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ON SOME CONTINUITY AND DIFFERENTIABILITY PROPERTIES OF PATHS OF GAUSSIAN PROCESSES[1]

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ABSTRACT

This paper considers some path properties of real separable Gaussian processes ξ with parameter set an arbitrary interval. The following results are established, among others. At every fixed point the paths of ξ are continuous, or differentiable, with probability zero or one. If ξ is measurable, then with probability one its paths have essentially the same points of continuity and differentiability. If ξ is measurable and not mean square continuous or differentiable at every point, then with probability one its paths are almost nowhere continuous or differentiable respectively. If ξ is mean square continuous and stationary, then its paths are differentiable with probability one if and only if ξ is mean square differentiable. If ξ is harmonizable, then its paths are absolutely continuous if and only if ξ is mean square differentiable. Also a class of harmonizable processes is determined for which the following are true: (i) with probability one paths are either continuous or unbounded on every interval, and (ii) path differentiability with probability one is equivalent to mean square differentiability.

ON SOME CONTINUITY AND DIFFERENTIABILITY PROPERTIES OF PATHS OF GAUSSIAN PROCESSES[1]

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1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper local and global path continuity and differentiability properties of Gaussian processes are studied. $\xi = \{\xi(t,\omega),\ t\in T\}$ is a real separable Gaussian process on the probability space (Ω, F, P) with arbitrary mean and covariance functions; T is any interval and F is the smallest σ-algebra of subsets of Ω with respect to which the random variables $\{\xi(t,\omega),\ t\in T\}$ are measurable.

It has been shown in [2] that the paths of ξ have any one of the following properties on T with probability zero or one: continuity, differentiability, absolute continuity, etc. A difficult problem is to obtain necessary and sufficient conditions for the two alternatives. Partial answers to this problem are given in Theorems 4 and 6, where necessary and sufficient conditions are obtained for almost sure path differentiability of stationary processes and almost sure path absolute continuity of harmonizable processes.

Zero-one type results for local path continuity and differentiability are given in Theorems 1 and 3, where it is also shown that the set of points of continuity and differentiability are essentially the same for almost all paths. Note that these results do not assume mean square continuity of ξ.

If in addition ξ is mean square continuous on T, then its modulus of continuity has been thoroughly studied by Ito and Nisio (1968). A similar local result is obtained under a local mean square continuity assumption (Theorem 1), and corresponding local and global results for path differentiability are obtained under local and global mean square differentiability assumptions (Theorem 3).

Also Belayev's (1961) alternatives for mean square continuous stationary Gaussian processes are shown to be valid for a slightly larger class of Gaussian processes (Corollary 3).

A remarkable feature of the proofs of all results is the fact that even when mean square continuity and stationarity are assumed, no use is made of the available process representations: the Karhumen-Loève representation and the spectral representation. Instead a recently established zero-one law for Gaussian processes is used [8, 9, 12].

All results are stated and discussed in this section and their proofs are given in Section 2.

In should be pointed out that, for simplicity, no special care has been taken for the endpoints of the interval T, when they belong to T; in every particular case it is easy to see what is the precise meaning of the results.

Finally, even though all results are stated and proven for T an interval in R^1 and for ξ a real valued process, they can be easily generalized to the case where T is an "interval" in R^n or a separable metric space, and ξ takes complex values or values in R^n (the n-dimensional case) or, for some of the results, in a compact metric space.

1.1. Continuity properties of paths of Gaussian processes. Path continuity properties can be studied by using the oscillation function $W_\xi(t,\omega)$ of the path $\xi(\cdot,\omega)$ at $t \in T$ defined by

$$W_\xi(t,\omega) = \lim_{\varepsilon \downarrow 0} \sup_{u,v \in (t-\varepsilon,t+\varepsilon) \cap T} |\xi(u,\omega) - \xi(v,\omega)|$$

for every $t \in T$ and $\omega \in \Omega$. Clearly the path $\xi(\cdot,\omega)$ is continuous at t if and only if $W_\xi(t,\omega) = 0$. The separability of ξ implies that for each $t \in T$, $W_\xi(t,\omega)$ is $\bar{F}$-measurable, where $\bar{F}$- is the completion of F with respect to P.

THEOREM 1. Let $\xi = \{\xi(t,\omega), t \in T\}$ be a real separable Gaussian process, where T is any interval.

(i) At every fixed point t in T the paths of ξ are continuous with probability zero or one.

(ii) Let T_c be the set of points t in T where the paths of ξ are continuous with probability one, and $T_c(\omega)$ be the set of points of continuity of the path $\xi(\cdot,\omega)$. If ξ is measurable then with probability one

$$\text{Leb}\{T_c(\omega) \,\Delta\, T_c\} = 0$$

i.e., almost all paths have essentially the same set of points of continuity.

(iii) If ξ is mean square continuous at the point $t \in T$, then with probability one

$$W_\xi(t,\omega) = \alpha_\xi(t)$$

where $\alpha_\xi(t)$ is a constant (extended real number).

The properties given in Theorem 1 are local continuity properties. One can similarly obtain the global property (the proof is similar to that of Theorem 3 (iv)):

(iv) If ξ is mean square continuous on T, then with probability one

$$W_\xi(t,\omega) = \alpha_\xi(t) \quad \text{for all} \quad t \in T.$$

This result was proven by Ito and Nisio (1968) for compact intervals T, by employing a global property of ξ, its Karhunen-Loève representation; they have also characterized the oscillation function α_ξ.

At a point of discontinuity it is important to know the value of the oscillation and in particular whether this value is finite or infinite. As in Theorem 1, the following are seen to be true.

(v) At every point t in T the paths of ξ have with probability one either finite or infinite oscillation.

(vi) Let T_o be the set of points t in T where the paths of ξ have infinite (resp. finite) oscillation with probability one, and $T_o(\omega)$ be the set of points of infinite (resp. finite) oscillation of the path $\xi(\cdot,\omega)$. If ξ is measurable, then with probability one $\text{Leb}\{T_o(\omega) \,\Delta\, T_o\} = 0$, i.e., almost all paths have essentially the same set of points of infinite (resp. finite) oscillation.

By introducing appropriate special oscillation functions one can study local properties such as left or right continuity at a point, existence of left or right limits at a point, and whether a point is a simple discontinuity. For instance, the oscillation functions appropriate for the study of left continuity and of the existence of left limits are respectively

$$W'_\xi(t,\omega) = \lim_{\varepsilon \downarrow 0} \sup_{u \in (t-\varepsilon,t)\cap T} |\xi(u,\omega) - \xi(t,\omega)|$$

$$W''_\xi(t,\omega) = \lim_{\varepsilon \downarrow 0} \sup_{u,v \in (t-\varepsilon,t)\cap T} |\xi(u,\omega) - \xi(v,\omega)|.$$

One can thus obtain the results of Theorem 1 (i) to (vi) with "continuity" replaced by:

left (or right) continuity

existence of left (or right) limits

simple discontinuity

and "oscillation" replaced by the corresponding special oscillation.

For Gaussian processes, almost sure path continuity on T implies mean square continuity on T, but the converse is not true. A relationship between mean square discontinuity and path discontinuity on T is given in the following theorem, whose inverse is not true.

THEOREM 2. If a real separable and measurable Gaussian process on an interval is not mean square continuous at any point, then with probability one its paths are almost nowhere continuous.

Here almost refers to the Lebesgue measure and also there is no fixed point of continuity of the paths.

1.2. Differentiability properties of paths of Gaussian processes. For the study of path differentiability properties we introduce the modulus of differentiability $\Delta_\xi(t,\omega)$ of the path $\xi(\cdot,\omega)$ at $t \in T$ defined by

$$\Delta_\xi(t,\omega) = \lim_{\varepsilon \downarrow 0} \sup_{u,v \in [(t-\varepsilon,t) \cup (t,t+\varepsilon)] \cap T} \left| \frac{\xi(u,\omega)-\xi(t,\omega)}{u-t} - \frac{\xi(v,\omega)-\xi(t,\omega)}{v-t} \right|$$

for every $t \in T$ and $\omega \in \Omega$. Clearly the path $\xi(\cdot,\omega)$ is differentiable at $t \in T$ if and only if $\Delta_\xi(t,\omega) = 0$. Also, the separability of ξ implies that for each $t \in T$, $\Delta_\xi(t,\omega)$ is $\bar{F}$-measurable. Differentiability properties similar to the continuity properties of Theorem 1 are given in the following theorem.

THEOREM 3. Let $\xi = \{\xi(t,\omega), t \in T\}$ be a real separable Gaussian process, where T is any interval.

(i) At every fixed point t in T the paths of ξ are differentiable with probability zero or one.

(ii) Let T_d be the set of points t in T where the paths of ξ are differentiable with probability one, and $T_d(\omega)$ be the set of points t in T where the path $\xi(\cdot,\omega)$ is differentiable. If ξ is measurable, then with probability one

$$\text{Leb}\{T_d(\omega) \,\Delta\, T_d\} = 0,$$

i.e., almost all paths have essentially the same set of points of differentiability.

(iii) If ξ is mean square differentiable at the point $t \in T$, then with probability one

$$\Delta_\xi(t,\omega) = \beta_\xi(t)$$

where $\beta_\xi(t)$ is a constant (extended real number).

(iv) If ξ is mean square differentiable on T, then with probability one

$$\Delta_\xi(t,\omega) = \beta_\xi(t) \quad \text{for all } t \in T.$$

The results of Theorem 3 remain valid if "differentiability" is replaced by "left (or right) differentiability". Also, one could possibly obtain a characterization of the modulus of differentiability β_ξ similar to the characterization of the modulus of continuity α_ξ given in [7].

An important question is the relationship between mean square differentiability and path differentiability. For Gaussian processes almost sure path differentiability on T implies mean square differentiability on T. The extent to which the converse is true for Gaussian processes is not known at present (see also Theorem 7). An interesting case where the converse is true is that of stationary Gaussian processes.

THEOREM 4. A real separable mean square continuous stationary Gaussian process has everywhere differentiable paths with probability one if and only if it is mean square differentiable, or equivalently, if and only if

$$\int_{-\infty}^{\infty} \lambda^2 \, dF(\lambda) < +\infty$$

where F is its spectral distribution.

For a real separable mean square continuous wide sense stationary (not necessarily Gaussian) process it is known that mean square differentiability implies absolute continuity of paths with probability one, and hence almost everywhere path differentiability with probability one [5, pp. 536-537]. Theorem 4 strengthens this result for Gaussian processes.

Theorem 4 is also valid for real separable mean square continuous Gaussian processes with stationary increments. In this case a necessary and sufficient condition for mean square differentiability is

$$\int_{-\infty}^{\infty} \lambda^2 \, dH(\lambda) < +\infty$$

where H is the spectral distribution as in [5, p. 552].

As it is clear from the proof of Theorem 4, if ξ is mean square differentiable, the path derivatives $\xi'(t,\omega)$ of $\xi(t,\omega)$ form a mean square continuous stationary Gaussian process with spectral distribution G such that $dG(\lambda) = \lambda^2 dF(\lambda)$. According to Belayev's alternatives then, the paths of ξ' are either continuous or unbounded on every interval with probability one. It now follows from Theorem 4 and from Theorem 3 of [1] that a sufficient condition for the latter alternative is that ξ has a spectral density f satisfying $\int_{-\infty}^{\infty} \lambda^2 f(\lambda) d\lambda < +\infty$ and such that for some $C > 0$ and $\lambda_0 > 0$:

$$f(\lambda) \geq \frac{C}{\lambda^3 (\log \lambda)^2} \quad \text{for all } \lambda \geq \lambda_0.$$

Theorem 4 can be used to derive sufficient conditions for the derivatives of the paths of a stationary Gaussian process to have certain properties, from known sufficient conditions for the paths of a stationary Gaussian process to have the same properties. For instance Hunt's (1951) sufficient condition for almost sure path continuity:

$$\int_{-\infty}^{\infty} [\log(1+|\lambda|)]^a \, dF(\lambda) < +\infty \quad \text{for some} \quad a > 1,$$

and Theorem 4 imply that a sufficient condition for almost all paths of ξ to be n-times continuously differentiable $(n = 1,2,\ldots)$ is

$$\int_{-\infty}^{\infty} \lambda^{2n} [\log(1+|\lambda|)]^a \, dF(\lambda) < +\infty \quad \text{for some} \quad a > 1.$$

This condition is well-known [4] and only its derivation here is different. Also, the sufficient condition for almost all paths of ξ to satisfy a Lipschitz condition of order α, $0 < \alpha < 1$,

$$\int_{-\infty}^{\infty} \lambda^{2\alpha} \log(1+|\lambda|) \, dF(\lambda) < +\infty$$

[1], and Theorem 4 imply that a sufficient condition for almost all paths of ξ to be n-times differentiable $(n = 1,2,\ldots)$ with n-th derivatives satisfying a Lipschitz condition of order α is

$$\int_{-\infty}^{\infty} \lambda^{2(n+\alpha)} \log(1+|\lambda|) \, dF(\lambda) < +\infty.$$

The implication of mean square non-differentiability to path non-differentiability on T is given in the following theorem.

THEOREM 5. If a real separable and measurable Gaussian process on an interval is not mean square differentiable at any point, then with probability one its paths are almost nowhere differentiable.

Again, almost refers to the Lebesgue measure and there are no fixed points of differentiability.

Theorem 5 includes as a particular case the well-known property of the paths of the Wiener process.

1.3. Some remarks on the absolute continuity of paths of Gaussian processes. As it is pointed out in Section 1.2, the implication of mean square differentiability to path properties is not fully known at present even for Gaussian processes. It is known however [3, pp. 186-187] that for a real separable (not necessarily Gaussian) process ξ on an interval T, mean square differentiability plus some rather mild assumptions imply that with probability one the paths of ξ are absolutely continuous on every compact subinterval of T. These additional assumptions are (i) the local Lebesgue integrability (in t and in t and s respectively) of the derivatives $\frac{\partial R(t,s)}{\partial t}$ and $\frac{\partial^2 R(t,s)}{\partial t \partial s}$ of the autocorrelation function R of ξ, and (ii) the existence of a measurable modification of the mean square derivative of ξ. The converse of this result does not seem to be true for the general Gaussian process. As it is seen from the proof of Theorem 6, almost sure path absolute continuity implies almost sure path differentiability and mean square differentiability on T except on an at most countable set of points, but we have not been able to strengthen this result. An interesting class of Gaussian processes for which the converse of this result is true is the harmonizable Gaussian processes.

THEOREM 6. A real separable harmonizable Gaussian process has with probability one paths absolutely continuous on every compact interval if and only if it is mean square differentiable, or equivalently, if and only if

$$\int_{-\infty}^{\infty}\int |\lambda\mu| \, dr(\lambda,\mu) < +\infty$$

where r is its two-dimensional spectral measure.

Since a mean square continuous stationary process is harmonizable (with a spectral measure supported by the diagonal of the plane), Theorems 4 and 6 give the following result.

COROLLARY 1. For a real separable mean square continuous stationary Gaussian process, everywhere differentiability of its paths with probability one is equivalent to the absolute continuity of its paths on every compact interval with probability one.

1.4. Some remarks for non-Gaussian processes. It is of interest to study the extent to which the results of Theorems 1 to 6 remain valid for non-Gaussian processes. It should be first remarked that the word "Gaussian" cannot be altogether deleted from the statements of these theorems. This is obvious for Theorems 1 and 3; for Theorems 5 and 6 it follows from the existence of a mean square continuous stationary process with almost surely absolutely continuous paths, which is not mean square differentiable [5, p.537]; and for Theorems 2 and 4 it is conjectured.

The following property opens a way of determining classes of non-Gaussian processes for which the results of Sections 1.1 to 1.3 apply.

THEOREM 7. Let ξ and η be two stochastic processes defined on the same probability space and time interval and let W_ξ and W_η be their oscillation processes defined in Section 1.1. Then if ξ and η are equivalent, so are W_ξ and W_η, and if W_ξ and W_η are singular, so are ξ and η.

Clearly the result of Theorem 7 is true for the left oscillation process, etc., and also for the modulus of differentiability Δ_ξ defined in Section 1.2. The following is an immediate consequence of Theorem 7.

COROLLARY 2. If a real separable process ξ is equivalent to a real separable Gaussian process η satisfying the assumptions of any one of Theorems 1 to 6, then the theorem applies to ξ.

As a practical method of determining non-Gaussian processes for which the results of the previous sections are valid, Corollary 2 is of limited interest because very little is known at present concerning characterizations of non-Gaussian processes equivalent to fixed Gaussian processes. In fact, the only complete result available in the literature is the characterization of all processes equivalent to the Wiener process.

On the other hand complete characterizations are known for all Gaussian processes equivalent to a fixed Gaussian process. Using these characterizations we obtain from Belayev's alternatives, Theorem 4 and Corollary 2, the following result, which in fact generalizes Belayev's alternatives and Theorem 4 to a subclass of the harmonizable Gaussian processes.

COROLLARY 3. Let ξ be a real separable harmonizable Gaussian process with zero mean and two-dimensional spectral measure r such that for all Borel sets B and C of the real line

$$r(B\times C) = q(B\cap C) + \int_B \int_C K(\lambda,\mu)dq(\lambda)dq(\mu)$$

where q is a finite nonnegative measure on the Borel sets of the real line and K is a symmetric function in $L_2(q\times q)$ that does not have -1 as an eigenvalue. Then

(i) the paths of ξ are either continuous or unbounded on every interval with probability one, and

(ii) the paths of ξ are everywhere differentiable with probability one if and only if ξ is mean square differentiable (or equivalently, if and only if $\int_{-\infty}^{\infty}\lambda^2 dq(\lambda) < +\infty$).

Implicit in the proof of Corollary 3 is the interesting fact that a zero mean Gaussian process is equivalent to a zero mean stationary Gaussian process with spectral measure q if and only if it is harmonizable with spectral measure r as described in Corollary 3. The case of nonzero mean ξ can be included in Corollary 3; (i) and (ii) remain valid if ξ has mean m of the form

$$m(t) = \int_{-\infty}^{\infty} \exp(it\lambda)\ f(\lambda)\ dq(\lambda)$$

for all $t \in T$, where $f \in L_2(q)$.

Corollary 3 raises the problem of characterizing the classes of Gaussian processes, and in particular the classes of harmonizable Gaussian processes, for which Belayev's alternatives and Theorem 4 are valid.

2. PROOFS

The following facts are repeatedly used in the proofs of the theorems. Let R^T be the set of all real functions on T and $\mathcal{U}(R^T)$ the σ-algebra of subsets of R^T generated by sets of the form $\{x \in R^T: (x(t_1),..,x(t_n)) \in B^n\}$, where $t_1,...,t_n \in T$ and B^n is an n-dimensional Borel set. The transformation $\Phi: (\Omega, \bar{F}, P) \to (R^T, \mathcal{U}(R^T))$ defined by $\Phi(\omega) = \xi(\cdot,\omega)$ is measurable and induces a probability measure $\mu = P \circ \Phi^{-1}$ on $\mathcal{U}(R^T)$. $\bar{\mathcal{U}}(R^T)$ denotes the completion of $\mathcal{U}(R^T)$ with respect to μ. The stochastic process $\{x(t), t \in T\}$ defined on the probability space $(R^T, \mathcal{U}(R^T), \mu)$ is clearly Gaussian with the same mean and covariance functions as ξ. Let also $H(\xi)$ be the reproducing kernel Hilbert space of ξ (or of its autocorrelation function), i.e., the subset of R^T which consists of all functions of the form $E[\xi(t,\omega)\eta(\omega)]$, where η is a random variable in the $L_2(\Omega,\bar{F},P)$-closure of the linear span of the random variables $\{\xi(t,\omega),\ t\in T\}$, and E denotes expectation. The following facts are known [8, 9, 12].

(i) If G is a $\bar{\mathcal{U}}(R^T)$-measurable subgroup of R^T then $\mu(G) = 0$ or 1.

(ii) If g is a $\bar{\mathcal{U}}(R^T)$-measurable real function on R^T and $g(x+m) = g(x)$ a.e. $[\mu]$ for all $m \in H(\xi)$, then $g(x) =$ constant a.e. $[\mu]$.

PROOF OF THEOREM 1. If S is a countable dense subset of T which is a separant of ξ, then for every $t\in T$ we have

$$W_\xi(t,\omega) = \lim_{\varepsilon \downarrow 0} \sup_{u,v\in(t-\varepsilon,t+\varepsilon)\cap S} |\xi(u,\omega)-\xi(v,\omega)| \quad \text{a.s. } [P].$$

For $x\in R^T$ and $t\in T$ set

$$W(t,x) = \lim_{\varepsilon \downarrow 0} \sup_{u,v\in(t-\varepsilon,t+\varepsilon)\cap S} |x(u)-x(v)|.$$

Then for every $t\in T$, $W(t,x)$ is $\mathcal{U}(R^T)$-measurable and $W_\xi(t,\omega) = W(t,\Phi(\omega))$ a.s. [P].

(i) For fixed $t\in T$ define the sets

$$F_t = \{\omega\in\Omega:\ W_\xi(t,\omega) = 0\}$$
$$G_t = \{x\in R^T:\ W(t,x) = 0\}.$$

Then G_t is $\mathcal{U}(R^T)$-measurable and since $G_t = \{x\in R^T$: the restriction of x to $S \cup \{t\}$ is continuous at $t\}$, G_t is a subgroup of R^T. It follows that $\mu(G_t) = 0$ or 1 and by $F_t = \Phi^{-1}(G_t)$, $P(F_t) = 0$ or 1.

(ii) Since ξ is measurable, so is W_ξ. Let

$$E = \{(t,\omega) \in T\times\Omega:\ W_\xi(t,\omega) = 0\}.$$

Then $E \in \mathcal{B}(T)\times\mathcal{F}$, where $\mathcal{B}(T)$ is the σ-algebra of Borel subsets of T, and its sections

$$E_\omega = \{t\epsilon T\colon W_\xi(t,\omega) = 0\} = T_c(\omega), \quad \omega \in \Omega$$

$$E_t = \{\omega\epsilon\Omega\colon W_\xi(t,\omega) = 0\}, \quad t \in T$$

satisfy $E_\omega \in \mathcal{B}(T)$ for all $\omega\epsilon\Omega$, and $E_t \epsilon \mathcal{F}$ for all $t\epsilon T$. Also

$$T_c = \{t\epsilon T\colon P(E_t) = 1\}$$

and $T_c \in \mathcal{B}(T)$, since $E \in \mathcal{B}(T)\times\mathcal{F}$ implies that $P(E_t)$ is $\mathcal{B}(T)$-measurable. Now let $F = T_c\times\Omega$. Then $(E\Delta F)_t = E_t \,\Delta\, F_t$ and this is equal to $\Omega - E_t$ for $t\epsilon T_c$ and to E_t for $t \not\epsilon T_c$. Thus

$$\begin{aligned}(\mathrm{Leb}\times P)(E\Delta F) &= \int_T P\{(E\Delta F)_t\}\,dt \\ &= \int_{T_c} P(\Omega - E_t)dt + \int_{T-T_c} P(E_t)dt \\ &= 0\end{aligned}$$

by the definition of T_c and (i), which implies that for $t \in T-T_c$, $P(E_t) = 0$. Now from $(E\Delta F)_\omega = E_\omega \,\Delta\, F_\omega = T_c(\omega) \,\Delta\, T_c$ and

$$0 = (\mathrm{Leb}\times P)(E\Delta F) = \int_\Omega \mathrm{Leb}\{(E\Delta F)_\omega\}dP(\omega)$$

it follows that $\mathrm{Leb}\{T_c(\omega) \,\Delta\, T_c\} = 0$ a.s. [P].

(iii) It is easily seen that for all $m,x \in R^T$

$$W(t,x+m) \le W(t,x) + W(t,m)$$

$$W(t,x) \le W(t,x+m) + W(t,m).$$

Since ξ is mean square continuous at the point $t\epsilon T$, every $m \in H(\xi)$ is continuous at t and thus $W(t,m) = 0$. It follows that for all $m \in H(\xi)$, $W(t,x+m) = W(t,x)$ for all $x\epsilon R^T$ and thus $W(t,x)$ is a constant a.s. [μ], $W(t,x) = \alpha_\xi(t)$ a.s. [μ]. Thus $W_\xi(t,\omega) = \alpha_\xi(t)$ a.s. [P].

PROOF OF THEOREM 2. Fix a point t in T. By Theorem 1. (i), the paths of ξ are continuous at t with probability zero or one. If they are continuous with probability one, since almost sure convergence of a sequence of Gaussian random variables implies convergence in the mean square, it follows that ξ is mean square continuous at t. Thus if ξ is mean square discontinuous at t, with probability one its paths are discontinuous at t. Hence, if ξ is not mean square continuous at any point of T, then the set T_c defined in Theorem 1. (ii) is empty. It follows, again from Theorem 1.(ii), that $\mathrm{Leb}\{T_c(\omega)\} = 0$ a.s. [P], i.e., with probability one the paths of ξ are continuous on a set of zero Lebesgue measure.

PROOF OF THEOREM 3. This theorem is proven as Theorem 1. We have for every $t \in T$,

$$\Delta_\xi(t,\omega) = \lim_{\varepsilon \downarrow 0} \sup_{u,v \in [(t-\varepsilon,t) \cup (t,t+\varepsilon)] \cap S} \left| \frac{\xi(u,\omega)-\xi(t,\omega)}{u-t} - \frac{\xi(v,\omega)-\xi(t,\omega)}{v-t} \right| \quad \text{a.s. [P]}$$

and if we define for $x \in R^T$ and $t \in T$,

$$\Delta(t,x) = \lim_{\varepsilon \downarrow 0} \sup_{u,v \in [(t-\varepsilon,t) \cup (t,t+\varepsilon)] \cap S} \left| \frac{x(u)-x(t)}{u-t} - \frac{x(v)-x(t)}{v-t} \right|$$

then for every fixed $t \in T$, $\Delta(t,x)$ is $\mathcal{U}(R^T)$-measurable and $\Delta_\xi(t,\omega) = \Delta(t,\Phi(\omega))$ a.s. [P].

Parts (i) and (ii) are shown exactly as parts (i) and (ii) of Theorem 1. For part (iii), it is easily seen that for all $m,x \in R^T$,

$$\Delta(t,x+m) \le \Delta(t,x) + \Delta(t,m)$$

$$\Delta(t,x) \le \Delta(t,x+m) + \Delta(t,m).$$

If ξ is mean square differentiable at $t \in T$, then every $m \in H(\xi)$ is differentiable at t [10, p.303] and so $\Delta(t,m) = 0$. The proof of (iii) is completed as in Theorem 1.

(iv) We introduce the modulus of differentiability on the closed interval $[s,t]$, $s < t$, by

$$\Delta_\xi([s,t],\omega) = \lim_{n\uparrow+\infty} \lim_{m\uparrow+\infty} \sup_{\substack{u,v,w\in(s-\frac{1}{n},t+\frac{1}{n})\cap T \\ u<v<w \\ |u-w|<\frac{1}{m}}} \{\max[\Delta_i(u,v,w;\omega)]_{i=1}^{3}\},$$

where $\Delta_1(u,v,w;\omega) = \left|\frac{\xi(u,\omega)-\xi(v,\omega)}{u-v} - \frac{\xi(u,\omega)-\xi(w,\omega)}{u-w}\right|$ and $\Delta_2(u,v,w;\omega) = \Delta_1(w,u,v;\omega)$, $\Delta_3(u,v,w;\omega) = \Delta_1(v,u,w;\omega)$. We also define $\Delta_\xi([t,t],\omega) = \Delta_\xi(t,\omega)$. It is easily seen that $\Delta_\xi([s,t],\omega)$, $s \le t$, is left continuous in s and right continuous in t. Also, it follows as in (iii), that if ξ is mean square differentiable on T then for fixed $s \le t$ in T,

$$\Delta_\xi([s,t],\omega) = \beta_\xi(s,t): \text{ constant}$$

with probability one. Hence

$$P\{\omega\in\Omega: \Delta_\xi([s,t],\omega) = \beta_\xi(s,t) \text{ for all rationals } s \le t \text{ in } T\} = 1.$$

Since $\Delta_\xi([s,t],\omega)$ is left continuous in s and right continuous in t, it follows that

$$P\{\omega\in\Omega: \Delta_\xi([s,t],\omega) = \beta_\xi(s,t) \text{ for all } s \le t \text{ in } T\} = 1$$

and thus

$$P\{\omega\in\Omega: \Delta_\xi(t,\omega) = \beta_\xi(t) \text{ for all } t \in T) = 1.$$

This argument is similar to the one given for W_ξ in [7].

PROOF OF THEOREM 4. Assume first that ξ is mean square differentiable. Then with probability one its paths are absolutely continuous on every compact subinterval of T [5, pp.536-537] and thus differentiable on T a.e. [Leb]. Hence with probability one

$$\Delta_\xi(t,\omega) = 0 \quad \text{on} \quad T \text{ a.e. [Leb]}.$$

On the other hand, Theorem 3.(iv) and the stationarity of ξ imply that

$$P\{\omega\in\Omega: \Delta_\xi(t,\omega) = \beta_\xi \quad \text{for all} \quad t\in T\} = 1.$$

It follows that $\beta_\xi = 0$ and thus with probability one the paths of ξ are differentiable on T.

Conversely, assume that with probability one the paths of ξ are differentiable on T. Let $\xi'(t,\omega)$ be the path derivative at $t\in T$, i.e.,

$$\xi'(t,\omega) = \lim_{s \to t} \frac{\xi(s,\omega)-\xi(t,\omega)}{s - t} \quad \text{a.s. [P]}.$$

Since for every fixed t, the convergence is almost sure along any sequence converging to t, and since a.s. convergence of Gaussian random variables implies convergence in the mean square, it follows that ξ is mean square differentiable at every t in T.

The process ξ' is defined except on a zero probability set and it is clearly stationary and Gaussian. Also for every fixed t, $\xi'(t,\omega)$ equals a.s. [P] the mean square derivative of ξ at t. Thus its spectral distribution G is equal to the spectral distribution of the mean square derivative of ξ, i.e., $dG(\lambda) = \lambda^2 dF(\lambda)$.

Theorem 5 is proven as Theorem 2, by using parts (i) and (ii) of Theorem 3.

PROOF OF THEOREM 6. If R is the autocorrelation function of the harmonizable process ξ, then for all t and s in T we have

$$R(t,s) = \int_{-\infty}^{\infty}\int \exp(i(t\lambda - s\mu))\, dr(\lambda,\mu)$$

where r is its spectral measure, a finite signed measure on the plane which is nonnegative definite on the measurable rectangles.

If ξ is mean square differentiable at a point t in T, then it is shown in [11, p. 282] that $\int_{-\infty}^{\infty}\int i\lambda \exp(it\lambda)\,(-i\mu)\exp(-it\mu)\, dr(\lambda,\mu)$ exists as an ordinary Lebesgue integral and thus

$$\int_{-\infty}^{\infty}\int |\lambda\mu|\, dr(\lambda,\mu) < +\infty.$$

Conversely, the finiteness of this integral implies mean square differentiability of ξ at t. Indeed

$$\frac{1}{hh'}\Delta_h\Delta_{h'} R(t,t) = \frac{1}{hh'}\{R(t+h,t+h') - R(t+h,t) - R(t,t+h') + R(t,t)\}$$

$$= \int_{-\infty}^{\infty}\int \exp(it(\lambda-\mu)) \cdot \frac{\exp(ih\lambda)-1}{h} \cdot \frac{\exp(-ih'\mu)-1}{h'}\, dr(\lambda,\mu)$$

and it follows from $\lim_{h\to 0} \frac{\exp(ih\lambda)-1}{h} = i\lambda$ for all λ, $\left|\frac{\exp(ih\lambda)-1}{h}\right| \le |\lambda|$ for all λ and $h \neq 0$, and the bounded convergence theorem that

$$\lim_{h,h' \to 0} \frac{1}{hh'}\Delta_h\Delta_{h'} R(t,t) = \int_{-\infty}^{\infty}\int \exp(it(\lambda-\mu))\, \lambda\mu\, dr(\lambda,\mu),$$

i.e., the limit exists and is finite. Thus ξ is mean square differentiable at t.

Hence a harmonizable process is mean square differentiable at one point if and only if it is everywhere mean square differentiable, and a necessary and sufficient condition is the r-integrability of the function $\lambda\mu$.

Assume now that ξ is mean square differentiable. Then

$$\frac{\partial R(t,s)}{\partial t} = \int_{-\infty}^{\infty}\int i\lambda \exp(i(t\lambda - s\mu))\, dr(\lambda,\mu)$$

$$\frac{\partial^2 R(t,s)}{\partial t \partial s} = \int_{-\infty}^{\infty}\int \lambda\mu \exp(i(t\lambda - s\mu))\, dr(\lambda,\mu)$$

are both continuous functions, hence locally Lebesgue integrable. Also, since $\frac{\partial^2 R(t,s)}{\partial t \partial s}$ is the autocorrelation of the mean square derivative $\hat{\xi}$ of ξ, $\hat{\xi}$ has a measurable modification. It follows that with probability one the paths of ξ are absolutely continuous on every compact interval [3, pp. 186-187].

Conversely, assume that almost all paths of ξ are absolutely continuous on every compact interval of T. Then, with probability one, $\xi(\cdot,\omega)$ is differentiable on T except at most on a countable set of points. Thus $\text{Leb}\{T-T_d(\omega)\} = 0$ and by Theorem 3.(ii), $\text{Leb}\{T-T_d\} = 0$. Now fix a point t in T_d. Then with probability one the paths of ξ are differentiable at t and thus ξ is mean square differentiable at t, and also on T from the previous remark.

PROOF OF THEOREM 7. If $\zeta = \{\zeta(t,\omega),\ t \in T\}$ is a stochastic process on $(\Omega, \bar{F}, P)$, define the map $\Phi_\zeta: (\Omega, \bar{F}, P) \to (R^T, \mathcal{U}(R^T))$ by $\Phi_\zeta(\omega) = \zeta(\cdot,\omega)$ and let $\mu_\zeta = P \circ \Phi_\zeta^{-1}$ be the induced probability measure on $(R^T, \mathcal{U}(R^T))$. ξ and η are called equivalent if μ_ξ and μ_η are equivalent: $\mu_\xi \sim \mu_\eta$ (i.e., mutually absolutely continuous) and singular if μ_ξ and μ_η are singular: $\mu_\xi \perp \mu_\eta$. Similarly for W_ξ and W_η.

Note that $W(t,x)$ as defined in the proof of Theorem 1 is such that $x \to W(\cdot,x)$ is a measurable map from $(R^T, \mathcal{U}(R^T))$ to $(R^T, \mathcal{U}(R^T))$. Hence for every $B \in \mathcal{U}(R^T)$ we have $C = \{x \in R^T: W(\cdot,x) \in B\} \in \mathcal{U}(R^T)$. Also

$$\begin{aligned}
\mu_{W_\xi}(B) &= P\circ\Phi_{W_\xi}^{-1}(B) = P\{\omega\in\Omega\colon W_\xi(\cdot,\omega)\in B\} \\
&= P\{\omega\in\Omega\colon W(\cdot,\Phi_\xi(\omega))\in B\} \\
&= P\{\omega\in\Omega\colon \Phi_\xi(\omega)\in C\} \\
&= P\circ\Phi_\xi^{-1}(C) = \mu_\xi(C).
\end{aligned}$$

Assume now $\mu_\xi \sim \mu_\eta$. If for some $B \in \mathcal{U}(R^T)$, $\mu_{W_\xi}(B) = 0$ then $\mu_\xi(C) = 0$ and since $\mu_\xi \sim \mu_\eta$, $\mu_\eta(C) = 0$ and $\mu_{W_\eta}(B) = 0$. Thus $\mu_{W_\eta} \ll \mu_{W_\xi}$ and similarly $\mu_{W_\xi} \ll \mu_{W_\eta}$, i.e., $\mu_{W_\xi} \sim \mu_{W_\eta}$. In a similar way it is seen that $\mu_{W_\xi} \perp \mu_{W_\eta}$ implies $\mu_\xi \perp \mu_\eta$.

PROOF OF COROLLARY 3. Let $\{\eta(t,\omega), -\infty < t < +\infty\}$ be a mean square continuous wide sense stationary Gaussian process with zero mean, autocorrelation function R_η and spectral measure q. Then for all t and s,

$$R_\eta(t,s) = \int_{-\infty}^{\infty} \exp(i(t-s)\lambda)\, dq(\lambda).$$

Let $L_2(\eta)$ be the $L_2(\Omega, \bar{F}, P)$-closure of the linear span of the random variables $\{\eta(t,\omega), -\infty < t < +\infty\}$ and denote $L_2(R, \mathcal{B}(R), q)$ by $L_2(q)$, where R is the real line and $\mathcal{B}(R)$ the Borel sets of R. Then it is well-known that there is an inner product preserving isomorphism between $L_2(\eta)$ and $L_2(q)$, denoted by $\leftrightarrow$ such that $\eta(t,\omega) \leftrightarrow \exp(it\lambda)$ for all $-\infty < t < +\infty$.

Now let $\eta = \{\eta(t,\omega), t\in T\}$ and let $\xi = \{\xi(t,\omega), t\in T\}$ be a Gaussian process with zero mean and autocorrelation function $R_\xi(t,s)$. Then [12] ξ is equivalent to η if and only if for all $t,s \in T$

$$R_\xi(t,s) = R_\eta(t,s) + \langle K\eta(t),\eta(s)\rangle_{L_2(\Omega,\bar{F},P)}$$

where K is a Hilbert-Schmidt operator in $L_2(\eta,T)$, the $L_2(\Omega,\bar{F},P)$-closure of the linear span of the set $\{\eta(t,\omega), t\in T\}$, that does not have -1 as an eigenvalue; K may be taken self-adjoint. Because of the isomorphism

$L_2(\eta,T) \subset L_2(\eta) \leftrightarrow L_2(q)$, and the fact that Hilbert-Schmidt operators in L_2 spaces are of integral type, there exists a symmetric function $K(\lambda,\mu)$ in $L_2(q\times q) = L_2(R^2,\mathcal{B}(R^2),q\times q)$ such that the integral type operator in $L_2(q)$ with kernek K does not have -1 as an eigenvalue, and

$$\langle K\eta(t),\eta(s)\rangle_{L_2(\Omega,\bar{F},P)} = \int_{\infty}^{\infty}\int K(\lambda,\mu)\exp(i(t\lambda-s\mu))\,dq(\lambda)\,dq(\mu)$$

for all $t, s \in T$. Hence

$$R_\xi(t,s) = \int_{-\infty}^{\infty} \exp(i(t-s)\lambda)\,dq(\lambda) + \int_{-\infty}^{\infty}\int K(\lambda,\mu)\exp(i(t\lambda-s\mu))\,dq(\lambda)\,dq(\mu)$$
$$= \int_{-\infty}^{\infty}\int \exp(i(t\lambda-s\mu))\,dr(\lambda,\mu)$$

where the measure r is defined on $(R^2, \mathcal{B}(R^2))$ by

$$r(B\times C) = q(B\cap C) + \int_B\int_C K(\lambda,\mu)\,dq(\lambda)\,dq(\mu)$$

for all $B,C \in \mathcal{B}(R)$. Clearly r is a finite measure and thus ξ is harmonizable with two dimensional spectral measure r. Hence a Gaussian process ξ is equivalent to η if and only if it is harmonizable with spectral measure r as described above.

Now (i) is a consequence of Belayev's alternatives for η and Theorem 7. It follows from Corollary 2 and Theorem 4 that the paths of ξ are differentiable with probability one if and only if almost all paths of η are differentiable, i.e., $\int_{-\infty}^{\infty} \lambda^2 dq(\lambda) < +\infty$. Moreover, it is easily seen from the definition of r that $\int_{-\infty}^{\infty} \lambda^2 dq(\lambda) < +\infty$ is equivalent to $\int_{-\infty}^{\infty}\int |\lambda\mu|\,dr(\lambda,\mu) < +\infty$, and from Theorem 6, to the mean square differentiability of ξ.

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FOOTNOTES

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