p, \tag{3.11}$$

$$\underset{\sim}{T}_{\nu} = (T_{\nu 1},\ldots,T_{\nu p})', \tag{3.12}$$

where $c(u) = 1$, ½ or 0 according as $u >, =$ or <0,

$$a_{\nu}^{(j)*}(i) = E\phi_j^*(U_{\nu i}) \ [\text{or } \phi_j^*(i/(\nu+1))], \quad 1\le i\le \nu, \tag{3.13}$$

$$\phi_j^*(u) = \phi_j(\tfrac{1+u}{2}) \text{ and assume that } \phi_j(u)+\phi_j(1-u) = 0. \tag{3.14}$$

Some well-known cases of $\underset{\sim}{S}_{\nu}$ and $\underset{\sim}{T}_{\nu}$ are the normal scores and the Wilcoxon scores statistics which relate respectively to $\phi_j(u)$ as the inverse of the standard normal cdf and $\phi_j(u) = 2u-1$, $0<u<1$. Let us also define for later use

$$\underset{\sim}{\Gamma} = ((\gamma_{j\ell})), \ \gamma_{j\ell} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_j(F_{[j]}(x))\phi_\ell(F_{[\ell]}(y))dF_{[j\ell]}(x,y)-\mu_j\mu_\ell, \tag{3.15}$$

for $j,\ell=1,\ldots,p$, where $F_{[j]}$ is the jth marginal cdf and $F_{[j\ell]}$ is the bivariate (j,ℓ)th joint cdf in the joint cdf F, and

$$\mu_j = \int_0^1 \phi_j(u)du, \quad j=1,\ldots,p. \tag{3.16}$$

Assumption V. For every $j(=1,\ldots,p)$, the density function $f_{[j]}=F'_{[j]}$ and its first derivative $f'_{[j]}$ exist and are bounded for almost all x (a.a.x), and

$$\lim_{x\to\pm\infty} |\phi'_j(F_{[j]}(x))f_{[j]}(x)| \text{ is finite.} \tag{3.17}$$

Let us then denote by

$$B_j = B(F_{[j]},\phi_j) = \int_{-\infty}^{\infty} (d/dx)\phi_j(F_{[j]}(x))dF_{[j]}(x), \quad j=1,\ldots,p; \tag{3.18}$$

$$\underset{\sim}{T} = ((\tau_{j\ell})), \quad \tau_{j\ell} = \gamma_{j\ell}/[B_jB_\ell]; \quad j,\ell=1,\ldots,p \tag{3.19}$$

Note that $B_j>0 \ \forall j$, and $\underset{\sim}{T}$ is positive semi-definite.

4. ASYMPTOTIC PROPERTIES OF ROBUST SEQUENTIAL POINT ESTIMATORS OF $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$

We find it more convenient to consider separately the following three problems:

(I) Estimation of $\underset{\sim}{\alpha}$ assuming that $\underset{\sim}{\beta}=\underset{\sim}{0}$ (one-sample model),

(II) Estimation of $\underset{\sim}{\beta}$ treating $\underset{\sim}{\alpha}$ to be a nuisance parameter,

(III) Joint estimation of $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$.

In the first problem, assume that $F\varepsilon F_p^o$, and denote by $R_{\nu i}^{(j)+}(a_j)$, the rank of $|X_{ij}-a|$ among $|X_{1j}-a|,\ldots,|X_{\nu j}-a|$, $1\le i\le\nu$, $1\le j\le p$; the resulting rank statistics, defined in (3.11) are denoted by $T_{\nu j}(a)$, $j=1,\ldots,p$. Note that

$$T_{\nu j}(a) \text{ is } \downarrow \text{ in a for all } j=1,\ldots,p. \tag{4.1}$$

Define for each positive integer ν,

$$\hat{\alpha}_{\nu j}^{(1)} = \sup\{a: T_{\nu j}(a)>0\}, \quad \hat{\alpha}_{\nu j}^{(2)} = \inf\{a: T_{\nu j}(a)<0\}; \tag{4.2}$$

$$\hat{\alpha}_{\nu j} = \tfrac{1}{2}(\hat{\alpha}_{\nu j}^{(1)} + \hat{\alpha}_{\nu j}^{(2)}), \quad j=1,\ldots,p; \tag{4.3}$$

$$\hat{\underset{\sim}{\alpha}}_{\nu} = (\hat{\alpha}_{\nu 1},\ldots,\hat{\alpha}_{\nu p})'. \tag{4.4}$$

We intend to study various asymptotic properties of $\hat{\underset{\sim}{\alpha}}_{N_\nu}$, and towards this goal, we have the following.

<u>Theorem 4.1</u>. <u>When</u> $F\varepsilon F_p^o$ <u>and</u> $\underset{\sim}{\beta}=\underset{\sim}{0}$, <u>under</u> (2.1), (2.2), (2.3), (3.9), (3.13), (3.14) <u>and</u> (3.17), <u>as</u> $\nu\to\infty$

$$\mathcal{L}(N_\nu^{\frac{1}{2}}[\hat{\underset{\sim}{\alpha}}_{N_\nu}-\underset{\sim}{\alpha}]) \to N_p(\underset{\sim}{0},\underset{\sim}{T}), \tag{4.5}$$

<u>where</u> $\underset{\sim}{T}$ <u>is defined by</u> (3.19).

<u>Proof</u>. We use a recent powerful result of Mogyorodi [10] (Theorem 2), according to which we are only to show that for non-stochastic ν,

$$\mathcal{L}(\nu^{\frac{1}{2}}[\hat{\underset{\sim}{\alpha}}_{\nu}-\underset{\sim}{\alpha}]) \to N_p(\underset{\sim}{0},\underset{\sim}{T}), \text{ as } \nu\to\infty, \tag{4.6}$$

and for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an $n_o = n_o(\varepsilon,\eta)$, such that for $n\geq n_o$,

$$P\{\max_{k:\ |n-k|<\delta n} |\sqrt{n}\|\hat{\underset{\sim}{\alpha}}_k-\hat{\underset{\sim}{\alpha}}_n\| > \varepsilon\} < \eta, \tag{4.7}$$

where $\|\underset{\sim}{x}\| = \max_{1\leq j\leq p}|x_j|$, $\underset{\sim}{x}=(x_1,\ldots,x_p)'$. Now, (4.6) has already been proved in Theorem 6.2.3 (on page 226) of Puri and Sen [11]. On the other hand, the left hand side of (4.7) is bounded above by

$$\sum_{j=1}^{p} P\{\max_{k:\ |k-n|<\delta n} \sqrt{n}|\hat{\alpha}_{k,j}-\hat{\alpha}_{n,j}| > \varepsilon\}, \tag{4.8}$$

and hence, by the same technique as in Lemma 5.3 of Sen and Ghosh [13], it can be shown that (4.8) can be bounded by $\eta(>0)$ but a proper choice of $\delta(>0)$ and n. For brevity, the proof is therefore omitted.

Since λ, defined by (2.2), is a positive random variable, for every $0<\varepsilon<1$, there exists a $\lambda_\varepsilon(>0)$, such that $P\{\lambda\geq\lambda_\varepsilon\}\geq 1-\varepsilon$, and hence, $N_\nu\to\infty$, in probability, as $\nu\to\infty$. Consequently, by (4.5)

$$\hat{\underset{\sim}{\alpha}}_{N_\nu}\to\underset{\sim}{\alpha},\text{ in probability, as }\nu\to\infty. \tag{4.9}$$

Consider now the problem of estimating $\underset{\sim}{\beta}$ treating $\underset{\sim}{\alpha}$ as a nuisance parameter. Assume that $F\varepsilon\mathcal{F}_p$ and that II and III hold. Let $R^{(j)}_{\nu i}(b)$ be the rank of $X_{ij}-bc_i$ among $X_{1j}-bc_1,\ldots,X_{nj}-bc_n$ ($1\leq i\leq\nu$; $1\leq j\leq p$), b real; the resulting rank statistics defined by (3.6) are then denoted by $S_{\nu j}(b)$ ($1\leq j\leq p$; $\nu\geq 1$). It follows from Sen ([12], section 6) that

$$S_{\nu j}(b)\text{ is }\downarrow\text{ in b for all }j=1,\ldots,p. \tag{4.11}$$

Define for each $\nu\geq 1$,

$$\hat{\beta}^{(1)}_{\nu j}=\sup\{b:\ S_{\nu j}(b)>0\},\qquad \hat{\beta}^{(2)}_{\nu j}=\inf\{b:\ S_{\nu j}(b)<0\},\quad 1\leq j\leq p; \tag{4.12}$$

$$\hat{\beta}_{\nu j}=\tfrac{1}{2}(\hat{\beta}^{(1)}_{\nu j}+\hat{\beta}^{(2)}_{\nu j}),\quad 1\leq j\leq p; \tag{4.13}$$

$$\hat{\underset{\sim}{\beta}}_\nu=(\hat{\beta}_{\nu 1},\ldots,\hat{\beta}_{\nu p})' \tag{4.14}$$

Then, parallel to theorem 4.1, we have the following.

<u>Theorem 4.2</u>. <u>For</u> $F\varepsilon\mathcal{F}_p$, <u>when</u> (2.1), (2.2), (2.3), (3.2), (3.3), (3.9), <u>and</u> (3.17) <u>hold, as</u> $\nu\to\infty$,

$$\mathcal{L}(C_{N_\nu}[\hat{\underset{\sim}{\beta}}_{N_\nu}-\underset{\sim}{\beta}])\to\mathcal{N}_p(\underset{\sim}{0},\underset{\sim}{T}), \tag{4.15}$$

where $\underset{\sim}{T}$ *is defined by* (3.19).

Proof. As in the proof of theorem 4.1, we require only to show that for non-stochastic ν,

$$\mathcal{L}(C_\nu[\hat{\underset{\sim}{\beta}}_\nu - \underset{\sim}{\beta}]) \to N_p(0, \underset{\sim}{T}) \text{ as } \nu\to\infty, \tag{4.16}$$

and for every $\varepsilon>0$ and $\eta>0$, there exist a $\delta>0$ and an $n_o=n_o(\varepsilon,\eta)$, such that for $n\geq n_o$,

$$P\{\max_{k:\ |n-k|<\delta n} C_n \|\hat{\underset{\sim}{\beta}}_k - \hat{\underset{\sim}{\beta}}_n\| > \varepsilon\} < \eta. \tag{4.17}$$

Now, (4.16) has already been proved in theorem 5.1 of Sen and Puri [14], while (4.17), by virtue of an inequality similar to that in (4.7)-(4.8), follows from Lemma 4.4 of Ghosh and Sen [6]. Hence, the details are omitted.

By (4.16), (3.3) and the discussion preceding (4.9), as $\nu\to\infty$,

$$\hat{\underset{\sim}{\beta}}_{N_\nu} \to \underset{\sim}{\beta}, \text{ in probability.} \tag{4.18}$$

Finally, consider the joint estimation of $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. Assume that $F\varepsilon F_p^o$ and assumptions II, III', IV and V hold. Define the estimators $\hat{\underset{\sim}{\beta}}_\nu$ as in (4.12)-(4.14), and then for estimating $\underset{\sim}{\alpha}$, consider the following aligned rank statistics.

Let $\tilde{R}_{\nu i}^{(j)+}(a)$ be the rank of $|X_{ij}-a-\hat{\beta}_{\nu j}c_i|$ among $|X_{1j}-a-\hat{\beta}_{\nu j}c_1|,\ldots,|X_{\nu j}-a-\hat{\beta}_{\nu j}c_\nu|$, $(1\leq i\leq \nu,\ 1\leq j\leq p)$. The resulting one-sample rank-order statistics defined by (3.11) are denoted by $\tilde{T}_{\nu j}(a)$, $(1\leq j\leq p,\ \nu\geq 1)$. Define

$$\tilde{\alpha}_{\nu j}^{(1)} = \sup\{a: \tilde{T}_{\nu j}(a)>0\},\ \tilde{\alpha}_{\nu j}^{(2)} = \inf\{a: \tilde{T}_{\nu j}(a)<0\},\ \nu\geq 1,\ 1\leq j\leq p; \tag{4.19}$$

$$\tilde{\alpha}_{\nu j} = \tfrac{1}{2}(\tilde{\alpha}_{\nu j}^{(1)} + \tilde{\alpha}_{\nu j}^{(2)}), \quad 1\leq j\leq p,\ \nu\geq 1; \tag{4.20}$$

$$\tilde{\underset{\sim}{\alpha}} = (\tilde{\alpha}_{\nu 1},\ldots,\tilde{\alpha}_{\nu p})', \quad \nu\geq 1. \tag{4.21}$$

For notational simplicity, let $\underset{\sim}{\theta}=(\underset{\sim}{\alpha}',\underset{\sim}{\beta}')'$ and $\hat{\theta}_{\nu}=(\underset{\sim}{\tilde{\alpha}}_{\nu}',\underset{\sim}{\hat{\beta}}_{\nu}')'$. Then, we have the following theorem.

Theorem 4.3. Under (2.1)-(2.3), (3.9), (3.13), (3.14) and assumptions I, II, III', IV and V, as $\nu\to\infty$,

$$\mathcal{L}(N_{\nu}^{\frac{1}{2}}(\underset{\sim}{\tilde{\theta}}_{N_{\nu}}-\underset{\sim}{\theta})) \to N_{2p}(\underset{\sim}{0},\ \underset{\sim}{\Delta}\otimes\underset{\sim}{T}), \tag{4.22}$$

where $\underset{\sim}{T}$ is defined by (3.19) and

$$\underset{\sim}{\Delta} = \begin{pmatrix} 1+\bar{c}^2/C^2 & -\bar{c}/C^2 \\ -\bar{c}/C^2 & 1/C^2 \end{pmatrix} \tag{4.23}$$

Proof. First note that by the same technique as in the proofs of results in section 7 of Sen and Puri [14] (who considered the particular case of $\bar{c}_{\nu}=0$ for all $\nu\geq 1$), one gets,

$$\mathcal{L}(\nu^{\frac{1}{2}}(\underset{\sim}{\tilde{\theta}}_{\nu}-\underset{\sim}{\tilde{\theta}})) \to N_{2p}(\underset{\sim}{0},\ \underset{\sim}{\Delta}\otimes\underset{\sim}{T}). \tag{4.24}$$

Hence, similarly as in theorems 4.1 and 4.2, one needs to show that for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an $n_o=n_o(\varepsilon,\eta)$ such that for $n\geq n_o$,

$$P\{\max_{k:\ |k-n|<\delta n} \| n^{\frac{1}{2}}(\underset{\sim}{\tilde{\theta}}_k-\underset{\sim}{\tilde{\theta}}_n) \| > \varepsilon\} < 2\eta. \tag{4.25}$$

Now, the left hand side of (4.25) is bounded above by

$$P\{\max_{k:\ |k-n|<\delta n}\| n^{\frac{1}{2}}(\underset{\sim}{\tilde{\alpha}}_k-\underset{\sim}{\tilde{\alpha}}_n)\|>\varepsilon\} + P\{\max_{k:\ |k-n|<\delta n}\| n^{\frac{1}{2}}(\underset{\sim}{\hat{\beta}}_k-\underset{\sim}{\hat{\beta}}_n)\|>\varepsilon\} \tag{4.26}$$

By virtue of (4.17) and Bonferroni inequality, it suffices to show now that

$$\sum_{j=1}^{p} P\{\max_{k:\ |k-n|<\delta n}\| n^{\frac{1}{2}}(\underset{\sim}{\tilde{\alpha}}_k-\underset{\sim}{\tilde{\alpha}}_n)\|>\varepsilon\} < \eta \tag{4.27}$$

For simplicity, instead of proving (4.27) we shall consider the following: Let $\tilde{\tilde{R}}_{\nu i}^{(j)+}$ be the rank of $|X_{ij}-a-\hat{\beta}_{\nu j}(c_i-\bar{c}_\nu)|$ among $|X_{1j}-a-\hat{\beta}_{\nu j}(c_1-\bar{c}_\nu)|,\ldots,$ $|X_{\nu j}-a-\hat{\beta}_{\nu j}(c_\nu-\bar{c}_\nu)|$, $1\le i\le\nu$, $1\le j\le p$. The resulting one sample rank order statistics defined by (3.11) will now be denoted by $\tilde{\tilde{T}}_{\nu j}(a)$ $(1\le j\le p;\ \nu\ge 1)$. Define

$$\hat{\delta}_{\nu j}^{(1)} = \sup\{a:\ \tilde{\tilde{T}}_{\nu j}(a)>0\},\quad \hat{\delta}_{\nu j}^{(2)} = \inf\{a:\ \tilde{\tilde{T}}_{\nu j}(a)<0\},\quad \nu\ge 1,\ 1\le j\le p; \tag{4.28}$$

$$\hat{\delta}_{\nu j} = \tfrac{1}{2}(\hat{\delta}_{\nu j}^{(1)} + \hat{\delta}_{\nu j}^{(2)}),\quad 1\le j\le p,\ \nu\ge 1; \tag{4.29}$$

$$\hat{\underset{\sim}{\delta}}_\nu = (\hat{\delta}_{\nu 1},\ldots,\hat{\delta}_{\nu p})',\quad \nu\ge 1. \tag{4.30}$$

It follows from the results of Adichie [1] that

$$\hat{\delta}_{\nu j} = \tilde{\alpha}_{\nu j} + \hat{\beta}_{\nu j}\bar{c}_\nu \quad (1\le j\le p;\ \nu\ge 1), \tag{4.31}$$

i.e.,

$$\hat{\delta}_\nu = \tilde{\underset{\sim}{\alpha}}_\nu + \hat{\underset{\sim}{\beta}}_\nu\bar{c}_\nu \quad (\nu\ge 1). \tag{4.32}$$

In view of (4.25)-(4.27) and (4.31)-(4.32) it now suffices to show that for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an $n_o = n_o(\varepsilon,\eta)$ such that for $n\ge n_o$,

$$\sum_{j=1}^{p} P\{\max_{k:\ |k-n|<\delta n}\|n^{\frac{1}{2}}(\hat{\underset{\sim}{\delta}}_k-\hat{\underset{\sim}{\delta}}_n)\|>\varepsilon\} < \eta. \tag{4.33}$$

To prove (4.33) we prove the following two lemmas. Since $\tilde{\underset{\sim}{\alpha}}_\nu$ and $\hat{\underset{\sim}{\beta}}_\nu$ (and hence $\hat{\underset{\sim}{\delta}}_\nu$) are translation invariant for every ν (see [14]), we may, for proving these lemmas, assume that $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$.

<u>Lemma 4.4</u>. <u>Under the assumptions of Theorem 4.3, for every</u> $s>0$, <u>there exist positive constants</u> $c_s^{(1)}$ <u>and</u> $c_s^{(2)}$ <u>and a positive integer</u> ν_s <u>such that for</u> $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$, <u>and all</u> $\nu\ge\nu_s$,

$$P\{\sup_{|a|\le K_o(\log \nu)^k \nu^{-\frac{1}{2}}} \nu^{-\frac{1}{2}}|\tilde{\tilde{T}}_{\nu_j}(a)-\tilde{T}_{\nu j}(a)|>c_s^{(1)}\nu^{-\delta}(\log \nu)^{k+1}\} \le c_s^{(2)}\nu^{-s}, \tag{4.34}$$

where K_o is a positive constant, k any positive integer and δ fixed $(0<\delta<\frac{1}{4})$.

Before proving the above lemma, we may note that taking s>1 and on using the Borel-Cantelli Lemma, (4.34) implies that

$$\sup_{|a|\le K_o\nu^{-\frac{1}{2}}(\log \nu)^k} \nu^{-\frac{1}{2}}|\tilde{\tilde{T}}_{\nu j}(a)-\tilde{T}_{\nu j}(a)| \to 0 \text{ a.s. as } \nu\to\infty. \tag{4.35}$$

The proof of the lemma is accomplished in several steps. First we show that for any real b, defining $T_{\nu j}(a,b)$ as similar to $\tilde{\tilde{T}}_{\nu j}(a)$ with $\hat{\beta}_\nu$ replaced by b, for $\nu\ge\nu_s^{(1)}$ (depending on s),

$$P\{\sup_{|a|\le K_o\nu^{-\frac{1}{2}}(\log \nu)^k} \sup_{|b|\le K_1\nu^{-\frac{1}{2}}(\log \nu)^k} \nu^{-\frac{1}{2}}|T_{\nu j}(a,b)-\tilde{T}_{\nu j}(a)|$$
$$> c_s^{(3)}\nu^{-\delta}(\log \nu)^{k+1}\} < c_s^{(4)}\nu^{-s}, \tag{4.36}$$

where K_1 is a positive constant, $c_s^{(3)}$, $c_s^{(4)}$ are positive constants depending on s. Next, in analogy to lemma 4.1 of Ghosh and Sen [6], one can show that for every s(>0), there exist positive constants $c_s^{(5)}$ and $c_s^{(6)}$ and a positive integer ν_{s2} such that for $\nu\ge\nu_{s2}$,

$$P_{\beta=0}\ \{C_\nu|\hat{\beta}_{\nu j}| > c_s^{(5)}(\log \nu)^2\} \le c_s^{(6)}\nu^{-s}. \tag{4.37}$$

Defining now $c_s^{(1)}$, $c_s^{(2)}$ and ν_s appropriately on the basis of $c_s^{(i)}$ (i=3,4,5,6), $\nu_s^{(1)}$ and $\nu_s^{(2)}$, one gets (4.34) from (4.35), (4.37) and (3.4). Let $H_{\nu,j,a,b}(x) = \nu^{-1}\sum_{i=1}^{\nu} u(x-(X_{ij}-a-b(c_i-\bar{c}_\nu)))$ be the sample df of $X_{ij}-a-b(c_i-\bar{c}_\nu)$'s, and let $G_{\nu,j,a,b}(x) = \nu^{-1}\sum_{i=1}^{\nu} u(x-|X_{ij}-a-b(c_i-\bar{c}_\nu)|) = H_{\nu,j,a,b}(x)-H_{\nu,j,a,b}(-x-)$ be the sample cdf for $|X_{ij}-a-b(c_i-\bar{c}_\nu)|$'s. The corresponding population cdf's are denoted respectively by $\bar{F}_{\nu,j,a,b}(x) = \nu^{-1} \sum_{i=1}^{\nu} F_{[j]}(x+a+b(c_i-\bar{c}_\nu))$ and $\bar{D}_{\nu,j,a,b}(x) = \bar{F}_{\nu,j,a,b}(x)-\bar{F}_{\nu,j,a,b}(-x-)$. Writing $\phi^*_{\nu j}(i/(\nu+1)) = a_\nu^{(j)*}(i)$ $(1\le i\le \nu,\ \nu\ge 1)$, one can now write

$$\nu^{-1}[T_{\nu,j}(a,b)-\tilde{T}_{\nu j}(a)] = \int_0^\infty \phi^*_{\nu j}(\tfrac{\nu}{\nu+1}G_{\nu,j,a,b}(x))dH_{\nu,j,a,b}(x)$$

$$- \int_0^\infty \phi^*_{\nu j}(\tfrac{\nu}{\nu+1} G_{\nu,j,a,o}(x))dH_{\nu,j,a,o}(x).$$

A result analogous to theorem 3.6.6 of Puri and Sen [11] give

$$\max_{1\le i\le \nu}|\phi^*_{\nu j}(i/(\nu+1))-\phi^*_j(i/(\nu+1))| = O(\nu^{-\frac{1}{2}-\delta})$$

for some $\delta>0$, $j=1,2,\ldots,p$. Hence, one can write,

$$\nu^{-1}[T_{\nu,j}(a,b)-\tilde{T}_{\nu j}(a)] = I_{\nu j1}(a,b) + I_{\nu j2}(a,b) + O(\nu^{-\frac{1}{2}-\delta}), \tag{4.38}$$

where

$$I_{\nu j1}(a,b) = \int_0^\infty [\phi^*_j(\tfrac{\nu}{\nu+1}G_{\nu,j,a,b}(x))-\phi^*_j(\tfrac{\nu}{\nu+1}G_{\nu,j,a,o}(x))]dH_{\nu,j,a,b}(x), \tag{4.39}$$

$$I_{\nu j2}(a,b) = \int_0^\infty \phi^*_j(\tfrac{\nu}{\nu+1}G_{\nu,j,a,o}(x))d[H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)]. \tag{4.40}$$

On integration by parts, one can write, using (3.9), (3.13) and (3.14),

$$I_{\nu j2}(a,b) = \int_0^\infty [H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)]\phi^{*\prime}_j(\tfrac{\nu}{\nu+1}G_{\nu,j,a,o}(x))\tfrac{\nu}{\nu+1}G_{\nu,j,a,o}(x). \tag{4.41}$$

We shall now state a lemma. The proof follows the same line as lemma 4.1 of Sen and Ghosh [13] and theorem 3.1 of Ghosh and Sen [6]. For brevity, the details are omitted.

Lemma 4.5. For every $s(>0)$, there exist two positive constants $K_s^{(1)}$ and $K_s^{(2)}$, and a positive integer ν_s^* (all of which may depend on s) such that for $\nu \geq \nu_s^*$, $k \geq 1$ and $0<\delta<\frac{1}{4}$

$$P\{\sup_{-\infty<x<\infty}\ \sup_{|a|\leq K_o\nu^{-\frac{1}{2}}(\log\nu)^k}\ \sup_{|b|\leq K_1\nu^{-\frac{1}{2}}(\log\nu)^k} |H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)$$

$$-\bar{F}_{\nu,j,a,b}(x)+\bar{F}_{\nu,j,a,o}(x)| > K_s^{(1)}\nu^{-\frac{1}{2}-\delta}(\log\nu)^k\} \leq K_s^{(2)}\nu^{-s}. \tag{4.42}$$

Using also the fact that $\bar{F}_{\nu,j,a,b}(x)-\bar{F}_{\nu,j,a,o}(x) = \nu^{-1}\sum_{i=1}^{\nu}[F(x+a+b(c_i-\bar{c}_\nu)) -F(x+a)] = O(\nu^{-1}(\log\nu)^{2k})$, uniformly in x, a and $|b|\leq K_1\nu^{-\frac{1}{2}}(\log\nu)^k$, one gets,

$$P\{\sup_{-\infty<x<\infty}\ \sup_{|a|\leq K_o\nu^{-\frac{1}{2}}(\log\nu)^k}\ \sup_{|b|\leq K_1\nu^{-\frac{1}{2}}(\log\nu)^k} |H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)|$$

$$> K_s^{(1)}\nu^{-\frac{1}{2}-\delta}(\log\nu)^k\} \leq K_s^{(2)}\nu^{-s} \text{ for } \nu\geq\nu_s^{**}, \text{ say.} \tag{4.43}$$

Thus, by (4.41) and (4.43), one gets by using (3.9), (3.13) and (4.14) that

$$\sup_{|a|\leq K_o\nu^{-\frac{1}{2}}(\log\nu)^k}\ \sup_{|b|\leq K_1\nu^{-\frac{1}{2}}(\log\nu)^k} |I_{\nu j2}(a,b)|$$

$$\leq [O(\nu^{-\frac{1}{2}-\delta}(\log\nu)^k)]\ \frac{\nu}{\nu+1}\sum_{i=1}^{\nu} K[1-i/(\nu+1)]^{-1}$$

$$= O(\nu^{-\frac{1}{2}-\delta}(\log\nu)^{k+1}), \tag{4.44}$$

with probability $\geq 1-K_s^{(2)}\nu^{-s}$, for $\nu\geq\nu_s^{**}$.

Again, write

$$I_{\nu j1}(a,b) = \frac{\nu}{\nu+1}\int_0^\infty [G_{\nu,j,a,b}(x)-G_{\nu,j,a,o}(x)]\phi_j^{*\prime}(\frac{\nu}{\nu+1}[\theta G_{\nu,j,a,b}(x) + (1-\theta)G_{\nu,j,a,o}(x)])dH_{\nu,j,a,b}(x), \quad (0<\theta<1). \tag{4.45}$$

Since $G_{\nu,j,a,b}(x)-G_{\nu,j,a,o}(x) = [H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)]-[H_{\nu,j,a,b}(-x)-H_{\nu,j,a,o}(-x)]$, it follows from (4.43) that

$$\sup_{-\infty<k<\infty}\ \sup_{|a|\le K_o\nu^{-\frac{1}{2}}(\log\nu)^k}\ \sup_{|b|\le K_o\nu^{-\frac{1}{2}}(\log\nu)^k} |G_{\nu,j,a,b}(x)-G_{\nu,j,a,o}(x)| \le K_s^{(1)}\nu^{-\frac{1}{2}-\delta}(\log\nu)^k$$

with probability $\ge 1-K_s^{(2)}\nu^{-s}$ for large ν. Using arguments analogous to (4.20)-(4.27) of theorem 4.3 in Sen and Ghosh [13], one can prove now that

$$\sup_{|a|\le K_o\nu^{-\frac{1}{2}}(\log\nu)^k}\ \sup_{|b|\le K_1\nu^{-\frac{1}{2}}(\log\nu)^k} |I_{\nu j1}(a,b)| \le K_s^{(3)}\nu^{-\frac{1}{2}-\delta}(\log\nu)^{k+1},$$

with probability $\ge 1-K_s^{(4)}\nu^{-s}$ for large ν. Hence, the lemma.

For proving (4.33) we need another lemma which we prove below. For proving this lemma, we take $a_\nu^{(j)*}(i) = \phi_{\nu j}^*(i/(\nu+1)) = E\phi_j^*(U_{\nu i})$ $(1\le i\le\nu,\ 1\le j\le p)$.

Lemma 4.6. For $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$, for every $s>0$, there exist positive constants $d_s^{(1)}$ and $d_s^{(2)}$ and a positive integer ν_{so} such that for $\nu\ge\nu_{so}$,

$$P\{|\hat{\delta}_{\nu j}| \ge d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k\} \le d_s^{(2)}\nu^{-s}. \tag{4.46}$$

Proof. We prove only the case of $P\{\hat{\delta}_{\nu j} \ge d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k\}$ as the other case follows similarly. Note that

$$P\{\hat{\delta}_{\nu j} \geq d_s^{(1)} \nu^{-\frac{1}{2}} (\log \nu)^k\} \leq P\{\hat{\delta}_{\nu j}^{(2)} \geq d_s^{(1)} \nu^{-\frac{1}{2}} (\log \nu)^k\}$$

$$= P\{\tilde{T}_{\nu j}(d_s^{(1)} \nu^{-\frac{1}{2}} (\log \nu)^k) \geq 0\} = P\{\nu^{-\frac{1}{2}} \tilde{T}_{\nu j}(d_s^{(1)} \nu^{-\frac{1}{2}} (\log \nu)^k) \geq 0\} . \tag{4.47}$$

It follows from Lemma 4.5 that for every $s>0$, for large ν, with probability $\geq 1-c_s^{(2)} \nu^{-s}$,

$$\nu^{-\frac{1}{2}} [\tilde{T}_{\nu j}(d_s^{(1)} \nu^{-\frac{1}{2}} (\log \nu)^k) - T_{\nu j}(d_s^{(1)} \nu^{-\frac{1}{2}} (\log \nu)^k)] \leq c_s^{(1)} \nu^{-\delta} (\log \nu)^k. \tag{4.48}$$

Further, from theorem 4.3 of Sen and Ghosh [6], we have

$$\nu^{-\frac{1}{2}} [T_{\nu j}(d_s^{(1)} \nu^{-\frac{1}{2}} (\log \nu)^k) - T_{\nu j}(0)] - d_s^{(1)} (\log \nu)^k \leq d_s^{(2)} \nu^{-\delta} (\log \nu)^k, \tag{4.49}$$

with probability $\geq 1-c_s^{(2)} \nu^{-s}$, for large ν. Hence, from (4.47)-(4.49), it suffices to show that for large ν, for every $s>0$, there exist constants $d_s^{(1)}$ and $d_s^{(2)}$ such that

$$P\{\nu^{-\frac{1}{2}} T_{\nu j}(0) > d_s^{(1)} (\log \nu)^k\} \leq d_s^{(2)} \nu^{-\delta} (\log \nu)^k. \tag{4.50}$$

Since, $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$, for every ν, $\underset{\sim}{R}_{\nu}^{(j)+} = (R_{\nu 1}^{(j)+}, \ldots, R_{\nu\nu}^{(j)+})'$ is independent of $\underset{\sim}{s}_{\nu}^{(j)} = (s(X_{1j}), \ldots, s(X_{\nu j}))$, where $s(u) = 2c(u)-1$ i.e., $s(u) = 1$, 0 or -1, according as $u>$, $=$, or <0. Now

$$\nu^{-\frac{1}{2}} T_{\nu j}(0) = \nu^{-\frac{1}{2}} \sum_{i=1}^{\nu} \frac{1+s(X_{ij})}{2} E\, \phi_j^*(U_\nu R_{\nu i}^{(j)+})$$

$$= \tfrac{1}{2} \nu^{\frac{1}{2}} \int_0^1 \phi_j^*(u) du + \tfrac{1}{2} \nu^{-\frac{1}{2}} \sum_{i=1}^{\nu} s(X_{ij}) E \phi_j^*(U_\nu R_{\nu i}^{(j)+})$$

Since, (3.10) holds, we get from (3.13) and (3.14) the first term to be $O(\nu)$. Hence, (4.49) will be proved if one can show that for large ν

$$P\{\nu^{-\frac{1}{2}}T_{\nu jo}(0)>2d_s^{(1)}(\log\nu)^k\} \le d_s^{(2)}\nu^{-s}, \tag{4.51}$$

where

$$T_{\nu jo}(0) = \sum_{i=1}^{\nu} s(X_{ij})E\phi_j^*(U_\nu R_{\nu i}^{(j)+}), \quad 1\le j\le p.$$

Writing $g_\nu = 2d_\nu^{(1)}\nu^{\frac{1}{2}}(\log\nu)^k$, and using the Bernstein inequality, one gets,

$$P\{T_{\nu jo}(0)\ g_\nu\} \le \inf_{t>0} E[\exp\{t(T_{\nu jo}(0)-g_\nu)\}] \tag{4.52}$$

Now,

$$E[\exp\{t(T_{\nu jo}(0)-g_\nu)\}] = \exp(-tg_\nu)E[\exp(tT_{\nu jo}(0))].$$

Again,

$$E[\exp(t\ T_{\nu jo}(0))] = E\ E[\Pi_{i=1}^{\nu}\exp(ts(X_{ij})E\phi_j^*(U_\nu R_{\nu i}^{(j)+}))|\underset{\sim}{R}_\nu^{(j)+}]$$

Using the independence of $\underset{\sim}{s}_\nu^{(j)}$ and $\underset{\sim}{R}_\nu^{(j)+}$ and also the elementary inequality $\frac{1}{2}(e^x + e^{-x}) \le \exp(x^2/2)$, one gets,

$$\begin{aligned}
E[\exp(tT_{\nu jo}(0))] &= E[\Pi_{i=1}^{\nu}\{\tfrac{1}{2}\exp(tE\phi_j^*(U_\nu R_{\nu i}^{(j)+})) + \tfrac{1}{2}\exp(tE\phi_j^*(U_\nu R_{\nu i}^{(j)+}))\}] \\
&\le E\ \Pi_{i=1}^{\nu}\ \exp(\tfrac{t^2}{2}\ (E\phi_j^*(U_\nu R_{\nu i}^{(j)+}))^2) \\
&\le E\ \Pi_{i=1}^{\nu}\ \exp(\tfrac{t^2}{2}\ E\phi_j^{*2}(U_\nu R_{\nu i}^{(j)+})) \\
&= E\ \exp(\tfrac{t^2}{2}\sum_{i=1}^{\nu} E\phi_j^{*2}(U_\nu R_{\nu i}^{(j)+})) \\
&= E\ \exp(\tfrac{t^2}{2}\sum_{i=1}^{\nu} E\phi_j^{*2}(U_{\nu i})) = \exp(\frac{\nu t^2A_j^2}{2}),
\end{aligned}$$

where $A_j^2 = \int_0^1 \phi_j^{*2}(u)du$. Thus, from (4.52),

$$\begin{aligned}
P\{T_{\nu jo}(0)>g_\nu\} &\le \inf_{t>0}\exp(-tg_\nu + \frac{\nu t^2A_j^2}{2}) = \exp(-\frac{g_\nu^2}{2\nu A_j^2}) \\
&= \exp(-\frac{2d_s^{(1)2}}{A_j^2}(\log\nu)^{2k}),
\end{aligned}$$

and hence, (4.50) follows. Hence, the lemma.

It follows from Lemmas 4.4 and 4.6 that for large ν, $\nu^{-\frac{1}{2}}[\tilde{T}_{\nu j}(\hat{\delta}_\nu) - T_{\nu j}(\hat{\delta}_\nu)] = O(\nu^{-\delta}(\log \nu)^k)$ with probability $\geq$ 1-const. ν^{-s}. Again, it follows from theorem 4.3 of Sen and Ghosh [13] that $\nu^{-\frac{1}{2}}[T_{\nu j}(\hat{\delta}_\nu)-T_{\nu j}(0)] + \nu^{\frac{1}{2}}\hat{\delta}_\nu B_j = O(\nu^{-\delta}(\log \nu)^k)$ with probability $\geq$ 1-const. ν^{-s}. Hence, with probability $\geq$ 1-const. ν^{-s}, $\nu^{-\frac{1}{2}}[\tilde{T}_{\nu j}(\hat{\delta}_\nu)-T_{\nu j}(0)] + \nu^{\frac{1}{2}}\hat{\delta}_\nu B_j = O(\nu^{-s}(\log \nu)^k)$. i.e., $\nu^{\frac{1}{2}}\hat{\delta}_\nu B_j - \nu^{-\frac{1}{2}}T_{\nu j}(0) = O(\nu^{-\delta}(\log \nu)^k)$, noting that $\tilde{T}_{\nu j}(\hat{\delta}_\nu) = 0$. (4.33) now follows from theorem 4.5 of Sen and Ghosh [13]. Hence the theorem.

5. BOUNDED LENGTH (SEQUENTIAL) CONFIDENCE BANDS FOR $\underset{\sim}{\theta}$

Parallel to problems (I)-(III) of section 4, we consider here the following three problems.

<u>Problem I'</u> Confidence estimation of $\underset{\sim}{\alpha}$ assuming that $\underset{\sim}{\beta}=\underset{\sim}{0}$. More specifically we want a p-dimensional confidence rectangle for $\underset{\sim}{\alpha}$ such that the length of each side $\leq$ 2d (d>0, preassigned) and the confidence coefficient $\geq 1-\alpha$. This can be achieved by a direct extension of the results of Sen and Ghosh [13].

To see this, first note that under $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$, $\underset{\sim}{T}_{\nu o} = (T_{\nu 1 0},\ldots,T_{\nu p o})'$, ($T_{\nu j o}$'s defined after (4.51)) has a distribution independent of F diagonally symmetric about $\underset{\sim}{0}$. Hence, there exists a known constant $T_{\nu,\varepsilon}$ such that

$$P_{\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}}\{\max_{1\leq j\leq p}|T_{\nu j o}| \leq T_{\nu,\varepsilon}\} = 1-\varepsilon_\nu \to 1-\varepsilon \text{ as } \nu\to\infty. \tag{5.1}$$

For large ν, $\sqrt{\nu}\, T_{\nu,\varepsilon} \to \chi^*_{p,\varepsilon}$ where $\chi^*_{p,\varepsilon}$ is the upper 100ε% point of the distribution of the maximum of $\gamma_1,\ldots,\gamma_p$ where $\underset{\sim}{\gamma} = (\gamma_1,\ldots,\gamma_p)'$ is $N(\underset{\sim}{0},\underset{\sim}{\nu})$. Define now

$$\hat{\alpha}_{L,j,\nu} = \sup\{a: T_{\nu j o}(a) > T_{\nu,\varepsilon}\}, \tag{5.2}$$

$$\hat{\alpha}_{U,j,\nu} = \inf\{a: T_{\nu j o}(a) < -T_{\nu,\varepsilon}\}, \tag{5.3}$$

where $T_{\nu jo}(a)$ is defined in the same way as $T_{\nu jo} = T_{\nu jo}(0)$, replacing X_i's by X_i-a's ($1 \le j \le p$, $1 \le i \le \nu$). Then, $P_{\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}}\{\hat{\alpha}_{L,j,\nu} \le \alpha_j \le \hat{\alpha}_{U,j,\nu} \forall 1 \le j \le p\} =$ $P_{\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}}\{-T_{\nu,\varepsilon} \le T_{\nu jo} \le T_{\nu,\varepsilon} \forall 1 \le j \le p\} = 1-\varepsilon_\nu \to 1-\varepsilon$ as $\nu \to \infty$.

We define the stopping variable $N = N(d)$ to be the least positive integer $n(\ge n_o)$ such that $\max_{1 \le j \le p} (\hat{\alpha}_{U,j,n} - \hat{\alpha}_{L,n,n}) \le 2d$. Now, using Theorem 4.3 and Lemma 5.1 of Sen and Ghosh [13],

$$\sqrt{\nu}\ [T_{\nu j}(\hat{\alpha}_{U,j,\nu}) = T_{\nu j}(0) + \tfrac{1}{2}\hat{\alpha}_{U,j,\nu}B_j] = 0(\nu^{-\frac{1}{4}}(\log \nu)^4) \tag{5.4}$$

with probability $\ge$ 1-const. ν^{-s}, for every $s>0$, large ν. Thus noting that when $a_\nu^{(j)*}(i) = E[\phi_j^*(U_{\nu i})]$, $1 \le i \le \nu$, $T_{\nu jo}(a) = 2T_{\nu j}(a) - \int_0^1 \phi_j^*(u)du$, for all real a, it follows from (5.3) and (5.4) that

$$-\chi^*_{p,\varepsilon} - \sqrt{\nu}\ T_{\nu jo}(0) + \sqrt{\nu}\ \hat{\alpha}_{U,j,\nu}B_j \to 0 \text{ a.s. as } \nu \to \infty.$$

Similarly,

$$\chi^*_{p,\varepsilon} - \sqrt{\nu}\ T_{\nu jo}(0) + \sqrt{\nu}\ \hat{\alpha}_{L,j,\nu}B_j \to 0 \text{ a.s. as } \nu \to \infty$$

Thus,

$$\sqrt{\nu}\ (\hat{\alpha}_{U,j,\nu} - \hat{\alpha}_{L,j,\nu}) \to \frac{2\chi^*_{p,\varepsilon}}{B_j} \text{ a.s. as } \nu \to \infty.$$

Hence,

$$\max_{1 \le j \le p} \sqrt{\nu}\ (\hat{\alpha}_{U,j,\nu} - \hat{\alpha}_{L,j,\nu}) \to \frac{2\chi_{p,\varepsilon}}{\min_{1 \le j \le p} B_j} \text{ a.s. as } \nu \to \infty. \tag{5.5}$$

It follows now from the definition of N that $\lim_{d \to 0} N(d)/s(d) = 1$ a.s., where $s(d) = \chi^{*2}_{p,\varepsilon}/d^2 \min_{1 \le j \le p} B_j^2$, and as to the rate of convergence, we can make a similar statement as (5.4). Thus, generalizing the results of Sen and Ghosh [13], we get the following theorem.

<u>Theorem 5.1</u>. <u>Under the assumptions</u> $F \varepsilon \mathcal{F}_p^o$, (2.1)-(2.3), (3.9), (3.13)-(3.14) <u>and</u>

(3.17),

$$N(=N(d))\ \underline{\text{is a non-increasing function of}}\ d;\ N(d)<\infty\ \underline{\text{with probability}}\ 1,\ EN(d)<\infty\ \underline{\text{for all}}\ d>0,\ \lim_{d\to 0} N(d)=\infty \text{ a.s., } \underline{\text{and}} \lim_{d\to 0} EN(d)=\infty. \tag{5.6}$$

$$\lim_{d\to 0} N(d)/s(d) = 1 \text{ a.s.} \tag{5.7}$$

$$\lim_{d\to 0} P_{\underset{\sim}{\alpha}}\{\hat{\alpha}_{L,j,N} \leq \alpha_j \leq \hat{\alpha}_{U,j,N}\ \forall\ 1\leq j\leq p\} = 1-\varepsilon. \tag{5.8}$$

$$\lim_{d\to 0} EN(d)/s(d) = 1. \tag{5.9}$$

We now suggest an alternate procedure for the same problem. We find a confidence region R_N for $\underset{\sim}{\alpha}$ such that the maximum diameter of $R_N \leq 2d$. Our procedure is analogous to the one proposed by Srivastava [15].

We define

$$\hat{\gamma}_{j\ell}^{(n)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_j\left(\frac{n}{n+1}F_{[j]n}(x)\right)\phi_\ell\left(\frac{n}{n+1}F_{[\ell]n}(y)\right)dF_{[j,\ell]n}(x,y)-\mu_j\mu_\ell, \tag{5.10}$$

for $1\leq j\neq\ell\leq p$ where $F_{[j]n}(x)$ and $F_{[j,\ell]n}(x,y)$ are the empirical df's corresponding to the true df's $F_{[j]}(x)$ and $F_{[j,\ell]}(x,y)$ respectively, for $j=\ell$, $\hat{\gamma}_{jj}^{(n)} = \gamma_{jj} = \int_0^1 \phi_j^2(u)du-\mu_j^2$, $1\leq j\leq p$. Also, define $\hat{B}_{j,n}$ as the estimator of B_j $(1\leq j\leq p)$ as in Lemma 4.2 of Ghosh and Sen [6]. Define then

$$\hat{\underset{\sim}{T}}_n = ((\hat{\tau}_{j\ell}^{(n)}));\quad \hat{\tau}_{j\ell}^{(n)} = \hat{\gamma}_{j\ell}^{(n)}/\hat{B}_{j,n}\hat{B}_{\ell,n},\quad j,\ell=1,\ldots,p. \tag{5.11}$$

We denote by

$$\hat{\lambda}_n = \text{max. ch. root of } \hat{\underset{\sim}{T}}_n;\quad \lambda = \text{max. ch. root of } \underset{\sim}{T}, \tag{5.12}$$

where $\underset{\sim}{T}$ is defined by (3.19); finally, $\chi^2_{p,\varepsilon}$ is defined as the upper $100\varepsilon\%$ point of the chi square distribution with p degrees of freedom. Our procedure is as follows.

Starting with an initial sample of size $n_o(>p)$, we continue drawing observations one at a time according to a stopping time N defined by

$$N[=N(d)] = \text{smallest } n \geq n_o \text{ such that } \hat{\lambda}_n \leq d^2 n / \chi^2_{p,\alpha} \tag{5.13}$$

When sampling is stopped at N=n, construct the region R_n defined by

$$R_n = \{\underset{\sim}{z}: \quad (\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{z})'(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{z}) \leq d^2\} \tag{5.14}$$

Then, we have the following theorem.

<u>Theorem 5.2</u>. <u>Under the assumption that</u> $0<\lambda<\infty$ <u>and the hypothesis of Theorem 5.1</u>, <u>the results of Theorem 5.1 all hold for the stopping variable</u> N(d), <u>defined by</u> (5.13) <u>and</u> R_N, <u>defined by</u> (5.14), <u>provided we replace</u> s(d) <u>in</u> (5.7) <u>and</u> (5.9) <u>by</u>

$$\nu(d) = \chi^2_{p,\alpha} \; \lambda / d^2. \tag{5.15}$$

<u>Proof</u>. Running down the proof of Srivastava [16], it suffices to show that $\hat{\lambda}_n \to \lambda$ a.s., as $n \to \infty$; by the Courant Theorem, it thus suffices to show that

$$\hat{\underset{\sim}{T}}_n \to \underset{\sim}{T} \text{ a.s., as } n \to \infty. \tag{5.16}$$

Since, $\hat{B}_{j,n}$, j=1,...,p, converge a.s. to B_j, j=1,...,p as $n \to \infty$ (See [13]), it suffices to prove the following lemma.

<u>Lemma 5.3</u>. <u>Under</u> (3.4), (3.17), (3.18) <u>and</u> (3.19),

$$\hat{\gamma}^{(n)}_{j\ell} \to \gamma_{j\ell} \text{ a.s., as } n \to \infty, \text{ for all } 1 \leq j \neq \ell \leq p. \tag{5.17}$$

<u>Proof</u>. Since $\phi_j(u)$ is assumed to be non-decreasing, absolutely continuous and square integrable inside [0,1], by Lemma 5.1 of Hájek [9], we may write for $0<u<1$,

$$\phi_j(u) = \phi_j^{(1)}(n) - \phi_j^{(2)}(u) + \phi_j^{(3)}(u), \tag{5.18}$$

where $\phi_j^{(1)}(n)$ is a polynomial (i.e., has bounded second derivative) and

$$\int_0^1 \{\phi_j^{(k)}(u)\}^2 du < \tfrac{1}{2} [\int_0^1 \phi_j^2(u)du], \quad k=2,3, \tag{5.19}$$

where $\varepsilon>0$ is arbitrarily small. By (3.8), we may decompose the scores $a_\nu^{(j)}(i)$, $1\le i\le\nu$, also in three parts. On the first part, involving $\phi_j^{(1)}$, almost sure convergence of $F_{[j]n}$ and $F_{[j,\ell]n}$ to $F_{[j]}$ and $F_{[j,\ell]}$ (respectively) implies the a.s. convergence of the corresponding component of $\hat{\gamma}_{j\ell}^{(n)}$ to that of $\gamma_{j\ell}$; on the other components, the Schwarz inequality and (5.19) imply that the same can be made arbitrarily small by proper choice of $\varepsilon(>0)$. Q.E.D.

Remark. In (5.14), we could have taken a region $\{\underset{\sim}{z}: (\hat{\underset{\sim}{\alpha}}_n-\underset{\sim}{z})'\underset{\sim}{A}^{-1}(\hat{\underset{\sim}{\alpha}}_n-\underset{\sim}{z}) \le d^2\}$, where $\underset{\sim}{A}$ is any given positive definite matrix. In that case, we need to define $\hat{\lambda}_n$ = max. ch. root of $\underset{\sim}{A}^{-1}\underset{\sim}{T}_n$ and λ = max. ch. root of $\underset{\sim}{A}^{-1}\underset{\sim}{T}$. The proofs follows on parallel lines.

Problem II'. Confidence band for $\underset{\sim}{\beta}$ treating $\underset{\sim}{\alpha}$ as a nuisance parameter

(i) Rectangular regions. Note that under $\underset{\sim}{\beta}=\underset{\sim}{0}$, $s_{\nu j}$'s have a completely specified distribution generated by $(n!)^p$ equally likely realizations of the ranks. Hence, there exists a known $s_{\nu,\varepsilon}$ such that

$$P_{\underset{\sim}{\beta}=\underset{\sim}{0}}\{\max_{1\le j\le p} |S_{\nu j}| \le S_{\nu,\varepsilon}\} = 1-\varepsilon_\nu \to 1-\varepsilon \text{ as } \nu\to\infty.$$

For large ν, $\sqrt{S}_{\nu,\varepsilon} \to \chi^*_{p,\varepsilon}$, the upper 100$\varepsilon$% point of the distribution of the maximum of $\gamma_1,\ldots,\gamma_p$ where $\underset{\sim}{\gamma} = (\gamma_1,\ldots,\gamma_p)'$ is $N(\underset{\sim}{0},\underset{\sim}{\nu})$. Define now

$$\hat{\beta}_{L,j,\nu} = \sup\{b: S_{\nu j}(b) > S_{\nu,\varepsilon}\}$$

$$\hat{\beta}_{U,j,\nu} = \inf\{b: \; S_{\nu j}(b) < -S_{\nu,\varepsilon}\}$$

Then,

$$P_{\underset{\sim}{\beta}=\underset{\sim}{0}}\{\hat{\beta}_{L,j,\nu} \le \beta_j \le \hat{\beta}_{U,j,\nu} \; \forall \, 1 \le j \le p\}$$
$$= 1-\varepsilon_\nu \to 1-\varepsilon \text{ as } \nu\to\infty. \tag{5.20}$$

We define the stopping variable N=N(d) to be the least positive integer $n(\ge n_o)$ such that $\max_{1\le j\le p} (\hat{\beta}_{U,j,n}-\hat{\beta}_{L,j,n}) \le 2d$. Using Lemma 4.2 of Ghosh and Sen [6], we can now prove the following theorem. The proof is omitted because of its obvious analogy to Theorem 5.1.

Theorem 5.4. If $F\varepsilon \mathcal{F}_p$, then under (2.1)-(2.3), (3.9) and (3.17), N(d) as defined above and the related confidence band for $\underset{\sim}{\beta}$ satisfy the results of Theorem 5.1 provided we define

$$s(d) = Q^{-1}(\chi^{*2}_{p,\varepsilon}/[d^2 \max_{1\le j\le p} B_j^2]), \tag{5.21}$$

where $Q(n) = C_n^2$ for $n\ge 1$ and is obtained by linear interpolation for non-integer $t(>0)$.

(ii) Sprerical or Ellipsoidal regions. Here, we start by taking $n_o(\ge p)$ observations $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_{n_o}$ and continue sampling one observation at a time in accordance with the stopping variable

$$N(d) = \text{smallest } n(\ge n_o) \text{ such that } \lambda_n \le d^2 C_n^2/\chi^2_{p,\varepsilon},$$

where $\hat{\lambda}_n$ and $\chi^2_{p,\varepsilon}$ are defined in (5.12) and after that. When sampling is stopped at N=n, we construct the region R_n defined by

$$R_n = \{\underset{\sim}{\beta}: \; (\underset{\sim}{\beta}_n-\underset{\sim}{\beta})'(\beta_n-\underset{\sim}{\beta}) \le d^2\}, \tag{5.22}$$

where $\hat{\underset{\sim}{\beta}}_n$ is defined by (4.14). Then, we have the following.

Theorem 5.5. The conclusions of Theorem 5.2 *holds for* N(d) *and* R_n, *defined as above, provided we let*

$$\nu(d) = Q^{-1}(\lambda\, \chi^2_{p,\varepsilon}/d^2).$$

The proof follows along the same line as in Theorems 5.1 and 5.2.

Problem III'. Confidence bands for $\underset{\sim}{\theta}$. Here also, we can have either a rectangular or an ellipsoidal region for $\underset{\sim}{\theta}=(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. We need to change $\chi^*_{p,\varepsilon}$ and $\chi^2_{p,\varepsilon}$ to $\chi^*_{2p,\varepsilon}$ and $\chi^2_{2p,\varepsilon}$ respectively, and therefore, in view of the similarity with problems I' and II', the details are omitted.

REFERENCES

1. ADICHIE, J. N. (1967). Estimates of regression parameters based on rank tests. *Ann. Math. Statist.* *38*, 894-904.

2. ALBERT, A. (1966). Fixed size confidence ellipsoids for linear regression parameters. *Ann. Math. Statist.* *37*, 1602-1630.

3. BLUM, J. R., HANSON, D. L., and ROSENBLATT, J. I. (1963). On the central limit theorem for the sum of a random number of independent random variables. *Zeit Wahrseh. Verw. Geb.* *1*, 389-393.

4. CHOW, Y. S., and ROBBINS, H. (1965). On the asymptotic theory of fixed-width sequential confidence intervals for the mean. *Ann. Math. Statist.* *36*, 457-462.

5. GHOSH, M., and SEN, P. K. (1971). Sequential confidence interval for the regression coefficient based on Kendall's tau. *Calcutta Statist. Assoc. Bull.* *20*, 23-36.

6. GHOSH, M., and SEN, P. K. (1972a). On bounded length confidence interval for the regression parameter based on a class of rank statistics. *Sankhyā, Ser. A.* *34*, 33-52.

7. GHOSH, M., and SEN, P. K. (1972b). On some sequential simultaneous confidence intervals procedures. *Ann. Inst. Statist. Math.* (to appear)

8. GLESER, L. J. (1965). On the asymptotic theory of fixed size sequential confidence bounds for linear regression parameters. *Ann. Math. Statist.* *36*, 463-467.

9. HÁJEK, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. Ann. Math. Statist. 39, 325-346.

10. MOGYORODI, J. (1967). Limit distributions for sequences of random variables with random indices. Trans. 4th Prague Confer. Infor. Th. Statist. Dec. Fns. Random Proc. 463-470.

11. PURI, M. L., and SEN, P. K. (1971). Nonparametric Methods in Multivariate Analysis. John Wiley: New York

12. SEN, P. K. (1969). On a class of rank order tests for the parallelism of several regression lines. Ann. Math. Statist. 40, 1668-1683.

13. SEN, P. K. and GHOSH, M. (1971). On bounded length sequential confidence intervals based on one sample rank order statistics. Ann. Math. Statist. 42, 189-203.

14. SEN, P. K., and PURI, M. L. (1969). On robust nonparametric estimation in some multivariate linear models. Multivariate Analysis - II (Ed: P. R. Krishnaiah), Academic Press, New York, pp. 33-52.

15. SRIVASTAVE, M. S. (1967). On fixed-width confidence bounds for regression parameters and mean vector. Jour. Roy. Statist. Soc. Ser. B. 29, 132-140.

16. SRIVASTAVA, M. S. (1971). On fixed-width confidence bounds for regression parameters. Ann. Math. Statist. 42, 1403-1411.