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3. and Department of Mathematics, University of North Carolina at Chapel Hill.

ELEMENTARY STRONG MAPS AND TRANSVERSAL GEOMETRIES[1,2]

T. A. Dowling *and* D. G. Kelly[3]

Department of Statistics
University of North Carolina at Chapel Hill

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T. A. Dowling *and* D. G. Kelly

Abstract

Let $H \to G$ be a strong map between two combinatorial geometries on the same set X. The rank function, flats, and independent sets of G are characterized in terms of a factorization of $H \to G$ into elementary strong maps. When H is the free geometry on X, these results lead to a representation of G as the basis intersection of a family of transversal geometries, and dually, as the basis intersection of a family of principal geometries.

Elementary Strong Maps and Transversal Geometries

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One of the more familiar classes of combinatorial geometries [6] is the class of transversal geometries of Edmonds and Fulkerson [7]. Dually related to these are the principal geometries of Brown [1,2,3] (called 'F-products' by him), which are defined in Section 1 below.

In this paper we obtain Brown's theorem on the orthogonality of transversal and principal geometries, as a special case of more general results (Sections 2 and 3) on the factorization of the closure map of a pregeometry into elementary maps. These results also lead to theorems on 'representation' of geometries in the following sense:

If, for a geometry G on a set X, there is a family $\{G_i\}$ of geometries on X such that a set B is a basis of G if and only if B is a basis of each G_i, then we say that G is the *basis intersection* of the family $\{G_i\}$. ('Basis-family intersection' would be more precise.) Similar definitions can be made for *spanning-set intersection, independent-set intersection,* etc. In Section 4 we show that an arbitrary pregeometry on a finite set X is the independent-set intersection (and thus also the basis intersection) of a finite family of principal pregeometries of the same rank on X. Dually, in Section 5 we show that an arbitrary pregeometry is the spanning set intersection (and again also the basis intersection) of transversal pregeometries on the same set.

1. Introduction. In this section we collect various definitions and results concerning pregeometries and transversals of families of sets. For further details the reader is referred to [6,9].

Let L be a finite lattice. An element y of L *covers* x if $x < y$ but no element z exists with $x < z < y$. An *atom* of L is an element covering the zero element. A *chain* in L is a subset C of L which is linearly ordered by the order relation of L. If C has minimal element x and maximal element y, C is an x-y chain. An x-y chain in L is *saturated* if it preserves the cover relation of L. A finite lattice L is *geometric* when y covers x if and only if $y = x \vee p$ for some atom $p \nleq x$. Every saturated x-y chain in a geometric lattice has the same cardinality.

Let X be a finite set and let $\mathcal{B}$ be the lattice of subsets of X, ordered by inclusion. A *pregeometry (matroid)* G on X is a set of subsets of X, which contains X, and which, when ordered by inclusion, is a geometric lattice. Members of G are *closed sets* or *flats*. The *closure* of any subset A of X is the minimal flat containing A, denoted $\bar{A}$ (or $\bar{A}^G$ if more than one pregeometry on X is under consideration). The map $A \mapsto \bar{A}$ from $\mathcal{B}$ to G is a closure operator on $\mathcal{B}$ (i.e., $A \subseteq \bar{A}$, $\bar{\bar{A}} = \bar{A}$, and $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$) which satisfies in addition the *exchange property*: if $a,b \in X$, $A \subseteq X$, and $a \in \overline{A \cup b}$ but $a \notin \bar{A}$, then $b \in \overline{A \cup a}$. A consequence is that the flats covering any flat A partition the elements of $X-A$. G is a *geometry* on X if the empty set and all singleton subsets of X are closed. $\mathcal{B}$ is the *free geometry* on X.

The *rank* $r(A)$ (or $r_G(A)$) of any subset A of X is the size, less one, of all saturated $\bar{\phi}$-$\bar{A}$ chains in G. The rank of G is the rank of X. Flats of rank one, two, three are points, lines, planes, respectively, of G.

The *nullity* of a subset A of X is $n(A) = |A| - r(A)$, where $|\cdot|$ is the cardinality function. The nullity of G is the nullity of X.

A subset A of X is *independent* (or G-independent) if $n(A) = 0$. All maximal independent subsets of any set B, called *bases* of B, have the same cardinality, equal to the rank of B. A basis of G is a basis of X. A subset A *spans* B if $\bar{A} \supseteq B$, or equivalently, if A contains a basis of B. A spanning set of G is a spanning set of X. An element contained in every basis of B is an *isthmus* of B. If none exists, B is *isthmus-free*. An isthmus of G is an isthmus of X. A *loop* of G is an element of $\bar{\phi}$, i.e., an element contained in no basis of G.

A pregeometry G on X is characterized by any of the following families of subsets of X: flats, independent sets, bases, spanning sets. A nonempty set $\mathcal{I}$ of subsets of X is the set of independent sets of a (unique) pregeometry G on X iff $\mathcal{I}$ is an order ideal in $\mathcal{B}$ and all maximal $\mathcal{I}$-sets contained in any set are of the same cardinality.

Corresponding to a pregeometry G on X is its *orthogonal pregeometry* G^* on X. The bases of G^* are the set complements of the bases of G. Clearly $(G^*)^* = G$ and the rank of G^* is the nullity of G. Other relations between G^* and G include

(1.1)

(a) A spans G^* iff X-A is G-independent,

(b) A is a G^*-flat iff X-A is isthmus-free in G,

(c) a is an isthmus of G^* iff a is a loop of G,

(d) $r^*(A) = n(X) - n(X-A) = |A| + r(X-A) - r(X)$,

where r^* is the rank function of G^*.

Let H, G be two pregeometries on X. If $H \geq G$, i.e., if every G-flat is an H-flat, then the identity function on X extends to a *strong map* [6,8] from H to G and G is called a *quotient* of H. An equivalent condition is that $\bar{A}^G \supseteq \bar{A}^H$ for every subset A of X. We denote this strong map, which takes each H-flat to the minimal G-flat containing it (i.e., to its G-closure), by $H \to G$. In particular, the canonical closure map of G is $\mathcal{B} \to G$.

If $H \to G$ is a strong map, then $r_G(X) \leq r_H(X)$ with equality if and only if $G = H$. The difference $r_H(X) - r_G(X)$ is the *nullity* of $H \to G$. Since $r_B(X) = |X|$, the nullity of the closure map $\mathcal{B} \to G$ is the nullity of G, as defined earlier. A strong map $H \to G$ of nullity zero is *trivial* (since $G = H$), and one of nullity one is *elementary*. Every strong map $H \to G$ of nullity n may be written as a composite

$$H = G_0 \to G_1 \to \dots \to G_n = G$$

of n elementary maps $G_{i-1} \to G_i$ $(i = 1,\dots,n)$, using for example the *lift construction* of Higgs [8]. Such a sequence we call an *elementary factorization* of $H \to G$. It is not unique in general.

Elementary maps may be factored further in the category of all pregeometries and strong maps. A theorem of Brylawski [4] and Higgs [8] provides that an elementary map $H \to G$ factors as an *injection* of H into a *single-element extension* [5,6] K of H, followed by a *contraction* [6] of the added element e. There is consequently a one-one correspondence between single-element extensions of H and quotients of H under elementary maps. We describe below the basic facts we need concerning single-element extensions [5] and the corresponding elementary maps.

A *modular cut* in a pregeometry H on X is a nonempty order filter M in H with the property that if A, B are in M and each covers their infimum $A \wedge B$, then $A \wedge B$ is in M. Given a modular cut M in H, let C_M be the set of H-flats not in M, but covered by flats in M. Then every flat of C_M is covered by a unique member of M. A single-element extension K on $X \cup e$ of H is determined by a modular cut M in H and conversely. The corresponding elementary quotient $G = K/e$ of H on X is then given by $G = H\text{-}C_M$. The elementary map $H \to G$ fixes all flats of $H\text{-}C_M$, and takes each flat of C_M to the unique M-flat covering it. One exception to the foregoing must be noted. If $M = H$, then C_M is empty and the map $H \to G$ is trivial, not elementary. The rank function of G (when $M \neq H$) is related to that of H by

$$(1.2) \qquad r_G(A) = \begin{cases} r_H(A) - 1 & \text{if } \bar{A}^H \text{ is in } M. \\ r_H(A) & \text{if not.} \end{cases}$$

We use the following notation to denote a trivial or elementary map. If $\mathcal{E}$ is any antichain of subsets of X such that the set of H-flats containing one or more members of $\mathcal{E}$ is a modular cut M of H, and if G is the elementary (or trivial) quotient of H under the corresponding map, we write $H \overset{\mathcal{E}}{\to} G$. We could of course always take $\mathcal{E}$ to be the set of miminal flats of M, but it will be convenient for our purposes not to assume the members of $\mathcal{E}$ are flats of H. Note that $H \overset{\mathcal{E}}{\to} G$ is trivial if and only if $\bar{\phi}^H$ contains a member of $\mathcal{E}$.

If the modular cut M defining an elementary (or trivial) map $H \to G$ is a principal filter of G, the map $H \to G$ will be called a *principal map*. Such a map is specified by any subset E spanning the generator of M in H, and may be written $H \overset{E}{\to} G$. We call a pregeometry G on X a *principal pregeometry* if the closure map $\mathcal{B} \to G$ admits an elementary factorization into principal maps. Such pregeometries were first investigated by Brown [1,2,3], who called them "F-products".

Let $\underline{E} = (E_1, E_2, \ldots, E_m)$ be a finite family of subsets of X. (The E_i need not be empty, nor need they be distinct.) A *transversal* of $\underline{E}$ is an m-subset $\{x_1, x_2, \ldots, x_m\}$ of X for which $x_i \in E_i$ for $i = 1, \ldots, m$. A *partial transversal* of $\underline{E}$ is a transversal of a subfamily of $\underline{E}$. A corollary [9] of the "marriage" theorem of P. Hall provides that a subset A of X contains a partial transversal of $\underline{E}$ of size $n \le m$ if and only if for every $k \le m$ and choice of $i_1, \ldots, i_k$ with $1 \le i_1 < \ldots < i_k \le m$, the inequality

$$|A \cap \bigcup_{j=1}^{k} E_{i_j}| \ge k - (m-n)$$

holds.

It was observed by Edmonds and Fulkerson [7] that the set of partial transversals of a family $\underline{E}$ satisfy the conditions given above for the independent sets of a pregeometry on X. Such a pregeometry is called a *transversal pregeometry*, and will be denoted here by $T(\underline{E})$, or $T(E_1, \ldots, E_m)$. The bases of $T(\underline{E})$ are the maximal partial transversals of $\underline{E}_0$.

2. Rank and closure for elementary factorizations. We consider throughout this section a fixed elementary factorization

$$(2.1) \qquad H = G_0 \xrightarrow{\mathcal{E}_1} G_1 \xrightarrow{\mathcal{E}_2} \cdots \xrightarrow{\mathcal{E}_n} G_n = G$$

of a strong map $H \to G$ of nullity n.

By (1.2) we have, for any $A \subseteq X$

$$(2.2.) \qquad r_H(A) - r_G(A) = |\{i\colon 1 \le i \le n \text{ and } \bar{A}^{G_{i-1}} \text{ contains a member of } \mathcal{E}_i\}|$$

(Observe that if $H = \mathcal{B}$, then $r_H - r_G$ is the nullity function of G.)

Define, for $A \subseteq X$,

(2.3) $\quad f(A) = |\{i\colon 1 \le i \le n \text{ and } A \text{ contains a member of } \mathcal{E}_i\}|.$

The following proposition is then immediate.

Proposition 2.1: Let A be any subset of X. Then

(i) $0 \le f(A) \le r_H(A) - r_G(A)$.

(ii) If A is a G-flat, then $f(A) = r_H(A) - r_G(A)$.

Theorem 2.2: For any subset A of X,

$$r_G(A) = r_H(\bar{A}) - f(\bar{A}) = \min_{B \supseteq A} [r_H(B) - f(B)],$$

where $\bar{A}$ is the G-closure of A.

Proof: $r_G(A) = r_G(\bar{A}) = r_H(\bar{A}) - f(\bar{A})$ by Proposition 2.1 (ii). And if $B \supseteq A$ is arbitrary, then $r_G(A) \le r_G(B) \le r_H(B) - f(B)$ by Proposition 2.1 (i).

□

Corollary 2.2a: $r_G(A) = r_H(A)$ if and only if

(2.4) $\quad$ for each $B \supseteq A$, $f(B) \le r_H(B) - r_H(A)$.

Proof: If $r_G(A) = r_H(A)$ and $B \supseteq A$, then $r_H(A) = r_G(A) \le r_H(B) - f(B)$. If $r_G(A) \ne r_H(A)$, then $r_H(A) > r_G(A) = r_H(\bar{A}) - f(\bar{A})$, and (2.4) is violated by $B = \bar{A}$. □

Corollary 2.2b: A is G-independent if and only if A is H-independent and

$$\text{for each } B \supseteq A, \quad f(B) \le r_H(B) - |A|.$$

Corollary 2.2c: If $H = \mathcal{B}$ in (2.1), then for any subset A of X

$$r_G(A) \;=\; |A| - f(\bar{A}) \;=\; \min_{B \supseteq A} \; [|B| - f(B)],$$

and A is G-independent if and only if

$$\text{for each } B \supseteq A, \;\; f(B) \le |B-A|.$$

Corollary 2.2d: If $(F_1,\ldots,F_n)$ is any permutation of $(E_1,\ldots,E_n)$ for which the sequence

$$(2.5) \qquad H = G'_0 \xrightarrow{F_1} G'_1 \xrightarrow{F_2} \cdots \xrightarrow{F_n} G'_n$$

is well-defined (in the sense that for $i = 1,\ldots,n$ the G'_{i-1}-flats above members of F_i form a modular cut which is not all of G'_{i-1}), then $G'_n = G$.

Proof: Theorem 2.2 provides that as long as (2.5) is a sequence of elementary maps, G'_n is determined uniquely by H and the set function f. Neither of these depends on the order of the sequence $(E_1,\ldots,E_n)$. □

The hypothesis of Corollary 2.2d is of course quite restrictive. But in the special case treated in later sections, in which each E_i is a singleton, the filter of flats above members of E_i will be a principal filter and hence a modular cut.

Theorem 2.3: A is a flat of G if and only if

$$(2.6) \qquad \text{for each } B \supset A, \;\; f(B) - f(A) < r_H(B) - r_H(A).$$

Proof: If A is a G-flat and $B \supset A$, then

$$f(A) \;=\; r_H(A) - r_G(A) \quad \text{by Proposition 2.3 (ii)},$$

$$f(B) \;\le\; r_H(B) - r_G(B) \quad \text{by Proposition 2.3 (i)},$$

and

$$r_G(B) - r_G(A) > 0.$$

Hence

$$f(B) - f(A) \leq r_H(B) - r_G(B) - r_H(A) + r_G(A) < r_H(B) - r_H(A).$$

If A is not a G-flat, then since $r_G(\bar{A}) = r_G(A)$ we have

$$r_H(\bar{A}) - r_H(A) = r_H(\bar{A}) - r_G(\bar{A}) - [r_H(A)-r_G(A)]$$

$$\leq f(\bar{A}) - f(A) \quad \text{by Proposition 2.1,}$$

so that (2.6) is violated by $B = \bar{A}$. □

Corollary 2.3a: If $H = \mathcal{B}$ in (2.1), then A is a G-flat if and only if

$$\text{for each } B \supset A, \ f(B) - f(A) < |B-A|.$$

When $H = \mathcal{B}$, the function $r_H - r_G$ defined by (2.2) is, as noted above, the nullity function of G. As such it is (lower) *semimodular*:

$$n(A \cup B) + n(A \cap B) \geq n(A) + n(B).$$

Although many of the results of this section for the function f defined by (2.3) are similar to the corresponding results in terms of the nullity function n, f is *not* in general a semimodular function.

3. Elementary maps and partial transversals.

Throughout this section, we consider an arbitrary pregeometry G on X of nullity n, and a fixed factorization of the closure map $\mathcal{B} \to G$. The proof of Theorem 3.3 will require consideration of the more general case, in which trivial maps as well as elementary maps are admitted in the factorization. The notation is simplified somewhat by the following observation: If $H \to G$ is any strong map, and $\mathcal{E}$ is an antichain in $\mathcal{B}$ defining the trivial map on H, then $\mathcal{E}$ defines the trivial map on G also. For $\mathcal{E}$ defines the trivial map on H if and only if $\bar{\phi}^H$ contains a member E of $\mathcal{E}$; since $\bar{\phi}^G \supseteq \bar{\phi}^H$, the conclusion follows at once. Accordingly, any trivial map appearing in a factorization commutes with any elementary map immediately following it; the defining antichains of the two maps may be left unchanged. Thus let

$$\mathcal{B} = G_0 \xrightarrow{\mathcal{E}_1} G_1 \xrightarrow{\mathcal{E}_2} \cdots \xrightarrow{\mathcal{E}_n} G_n \xrightarrow{\mathcal{E}_{n+1}} \cdots \xrightarrow{\mathcal{E}_m} G_m = G \tag{3.1}$$

be a fixed factorization of the closure map $\mathcal{B} \to G$ into n elementary and $m - n$ trivial maps, where $m \geq n$. By the above remarks, we may without loss of generality assume that $\mathcal{E}_1, \ldots, \mathcal{E}_n$ define elementary maps, and $\mathcal{E}_{n+1}, \ldots, \mathcal{E}_m$ define trivial maps. Thus $G_n = G_{n+1} = \cdots = G_m = G$, and we may apply the results of Section 2 to G with f defined by

$$f(A) = |\{i\colon 1 \leq i \leq n \text{ and } A \text{ contains a member of } \mathcal{E}_i\}|. \tag{3.2}$$

We denote by $\underline{E} = (E_1, \ldots, E_m)$ a vector of subsets of X, and by $\underline{\mathcal{E}}$ the product set $\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$.

Theorem 3.1: Suppose G is a pregeometry of nullity n on X and (3.1) is a factorization of the closure map $\mathcal{B} \to G$ into n elementary maps and $m - n$ trivial maps. Then a subset of X is independent in G if and only if its complement contains a partial transversal of size n of every $\underline{E}$ in $\underline{\mathcal{E}}$.

Proof: We can assume that $\mathcal{E}_{n+1}, \ldots, \mathcal{E}_m$ define the trivial map, so that (3.2) holds. By Corollary 2.3c, $X - A$ is G-independent if and only if for all $B \supseteq X - A$, (3.3) $\quad f(B) \le |A \cap B|$

As noted in the introduction, A contains a partial transversal of size n of $\underline{E}$ if and only if

(3.4) $\quad k - (m-n) \le \left|A \cap \bigcup_{j=1}^{k} E_{i_j}\right|$ for all $k \le m$ and

$$1 \le i_1 < \ldots < i_k \le m.$$

We show (3.3) holds for $B \supseteq X - A$ if and only if (3.4) holds for all $\underline{E}$ in $\mathcal{E}$. Suppose (3.3) holds for $B \supseteq X - A$ and let $\underline{E}, k, i_1, \ldots, i_k$ be given. Define

$$B = (X-A) \cup \bigcup_{j=1}^{k} E_{i_j} .$$

At most $m - n$ of the i_j exceed n, so by (3.2), $f(B) \ge k - (m-n)$. Then from (3.3),

$$k - (m-n) \le f(B) \le |A \cap B| = \left|A \cap \bigcup_{j=1}^{k} E_{i_j}\right|.$$

Conversely, suppose (3.4) holds for all $\underline{E}$ in $\mathcal{E}$. It is sufficient to prove (3.3) for B closed in G, for if (3.3) fails for B but holds for $\bar{B}$, then

$$f(\bar{B}) - f(B) < |\bar{B} - B|,$$

which contradicts Corollary 2.2 (c).

Denote $f(B)$ by ℓ. Then there exist subsets $E_{i_1}, \ldots, E_{i_\ell}$ of B with $1 \le i_1 < \ldots < i_\ell \le n$ and E_{i_j} in $\mathcal{E}_{i_j}$ for $j = 1, \ldots, \ell$. Further, since $\mathcal{E}_{n+1}, \ldots, \mathcal{E}_m$ define the trivial map on G, there exist subsets $E_{n+1}, \ldots, E_m$ of $\bar{\phi}$, and hence of B, with E_i in $\mathcal{E}_i$ for $i = n+1, \ldots, m$. Thus

$$B \supseteq \bigcup_{j=1}^{\ell} E_{i_j} \cup \bigcup_{i=n+1}^{m} E_i,$$

so by (3.4)

$$|A \cap B| \geq \left| A \cap \left(\bigcup_{j=1}^{\ell} E_{i_j} \cup \bigcup_{i=n+1}^{m} E_i \right) \right| \geq \ell = f(B). \qquad \square$$

Theorem 3.2: Let (3.1) be any factorization of the closure map $\mathcal{B} \to G$ into n elementary maps and $m - n$ trivial maps. Then the nullity n of G satisfies

$$n = \max\{j: \text{every } \underline{E} \text{ in } \underline{\mathcal{E}} \text{ has a partial transversal of size } j\}.$$

Proof: Certainly every $\underline{E}$ in $\underline{\mathcal{E}}$ has a partial transversal of size n; for Theorem 3.1 provides that the complement of any G-independent set contains one. So we need to show that there are $\underline{E}$ in $\underline{\mathcal{E}}$ with no partial transversals of size exceeding n.

Again assume that $\mathcal{E}_{n+1},\ldots,\mathcal{E}_m$ define trivial maps in (3.1), so each contains some $E_i \subseteq \bar{\phi}$. Denoting $|\bar{\phi}|$ by k, we have by Corollary 2.2c that

$$0 = r(\bar{\phi}) = k - f(\bar{\phi}).$$

Thus there exist subsets $E_{i_1},\ldots,E_{i_k}$ of $\bar{\phi}$ with $1 \leq i_1 <\ldots< i_k \leq n$ and E_{i_j} in $\mathcal{E}_{i_j}$, $j = 1,\ldots,k$. But then

$$k = |\bar{\phi}| \geq \left| \bigcup_{j=1}^{k} E_{i_j} \cup \bigcup_{i=n+1}^{m} E_i \right|.$$

So no $\underline{E}$ in $\underline{\mathcal{E}}$ containing $E_{i_1},\ldots,E_{i_k},E_{n+1},\ldots,E_m$ can have a partial transversal of size exceeding n. $\square$

The preceding two theorems provide a short proof of the following theorem, due to T. Brown [3].

Theorem 3.3: Let $\underline{E} = (E_1,\ldots,E_m)$ be an arbitrary vector of subsets of X, $T(\underline{E})$ its induced transversal pregeometry, and $P(\underline{E})$ the principal pregeometry defined by the sequence

$$(3.5) \qquad \mathcal{B} = G_0 \xrightarrow{E_1} P_1 \xrightarrow{E_2} \cdots \xrightarrow{E_m} P_m = P(\underline{E}).$$

Then $T(\underline{E})$ and $P(\underline{E})$ are orthogonal pregeometries.

Proof: Observe first that since the E_i are not required to be closed and trivial maps are admitted, the factorization (3.5) is well-defined. Let $P(\underline{E})$ have nullity n. Then by Theorem 3.2, n is the rank of $T(\underline{E})$. It follows from Theorem 3.1 that $X - A$ is independent in $P(\underline{E})$ if and only if A spans $T(\underline{E})$. □

Corollary 3.3a: If $\underline{F} = (F_1,\ldots,F_m)$ is any permutation of the members of $\underline{E} = (E_1,\ldots,E_m)$, then $P(\underline{F}) = P(\underline{E})$.

Two other easy consequences of Theorem 3.3 are the following results, well-known in transversal theory.

Corollary 3.3b: If $H = T(E_1,\ldots,E_m)$ is a transversal pregeometry of rank n, then there exist $i_1 < \ldots < i_n$ such that $H = T(E_{i_1},\ldots,E_{i_n})$.

Proof: In the factorization (3.5) of the closure map $\mathcal{B} \to H^*$, delete the $m - n$ E_i's defining trivial maps. □

Corollary 3.3c: If $H = T(E_1,\ldots,E_i,\ldots,E_m)$ is a transversal pregeometry, and I is any set of isthmuses of $X - E_i$, then $H = T(E_1,\ldots,E_i \cup I,\ldots,E_m)$.

Proof: We may assume $i = m$ by relabeling if necessary. Then the closure map $\mathcal{B} \to H^* = P(E_1,\ldots,E_m)$ may be factored

$$\mathcal{B} = G_0 \xrightarrow{E_1} P_1 \xrightarrow{E_2} \dots P_{m-1} \xrightarrow{F} P_m = H^*,$$

where F is any set between E_m and its P_{m-1}-closure. But the P_{m-1}-closure of E_m is the same as its H^*-closure, and the H^*-closure of E_m is the union of E_m with all isthmuses of $X - E_m$ in H. □

4. Basis intersections of principal pregeometries. In this section, we again omit trivial maps and consider an elementary factorization

$$(4.1) \qquad \mathcal{B} = G_0 \xrightarrow{\mathcal{E}_1} G_1 \xrightarrow{\mathcal{E}_2} \dots \xrightarrow{\mathcal{E}_n} G_n = G$$

of the closure map of a pregeometry G of nullity n on X. The results of Section 3, therefore, apply with $m = n$. In particular, by Theorem 3.2, every $\underline{E}$ in $\underline{\mathcal{E}}$ has a transversal, so every $T(\underline{E})$ has rank n and every $P(\underline{E})$ nullity n.

The next theorem states that every pregeometry on X is the independent-set intersection of principal pregeometries on X.

Theorem 4.1: If G is a pregeometry of nullity n on X, there exists a family $\mathcal{P}$ of principal pregeometries of nullity n on X, such that a subset is G-independent if and only if it is P-independent for every P in $\mathcal{P}$.

Proof: Let (4.1) be an elementary factorization of $\mathcal{B} \to G$. By Theorem 3.1, a subset $X - A$ is G-independent if and only if, for every $\underline{E}$ in $\underline{\mathcal{E}}$, A spans $T(\underline{E})$. But the orthogonal of $T(\underline{E})$ is the principal pregeometry $P(\underline{E})$, by Theorem 3.3, so $X - A$ is G-independent if and only if it is independent in every $P(\underline{E})$. Thus $\mathcal{P} = \{P(\underline{E}): \underline{E} \in \underline{\mathcal{E}}\}$ is the desired family. □

In what follows we use $\mathcal{P}(\underline{E})$, or simply $\mathcal{P}$, to denote the family $\{P(\underline{E}): \underline{E} \in \underline{E}\}$ of principal pregeometries.

Corollary 4.1a: A subset of X is a basis of G if and only if it is a basis of every P in $\mathcal{P}$.

The rank function of G is determined by the rank functions of $\mathcal{P}$ by means of

Theorem 4.2: The rank in G of any subset of X is its minimum rank over all P in $\mathcal{P}$.

Proof: Let r, r^* denote the rank functions of G, G* respectively, and for any $\underline{E}$ in $\underline{E}$, let $r_{\underline{E}}$, $r_{\underline{E}}^*$ denote the rank functions of $P(\underline{E})$, $T(\underline{E})$, respectively. For a subset A of X, by (1.1d),

$$r_{\underline{E}}(A) = |A| - n + r_{\underline{E}}^*(X-A),$$

so

$$\min_{\underline{E}} r_{\underline{E}}(A) = |A| - n + \min_{\underline{E}} r_{\underline{E}}^*(X-A). \tag{4.2}$$

For the transversal pregeometry $T(\underline{E})$, the rank function is given [9, 10] by

$$r_{\underline{E}}^*(X-A) = |X-A| - \max_{B \supseteq A} \delta_{\underline{E}}(X-B), \tag{4.3}$$

where $\delta_{\underline{E}}$ is the *deficiency function* [10]

$$\begin{aligned} \delta_{\underline{E}}(X-B) &= |X-B| - |\{i: (X-B) \cap E_i \neq \phi\}| \\ &= |X-B| - n + |\{i: B \supseteq E_i\}|. \end{aligned} \tag{4.4}$$

Thus by (4.2), (4.3) and (4.4),

$$\min_{\underline{E}} r_{\underline{E}}(A) = |X| - \max_{\underline{E}} \max_{B \supseteq A} (|X-B| + |\{i:B\supseteq E_i\}|)$$

$$= |X| - \max_{B \supseteq A} (|X-B| + \max_{\underline{E}} |\{i:B\supseteq E_i\}|).$$

Now for fixed B,

$$\max_{\underline{E}} |\{i: B \supseteq E_i\}| = f(B),$$

so

$$\min_{\underline{E}} r_{\underline{E}}(A) = |X| - \max_{B \supseteq A} (|X-B| + f(B))$$

$$= \min_{B \supseteq A} (|B| - f(B))$$

$$= r(A),$$

by Corollary 2.2c. □

Theorem 4.3: Every flat of G is a flat of the same rank in some P in $\mathcal{P}$.

Proof: Let A be a flat of G. Then $n(A) = f(A)$, and by Theorem 2.3,

(4.5) $|A| - f(A) < |B| - f(B)$

for all $B \supset A$. For any $\underline{E}$ in $\underline{\mathcal{E}}$, define $f_{\underline{E}}$ by (2.2) for the principal factorization

$$\mathcal{B} = G_0 \xrightarrow{E_1} P_1 \xrightarrow{E_2} \cdots \xrightarrow{E_n} P_n = P(\underline{E}).$$

Clearly $f_{\underline{E}}(B) \le f(B)$ for any subset B. Choose $\underline{E}$ in $\underline{\mathcal{E}}$ so that $f_{\underline{E}}(A) = f(A)$. Then if $B \supset A$, we have by (4.5),

$$|A| - f_{\underline{E}}(A) = |A| - f(A)$$

$$< |B| - f(B)$$

$$\le |B| - f_{\underline{E}}(B),$$

so A is a flat of $P(\underline{E})$, by Corollary 2.3a. Thus, $f_{\underline{E}}(A)$ is the nullity of A in $P(\underline{E})$. Since $f_{\underline{E}}(A) = f(A)$, $r_{\underline{E}}(A) = r(A)$. □

5. Basis intersections of transversal pregeometries. The results in this section are dual to those of Section 4. All notation will be as in that section. Our next theorem states that every pregeometry on X is the spanning-set intersection of transversal pregeometries on X.

Theorem 5.1: If H is a pregeometry of rank n on X, there exists a family $\mathcal{T}$ of transversal pregeometries of rank n on X, such that a subset spans H if and only if it spans every T in $\mathcal{T}$.

Proof: The orthogonal pregeometry $G = H^*$ of H has nullity n, so its closure map has a factorization of the form (4.1). Then a subset A of X spans H if and only if $X - A$ is independent in G, which by Theorem 3.1 is equivalent to A containing a transversal of every $\underline{E}$ in $\underline{\mathcal{E}}$. Thus X contains a transversal of every $\underline{E}$ in $\underline{\mathcal{E}}$, so each $T(\underline{E})$ has rank n, and $\mathcal{T} = \{T(\underline{E}): \underline{E} \in \underline{\mathcal{E}}\}$ is the desired family. □

As with $\mathcal{P}$, we let $\mathcal{T}(\underline{\mathcal{E}})$, or simply $\mathcal{T}$, denote $\{T(\underline{E}): \underline{E} \in \underline{\mathcal{E}}\}$.

Corollary 5.1a: A subset of X is a basis of H if and only if it is a basis of every T in $\mathcal{T}$.

The next two theorems are the analogues of Theorems 4.2 and 4.3.

Theorem 5.2: The rank in H of any subset of X is its minimum rank over every T in $\mathcal{T}$.

Proof: For any subset X - A of X, we have by (4.2) that

$$\min_{\underline{E}} r_{\underline{E}}^*(X-A) = n - |A| + \min_{\underline{E}} r_{\underline{E}}(A)$$

$$= n - |A| + r(A)$$

$$= |X-A| + r(A) - r(X)$$

$$= r^*(X-A). \qquad \square$$

Theorem 5.3: Every flat of H is a flat of the same rank in some T in $\mathcal{T}$.

Proof: By (1.1b), X - A is a flat of H if and only if A is an isthmus-free set of G = H*, that is, if and only if

(5.1) $\quad r(A-a) = r(A)$

for every a in A. By Corollary 2.2c, (5.1) holds if and only if for every $a \in A$, $B \supseteq A-a$, there exists $A_1 \supseteq A$ such that

(5.2) $\quad |B| - f(B) \geq |A_1| - f(A_1).$

(In this case, the G-closure of A serves as A_1.) Similarly, X - A is a flat of $T(\underline{E})$ if and only if for all $a \in A$, $B \supseteq A-a$, there exists $A_2 \supseteq A$ such that

$$|B| - f_{\underline{E}}(B) \geq |A_2| - f_{\underline{E}}(A_2).$$

Suppose X - A is a flat of H, and let $A_2 = \bar{A}$, the G-closure of A. Then $f(\bar{A}) = n(\bar{A})$, and we can choose $\underline{E}$ in $\underline{\mathcal{E}}$ such that $f_{\underline{E}}(\bar{A}) = f(\bar{A})$. Then since $f_{\underline{E}}(B) \leq f(B)$ for any subset B, we have, by the remark following (5.2), for any $a \in A$, $B \supseteq A-a$,

$$\begin{aligned} |B| - f_{\underline{E}}(B) &\geq |B| - f(B) \\ &\geq |\bar{A}| - f(\bar{A}) \\ &= |\bar{A}| - f_{\underline{E}}(\bar{A}). \end{aligned}$$

Hence $X - A$ is a flat of $T(\underline{E})$. Now

$$r^*(X-A) = r_{\underline{E}}^*(X-A)$$

if and only if

$$r(A) = r_{\underline{E}}(A).$$

By Theorem 4.2,

$$r(A) \leq r_{\underline{E}}(A),$$

but

$$\begin{aligned} r(A) &= r(\bar{A}) \\ &= |\bar{A}| - f(\bar{A}) \\ &= |\bar{A}| - f_{\underline{E}}(\bar{A}) \\ &\geq \min_{B \supseteq A} (|B| - f_{\underline{E}}(B)) \\ &= r_{\underline{E}}(A). \quad \square \end{aligned}$$

Observe that the statements of Corollary 5.1a, Theorem 5.2, and Theorem 5.3 are just those of the corresponding results of Section 4, with "principal" and "nullity" replaced by "transversal" and "rank", respectively. The same analogy does not hold for Theorem 5.1 with respect to Theorem 4.1. That is, it is not necessarily true that every pregeometry is the spanning-set intersection of principal pregeometries or the independent-set intersection of transversal pregeometries. For example, in Theorem 5.1, an H-independent set is T-independent for every T in $\mathcal{T}$, but the converse is not necessarily true.

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FOOTNOTES

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