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# ON THE TAILS OF QUEUEING-TIME DISTRIBUTIONS*

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## 1. INTRODUCTION

This paper grew from a desire to examine the relationship, in the familiar G/G/1 queue, between the "tail probabilities" of large service-times and the "tail probabilities" of large queueing-times. The investigation was led to a class of functions we call of *moderate growth* and an important sub-class of these we call *sub-exponential tail functions*. A significant part of this paper is necessarily devoted to a study of these classes of functions. The rest of the paper then establishes the validity of some very simple, but quite general, relationships between the probabilities of large values of *service-times*, *ladder variables* (see below), and *queueing-times*.

We shall write $V$ for a typical service-time, with d.f. $B(x)$, and $U$ for a typical inter-arrival time, with d.f. $A(x)$. We shall, moreover, suppose that $U$ and $V$ have finite expectations, that $\mathcal{E}U > \mathcal{E}V$ (so that the queue is "stable"), and we set $X = V-U$, write $F(x)$ for the d.f. of $X$ and denote $|\mathcal{E}X|$ by $\mu$. As we shall often be considering "tail probabilities", we shall adopt the convenient notation exemplified thus: $B^c(x) = 1-B(x)$.

If $Q(x)$ be the stationary d.f. of the queueing-time, then it is well-known (Lindley (1952)) that

$$(1.1) \quad Q(x) = \int_{0-}^{\infty} F(x-z)\, Q(dz), \quad x > 0.$$

However, this integral equation is frequently intractable, in spite of the

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existence of "theoretical" solutions. It is gratifying, therefore, to find, as we shall show in this paper, that for a wide and interesting class of G/G/1 queues there is a very simple asymptotic relation between $Q^c(x)$ and $B^c(x)$; for these cases

$$(1.2)\quad Q^c(x) \sim \frac{1}{\mu}\int_x^\infty B^c(u)du, \quad \text{as} \quad x \to \infty.$$

Before we can be more precise about the conditions under which (1.2) holds we must discuss one or two technical matters. We shall say that a function $g(x)$ defined on $x \geq 0$, is a *function of moderate growth* (fmg) if it is strictly positive and if, for every fixed real $\theta$, $g(x) \sim g(x+\theta)$. These functions will be discussed in some detail in §2. A function $g(x)$ is fmg if and only if $g(\log x)$ is a function of slow growth, a class of functions long familiar since their principle properties were laid bare in the classical papers of Karamata (1930a), (1930b) (1933). We shall see that a non-decreasing $g(x)$ is fmg if and only if

$$(1.3)\quad g(x) \sim g(0)\exp\int_0^x \alpha(u)du, \quad \text{as} \quad x \to \infty,$$

where $\alpha(u)$ is a bounded non-negative function and $\alpha(u) \to 0$ as $u \to \infty$.

If, in (1.3), $\alpha(u)$ can be chosen so as to be non-increasing $(\alpha(u) \downarrow 0)$, then we shall say $g(x)$ is a *moment function* (MF); *the reciprocal of a MF will be called a tail function* (TF).

Occasionally we find it necessary to impose some restraint on the rate of growth of a fmg. It is convenient to write

$$(1.4)\quad \Lambda(x) = \int_0^x \alpha(u)du,$$

and we shall say $g(x)$ is *sub-exponential* (SE) if

$$(1.5)\quad \limsup_{x \to \infty} \frac{\Lambda(2x)}{\Lambda(x)} < 2.$$

Feller (1971) makes great use of functions he calls *functions of regular variation;* we refer to his book for details. However, we shall see below that any function of regular variation, with non-zero index, is asymptotically equal to a monotone such function. It is then an easy matter to show that any of the functions of regular variation, with strictly positive index, is asymptotically equal to a SEMF. If $0 < \beta < 1$, $\alpha > 0$, the function $\exp \alpha x^\beta$ is a SEMF. On the other hand, $\exp\{x/\log x\}$ is a MF, but is not SE.

Let us say $\tau(x)$ is *sublinear* if $\tau(x)$ is non-decreasing and $\tau(x)/x$ is non-increasing. In much of this paper sub-exponential tail functions (SETF) play an important role. It is shown in §2, amongst other things, that any tail integral of the form

$$T(x) = \int_x^\infty \frac{\tau(u)}{u^\gamma} du$$

is necessarily a SETF.

The role of the "ladder variable", introduced by Blackwell (1953), in queueing theory is now well understood; see, e.g. Prabhu (1967). It is necessary to present a brief synopsis of this theory here to acquaint the reader with our present notation and viewpoint. Let $\{U_n\}$, $\{V_n\}$ be independent infinite sequences of iid inter-arrival times, service-times, respectively. Let $X_n = V_n - U_n$, $S_0 = 0$, $S_n = X_1 + X_2 + \ldots + X_n$ for $n = 1,2,\ldots$, and denote by $A_k$ the event: $\{S_1 \le 0, S_2 \le 0, \ldots, S_{k-1} \le 0, S_k > 0\}$. Then we can define an improper random variable $N_1$ by setting $N_1 = k$ if $A_k$ happens and $N_1 = \infty$ if no $A_k$ happens (for any finite $k$). Our problem is banal if $P\{V > U\} = 0$ so we shall always suppose $P\{V > U\} > 0$; thus, if we write $\pi = P\{N_1 < \infty\}$, then $\pi > 0$. Furthermore $EX_n = EV_n - EU_n < 0$, which implies that $n^{-1}S_n \to -\mu$ almost surely; from this it is easy to deduce that $\pi < 1$ and is the stationary probability a customer has to queue. Although Spitzer (1956) has shown that

$$\pi = \exp\left\{-\sum_{n=1}^{\infty} \frac{1}{n} P[S_n \le 0]\right\},$$

this formula is difficult to apply and there seem at present to be no straightforward procedures *of general application* that lead to useful bounds on $\pi$, let alone precise evaluations, except when arrivals are at random (and then $\pi = \rho$, as is well-known).

If $N_1 < \infty$ we shall set $Z_1 = S_{N_1}$ and $Z_1$ is the first "ladder" variable.

In his pioneering paper Lindley (1952) showed that

(1.6) $Q(x) = P\{\max_{0 \le n < \infty} S_n \le x\}$

and this relation, rather than the integral equation (1.1), is the more useful in the present investigation. Let $A_\infty$ denote the event complimentary to $\sum_1^\infty A_k$; then $P(A_\infty) = 1-\pi$. If $A_\infty$ happens then the maximum of the random walk $\{S_n\}$ is plainly zero. But if $\sum_1^\infty A_k$ happens the maximum of $\{S_n\}$ is *at least* $Z_1$. To study the further development of $\{S_n\}$ beyond $S_{N_1}$ we can, because of the independence involved, treat the walk $\{S_n - S_{N_1}\}$, $n > N_1$, as a newly-started walk. Either there is no finite $n > N_1$ such that $S_n - S_{N_1} > 0$ (an event with probability $(1-\pi)$), or there is a least integer $N_2$ and we can define a second ladder variable $Z_2 = S_{N_2} - S_{N_1}$. In short, the random walk $\{S_n\}$, for $n \ge 0$, will produce a random integer $J$, say, and $J$ iid ladder variables $Z_1, Z_2, \ldots, Z_J$. It is easy to see that $P\{J=j\} = (1-\pi)\pi^j$, $j \ge 0$, and that the maximum of the walk $\{S_n\}$ is equal to $Z_1+Z_2+\ldots+Z_J$ (taken as zero if $J = 0$).

Let us write $L_r(x)$ for the d.f. of $Z_1+Z_2+\ldots+Z_r$, although we shall occasionally write $L_1(x)$ more simply as $L(x)$. Then, for $x > 0$, we evidently have

$$(1.7)\quad Q(x) = (1-\pi) + \sum_{r=1}^{\infty} (1-\pi)\pi^{r} L_r(x).$$

This (well-known) equation is vital to our arguments in this paper. We shall show in §3 that if either $L^c(x)$ or $\int_x^\infty B^c(u)du$ is a fmg then

$$(1.8)\quad L^c(x) \sim \frac{1-\pi}{\mu\pi}\int_x^\infty B^c(u)du, \quad \text{as} \quad x \to \infty.$$

Conversely, if (1.8) holds, then both $L^c(x)$ and $\int_x^\infty B^c(u)du$ must be fmg. Then in §4, we shall deduce from (1.7) that, if either $L^c(x)$ or $Q^c(x)$ is a SETF,

$$(1.9)\quad L^c(x) \sim \frac{1-\pi}{\pi} Q^c(x), \quad \text{as} \quad x \to \infty.$$

We shall also show that if (1.9) holds then $L^c(x)$ and $Q^c(x)$ are necessarily fmg (though we cannot show they are necessarily SETF). It is apparent at this point that (1.2) will hold if either $Q^c(x)$ or $\int_x^\infty B^c(u)du$ is a SETF and that if (1.2) holds then both these functions are necessarily fmg.

From the applied viewpoint it is desirable to have usable conditions to ensure that $\int_x^\infty B^c(u)du$ is a SETF. In §2 we shall see that one sufficient condition is that $B^c(x)$ be a function of regular variation with strictly negative index ( in fact the index must be strictly negative, if $\int_0^\infty B^c(u)du$ is to exist). A more general sufficient condition is that $B^c(x) \sim \tau(x)/x^\gamma$, for $\gamma > 1$, where $\tau(x)$ is sub-linear (as described earlier in this section). Another sufficient condition, somewhat wider in scope, is that $B^c(x)$ be a Tail Function for which $x\alpha(x)$ is non-decreasing (where $\alpha(x)$ is the function introduced above in the discussion of "moderate growth"). We shall, in Theorem 2.5, prove this condition and provide the estimate

$$\int_x^\infty B^c(u)du \sim \frac{\lambda}{\lambda-1}\frac{B^c(x)}{\alpha(x)}, \quad x \to \infty$$

in which $\lambda = \lim_{x\to\infty} x\alpha(x)$ and $\lambda/(\lambda-1)$ is interpreted as unity if $\lambda = \infty$. By way of an easy example: suppose $B^c(x) \sim e^{-x^\alpha}$ for $0 < a < 1$. Then $\alpha(x) = \alpha/x^{1-\alpha}$, $\lambda = \infty$, and we infer

$$Q^c(x) \sim \frac{x^{(1-\alpha)}}{\mu\alpha} e^{-x^\alpha}.$$

In the final section (§5) of this paper, we look at the question of moments of a fairly general kind. For instance, if a certain integral $\int_0^\infty S(x)Q(dx)$ must converge, what condition has to be met by $B(x)$? This question has been discussed in a fairly general way by Kiefer and Wolfowitz (1956). Here we introduce even more general moments and also examine the associated matter of moments of $L(x)$. The main result in §5 is that, under certain conditions on $S$, $\int_0^\infty S(x)Q(dx)$ converges if and only if $\int_0^\infty S_J(x)B(dx)$ converges, where $S_J(x) = \int_0^x S(u)du$. The conditions on $S$ are that it be a non-decreasing fmg such that $S(x+y) \le S(x)S(y)$ for all $x \ge 0$, $y \ge 0$.

## 2. FUNCTIONS OF MODERATE GROWTH

Although it is possible to infer some relevant results from the work of Karamata (1930a), (1930b), (1933) on functions of slow growth, useful insight will be gained by developing the basic theory of functions of moderate growth *ab initio*. Fortunately we need only concern ourselves with *monotone* functions, and this restriction helps avoid much difficulty.

Let $g(x)$ be a non-decreasing function of moderate growth, and set

$$(2.1) \quad e^{a_n} = g(n)/g(n-1), \quad n = 1,2,\ldots .$$

Then $a_n \ge 0$, $a_n \to 0$ as $n \to \infty$, and

$$g(n) = g(0) \exp(a_1+a_2+\ldots+a_n), \quad n = 1,2,\ldots .$$

Let us now define, for $n = 1,2,\ldots,$

$$\alpha(u) = a_n, \quad (n-1) \le u < n.$$

Then

$$g(n) = g(0) \exp \int_0^n \alpha(u)du.$$

Moreover, if $(n-1) \le x \le n$,

$$\int_0^{n-1} \alpha(u)du \le \int_0^x \alpha(u)du \le \int_0^n \alpha(u)du$$

and so

$$g(n-1) \le g(0)\, e^{\int_0^x \alpha(u)du} \le g(n).$$

But $g(n-1) \le g(x) \le g(n)$ and $g(n) \sim g(n-1)$ as $n \to \infty$. Thus

$$(2.2) \quad g(x) \sim g(0) \exp \int_0^x \alpha(u)du$$

as $x \to \infty$, with $\alpha(u) \ge 0$, and $\alpha(u) \to 0$ as $u \to \infty$. This relation (2.2) is what we principally wished to demonstrate, but it might be noticed that (2.2) is an exact equation (rather than an asymptotic relation) whenever $x$ is an integer.

Let us recall that, by the definitions in §1, a *tail function* $T(x)$, say, will have the representation

$$T(x) \sim e^{-\Lambda(x)}, \quad x \to \infty$$

where

$$\Lambda(x) = \int_0^x \alpha(u)du$$

and $\alpha(u) \downarrow 0$. Suppose further that $T(x)$ is sub-exponential, so that (1.5) holds. Then we can find a $\rho$, $0 < \rho < 1$, such that for all large $x$

$$\frac{\int_{\frac{1}{2}x}^{x} \alpha(u)du}{\int_0^{\frac{1}{2}x} \alpha(u)du} \le \rho.$$

For $0 \le z \le \frac{1}{2}x$ let us set

$$\psi(z) = \frac{\int_{x-z}^{x} \alpha(u)du}{\int_0^z \alpha(u)du}.$$

Then, for almost all $z$,

$$\frac{\psi'(z)}{\psi(z)} = \frac{\alpha(x-z)}{\int_{x-z}^{x} \alpha(u)du} - \frac{\alpha(z)}{\int_0^z \alpha(u)du}.$$

But, since $\alpha(u)$ is non-increasing,

$$\frac{\alpha(z)}{\int_0^z \alpha(u)du} \le \frac{1}{z} \le \frac{\alpha(x-z)}{\int_{x-z}^{x} \alpha(u)du}.$$

Thus $\psi'(z) \ge 0$ and hence

$$\psi(z) \le \psi(\tfrac{1}{2}x), \quad 0 \le z \le \tfrac{1}{2}x.$$

This implies that

$$\int_{x-z}^{x} \alpha(u)du \le \rho \int_0^z \alpha(u)du,$$

or

$$\Lambda(x) - \Lambda(x-z) \le \rho\Lambda(z).$$

Therefore, for $0 \le z \le \frac{1}{2}x$, we have

$$\frac{e^{-\Lambda(x-z)}}{e^{-\Lambda(x)}} \le \left[\frac{1}{e^{-\Lambda(z)}}\right]^{\rho}.$$

Consequently we have established the following lemma.

*Lemma 2.1: If $T(x)$ is a SETF, then there exists finite constants $\rho$, $K$ such that $0 < \rho < 1$ and*

$$\frac{T(x-z)}{T(x)} \le \frac{K}{[T(z)]^{\rho}}$$

*for all $0 \le z \le \frac{1}{2}x$ and $x$ sufficiently large.*

Note that the constant $K$ arises in the above lemma because $T(x)$ and $e^{-\Lambda(x)}$ are only asymptotically equivalent.

Next suppose $A(x)$, for the moment, to be the arbitrary d.f. of two independent positive random variables $Y_1$ and $Y_2$. Choose a large $\Delta > 0$ and, for $x > 2\Delta$, set

$$I(x;\Delta,A) = \int_{\Delta}^{x-\Delta} A^c(x-z)A(dz)$$

$$= P\{\Delta < Y_1 \le x-\Delta,\ Y_1+Y_2 > x\}.$$

Consider the events:

$$E \equiv \{\Delta < Y_1 \le x-\Delta,\ Y_1+Y_2 > x\},$$
$$E_1 \equiv \{\Delta < Y_1 \le \tfrac{1}{2}x,\ Y_1+Y_2 > x\},$$
$$E_2 \equiv \{\tfrac{1}{2}x < Y_1,\ \tfrac{1}{2}x < Y_2\},$$
$$E_3 \equiv \{\Delta < Y_2 \le \tfrac{1}{2}x,\ Y_1+Y_2 > x\}.$$

Then it may be verified that $E_1$, $E_2$ and $E_3$ are disjoint and that

$$E \subset E_1 + E_2 + E_3.$$

Thus

$$I(x:\Delta,A) = P(E)$$
$$\le P(E_1) + P(E_2) + P(E_3)$$

i.e.

$$\text{(2.3)}\quad I(x;\Delta,A) \le [A^c(\tfrac{1}{2}x)]^2 + 2\int_{\Delta}^{\frac{1}{2}x} A^c(x-z)A(dz).$$

At this stage suppose $A^c(x)$ is a S.E.T.F.; say $A^c(x) \sim e^{-\Lambda(x)}$ as $x \to \infty$, where $\Lambda(x)$ is the function encountered earlier. Then, for all large $x$,

$$\text{(2.4)}\quad [A^c(\tfrac{1}{2}x)]^2 \sim e^{-2\Lambda(\frac{1}{2}x)}$$
$$\le \exp - \frac{2\Lambda(x)}{(1+\rho)},$$

by (1.5), with $\rho$ as before. But $\Lambda(x) \uparrow \infty$, since $A^c(x) \downarrow 0$, and

$2 > (1+\rho)$. Hence (2.4) gives

$$(2.5)\quad \frac{[A^c(\tfrac{1}{2}x)]^2}{A^c(x)} \to 0, \quad x \to \infty.$$

Also, by Lemma 2.1, for all large $x$,

$$\frac{1}{A^c(x)} \int_{\Delta}^{\frac{1}{2}x} A^c(x-z)A(dz) \le K \int_{\Delta}^{\frac{1}{2}x} \frac{dA(z)}{[1-A(z)]^{\rho}}$$
$$\le \frac{K}{(1-\rho)} [1-A(\Delta)]^{(1-\rho)}.$$

If we combine this inequality with (2.5) and (2.3) we have the following.

*Lemma 2.2: If $A(x)$ is any d.f. of a non-negative random variable such that $A^c(x)$ is a SETF then, for any $\varepsilon > 0$ we can find $\Delta(\varepsilon)$ such that*

$$\limsup_{x \to \infty} \frac{1}{A^c(x)} \int_{\Delta}^{x-\Delta} A^c(x-z)A(dz) \le \varepsilon.$$

As an example of the value of this lemma we prove the following useful result which generalizes a theorem of S. C. Port ( quoted on P. 272 of Feller (1971)) for functions of regular variation.

*Theorem 2.3: Let $A(x)$ and $B(x)$ be distribution functions and set*

$$C(x) = \int_{-\infty}^{+\infty} A(x-z)B(dz).$$

*Let $T(x)$ be a SETF and suppose that for some constants $\alpha \ge 0$, $\beta \ge 0$,*

$$\left.\begin{aligned} A^c(x)/T(x) &\to \alpha \\ B^c(x)/T(x) &\to \beta \end{aligned}\right\}, \quad x \to \infty.$$

*Then*

$$C^c(x)/T(x) \to \alpha + \beta.$$

*Proof:* To begin with, suppose $A$ and $B$ refer to non-negative random variables. Choose a large $\Delta > 0$ and write

$$\int_0^x A^c(x-z)B(dz) = \int_\Delta^{x-\Delta} \pm \int_{x-\Delta}^x + \int_0^\Delta$$
$$= I_1(x) + I_2(x) + I_3(x).$$

Plainly,

$$I_2(x) \le \int_{x-\Delta}^x B(dz)$$
$$= B^c(x-\Delta) - B^c(x)$$

and, since $T(x) \sim T(x-\Delta)$, we have

$$I_2(x)/T(x) \to 0, \quad x \to \infty.$$

Further

$$A^c(x)B(\Delta) \le I_3(x) \le A^c(x-\Delta)B(\Delta)$$

and so

$$I_3(x)/T(x) \to \alpha B(\Delta), \quad x \to \infty.$$

To deal with $I_1(x)$, let us set

$$G(x) = \tfrac{1}{2}\{A(x) + B(x)\},$$

so that $G^c(x)/T(x) \to \tfrac{1}{2}(\alpha+\beta)$. Then

$$I_1(x) \le 4 \int_\Delta^{x-\Delta} G^c(x-z)G(dz).$$

If $(\alpha+\beta) > 0$, $G^c(x)$ is a S E T F and Lemma 2.2 shows that, given any $\varepsilon > 0$ we can pick $\Delta$ so that

$$\limsup_{x\to\infty} I_1(x)/T(x) \le \varepsilon.$$

It then follows easily that

$$\frac{\int_0^x A^c(x-z)B(dz)}{T(x)} \to \alpha, \quad x \to \infty.$$

But

$$C^c(x) = B^c(x) + \int_0^x A^c(x-z)B(dz)$$

and hence $C^c(x) \sim (\alpha+\beta)T(x)$ as was to be proved. However we must deal with the case $\alpha+\beta = 0$. Now, given any small $\varepsilon > 0$ we can evidently construct distribution functions $A_1(x)$, $B_1(x)$ such that $A_1(x) \le A(x)$ for all $x$, and $A_1(x) = \varepsilon T(x) = B_1(x)$ for all large $x$. If

$$C_1(x) = \int_0^x A_1(x-z)B_1(dz)$$

then $C_1{}^c(x) \ge C^c(x)$ for all $x$ and, by the first part of the proof, $C_1{}^c(x) \sim 2\varepsilon T(x)$, as $x \to \infty$. Thus $C^c(x) = o(T(x))$ and the theorem is proved.

To extend the result to not-necessarily-non-negative random variables, we write

$$C^c(x) = \int_{-\infty}^{0+} A^c(x-z)B(dz) + \int_{0+}^{x} A^c(x-z)B(dz) + \int_x^{\infty} A^c(x-z)B(dz)$$

$$= J_1(x) + J_2(x) + J_3(x), \quad \text{say.}$$

It is trivial that $J_1(x)/\alpha T(x) \to B(0+)$ as $x \to \infty$ and the first part of the proof will show $J_2(x)/\alpha T(x) \to 1 - B(0+)$. Further, for $\Delta > 0$ large,

$$A^c(-\Delta)B^c(x+\Delta) \le J_3(x) \le B^c(x)$$

and, since $A^c(-\Delta)$ can be made arbitrarily near unity, this implies

$$J_3(x)/T(x) \to \beta.$$

Thus the theorem is established.

At this point it should be clear how one can also prove the following:

*Corollary 2.3.1: If A, B, C, T are as in Theorem 2.3.1, but one has only the weaker hypothesis*

$$\left.\begin{aligned} A^c(x) &= O(T(x)) \\ B^c(x) &= O(T(x)) \end{aligned}\right\}, \quad as \quad x \to \infty$$

*then one may conclude*

$$C^c(x) \;=\; O(T(x)).$$

For the remainder of this section we shall be concerned with displaying functions which belong to the various classes we have introduced. In particular, we shall be concerned with finding whether properties of a tail function $T(x)$ are preserved by $\int_x^\infty T(u)du$, when this tail integral exists, and we shall also look at the question of estimating this tail integral.

*Theorem 2.4: Let* $T(x)$ *be a non-negative non-increasing function in* $L_1(0,\infty)$ *and set* $T_I(x) = \int_x^\infty T(u)du$. *Then:*

*(i)* $T_I(x)$ *is a fmg if* $T(x)$ *is a fmg.*

*(ii)* $T_I(x)$ *is a tail function if* $T(x)$ *is a tail function.*

*Proof:* (i) If $T(x)$ is a fmg, then $T(x) \sim T(x+1)$ as $x \to \infty$ and hence $T_I(x) \sim \int_x^\infty T(u+1)du = T_I(x+1)$. This is enough to show $T_I(x)$ is a f.m.g.

(ii) If $T(x)$ is a tail function, then

$$T(x) \sim \exp - \int_0^x \;(u)du$$

where $\alpha(u) \downarrow 0$ as $u \to \infty$. Hence, if $\theta \ge 0$, $\nu \ge 0$,

$$\int_0^v \alpha(\theta+u)du \;\ge\; \int_0^v \alpha(\theta+1+u)du,$$

or

$$\int_\theta^{\theta+v} \alpha(u)du \;\ge\; \int_{\theta+1}^{\theta+1+v} \alpha(u)du.$$

Therefore

$$\int_0^\infty e^{-\int_\theta^{\theta+v}\alpha(u)du}\,dv \;\le\; \int_0^\infty e^{-\int_{\theta+1}^{\theta+1+v}\alpha(u)du}\,dv,$$

or

$$\int_\theta^\infty e^{-\int_\theta^{y}\alpha(u)du}\,dy \;\le\; \int_{\theta+1}^\infty e^{-\int_{\theta+1}^{y}\alpha(u)du}\,dy.$$

If we use the notation $\Lambda(x) = \int_0^x \alpha(u)du$, then we have:

$$(2.6)\qquad \frac{\int_\theta^\infty e^{-\Lambda(y)}\,dy}{e^{-\Lambda(\theta)}} \le \frac{\int_{\theta+1}^\infty e^{-\Lambda(y)}\,dy}{e^{-\Lambda(\theta+1)}} .$$

Let us set

$$\phi(\theta) = \frac{\int_{\theta+1}^\infty e^{-\Lambda(y)}\,dy}{\int_\theta^\infty e^{-\Lambda(y)}\,dy} .$$

Then $\phi(\theta) \to 1$ as $\theta \to \infty$, and

$$\frac{\phi'(\theta)}{\phi(\theta)} = \frac{e^{-\Lambda(\theta)}}{\int_\theta^\infty e^{-\Lambda(y)}\,dy} - \frac{e^{-\Lambda(\theta+1)}}{\int_{\theta+1}^\infty e^{-\Lambda(y)}\,dy} .$$

By (2.6) we see that $\phi'(\theta) \ge 0$, so the sequence $\phi(0), \phi(1), \phi(2), \ldots$ is non-increasing. If we refer to the discussion at the start of this section, leading to (2.2), we see that our results imply that

$$\int_x^\infty e^{-\Lambda(y)}\,dy \sim Ae^{-\int_0^x \beta(y)dy},$$

where $A$ is some constant and $\beta(u) \downarrow 0$. This is enough to show that $T_I(x)$ is, indeed, a tail function.

We shall occasionally need to know if $T_I(x)$ will be sub-exponential; the most we have achieved on this matter is the following.

*Theorem 2.5: Let $T(x)$ be a tail function such that $u\alpha(u)$ is non-decreasing; let $u\alpha(u) \uparrow \lambda \le \infty$ as $u \uparrow \infty$. Then, if $\lambda > 1$,*

$$T_I(x) \sim \frac{\lambda}{\lambda\ 1} \cdot \frac{T(x)}{\alpha(x)}, \quad \text{as } x \to \infty$$

*where $\lambda/(\lambda-1)$ is interpreted as unity if $\lambda = \infty$. Furthermore, (if $\lambda > 1$) $T_I(x)$ is sub-exponential.*

(Note that it will be shown later that $\lambda \ge 1$, necessarily, if $T_I(x)$ exists.)

*Proof:* Since $T(x)$ is a tail function,

$$T(x) \sim e^{-\Lambda(x)},$$

where

$$\Lambda(x) = \int_0^x \alpha(u)du$$

and $\alpha(u) \downarrow 0$. We are here to assume that $u\alpha(u)$ is non-decreasing from which it follows easily that $\Lambda(x) \to \infty$ as $x \to \infty$ (for we can show $\alpha(u) > a/u$, ultimately, for some $a > 0$). Thus, since $\Lambda(x)$ is strictly increasing, we can introduce an inverse function $\mu(y)$, say, such that

$$\Lambda(\mu(y)) = y.$$

Since, also, $\Lambda(x)$ is differentiable, it follows that $\mu(y)$ is likewise, and

$$\mu'(y) = \frac{1}{\alpha(\mu(y))}.$$

Let us set

$$\Theta(x) = e^{-\Lambda(x)}$$

and

$$\Theta_I(x) = \int_x^\infty \Theta(y)dy,$$

so that $T_I(x) \sim \Theta_I(x)$ as $x \to \infty$.

From its definition, one can see that $\mu(y)$ is strictly increasing and unbounded as $y \to \infty$. Thus a change of variable gives

$$\Theta_I(x) = \int_x^\infty e^{-\Lambda(u)}\, du$$
$$= \int_{\Lambda(x)}^\infty e^{-y} \frac{dy}{\alpha(\mu(y))}.$$

At this point, let us set

$$\chi(x) = \frac{e^{-\Lambda(x)}}{\alpha(x)} = \frac{1}{\alpha(x)} \int_{\Lambda(x)}^\infty e^{-y}\, dy.$$

Thus

$$(2.7)\quad \frac{\Theta_I(x)}{\chi(x)} - 1 = \int_{\Lambda(x)}^{\infty} e^{-y+\Lambda(x)} \left\{\frac{\alpha(x)}{\alpha(\mu(y))} - 1\right\} dy$$

$$= \int_0^{\infty} e^{-y} \left\{\frac{\alpha(x)}{\alpha(\mu(y+\Lambda(x)))} - 1\right\} dy.$$

The expression in braces can be written

$$\frac{\alpha(\mu(\Lambda(x)))}{\alpha(\mu(y+\Lambda(x)))} - 1 = \gamma_y(x), \quad \text{say,}$$

and, because $\mu(y)$ is increasing and $\alpha(x)$ is non-increasing, this expression is non-negative. Thus $\Theta_I(x) \geq \chi(x)$ for all $x$.

We wish to show that $\gamma_y(x) \to 0$ as $x \to \infty$ and that this convergence is dominated by some suitable function of $y$, thereby allowing us to take limits under the integral sign in (2.7).

Now, by definition of $\mu(y)$, we have

$$y = \int_0^{\mu(y)} \alpha(u)du.$$

Thus

$$(2.8)\quad 1 = \int_{\mu(y)}^{\mu(y+1)} \alpha(u)du.$$

However, we have that $u\alpha(u)$ is non-decreasing. Thus (2.8) yields

$$(2.9)\quad 1 \geq \mu(y)\ \alpha(\mu(y)) \int_{\mu(y)}^{\mu(y+1)} \frac{du}{u}$$

$$= \mu(y)\ \alpha(\mu(y)) \log \frac{\mu(y+1)}{\mu(y)}.$$

Let us first suppose that $\omega\alpha(\omega) \to \infty$ as $\omega \to \infty$. Then (2.9) and the fact that $\mu(y+1) \geq \mu(y)$, prove that

$$\log \frac{\mu(y+1)}{\mu(y)} \to 0, \quad \text{as} \quad y \to \infty,$$

i.e.

$$\mu(y+1) \sim \mu(y).$$

But $\mu(y)\alpha(\mu(y)) \le \mu(y+1)\alpha(\mu(y+1))$, so that

$$\limsup_{y \to \infty} \frac{\alpha(\mu(y))}{\alpha(\mu(y+1))} \le 1.$$

Since $\alpha(\mu(y))$ is non-increasing, it follows that

$$\frac{\alpha(\mu(y))}{\alpha(\mu(y+1))} \to 1, \quad \text{as} \quad y \to \infty,$$

and hence that $\alpha(\mu(y))$ is a function of mild growth. Consequently there is $\beta(u)$, say, such that $\beta(u) \ge 0$, $\beta(u) \to 0$ as $u \to \infty$, and

$$\alpha(\mu(y)) \sim e^{-\int_0^y \beta(u)du}, \quad \text{as} \quad y \to \infty.$$

Therefore, given an arbitrary $\varepsilon > 0$, for all $z \ge 0$,

$$\begin{aligned} \frac{\alpha(\mu(y))}{\alpha(\mu(y+z))} &\le (1+\varepsilon)\, e^{\int_y^{y+z} \beta(u)du} \\ &\le (1+\varepsilon)\, e^{\varepsilon z}, \end{aligned}$$

for all sufficiently large $y$.

From this inequality and the fact that $\alpha(\mu(y))$ is a fmg, we have that $\gamma_y(x) \to 0$ as $x \to \infty$, as desired; we also have that $0 \le \gamma_y(x) \le (1+\varepsilon)e^{\varepsilon y}$ for all large $y$. Thus we can appeal to dominated convergence to deduce from (2.7) that

$$\Theta_I(x) \sim \chi(x), \quad \text{as} \quad x \to \infty.$$

This proves the first part of the theorem if $\lambda = \infty$.

Suppose next that $u\alpha(u) \to \lambda < \infty$. If $\lambda < 1$ then $u\alpha(u) < \lambda' < 1$ for all large $u$. Thus $\Lambda(x) - \lambda' \log x \to -\infty$, and so $x^{\lambda'} T(x) \to \infty$, as $x \to \infty$. This contradicts the assumed convergence of $\int_1^\infty T(u)du$, so $\lambda \ge 1$, necessarily. With this established, let us suppose $\lambda > 1$. We have

$$\Lambda(x) \sim \lambda \log x, \quad \text{as} \quad x \to \infty,$$

so

$$\Lambda(\mu(y)) \sim \lambda \log \mu(y), \quad \text{as} \quad y \to \infty,$$

i.e.

$$\log \mu(y) \sim \frac{y}{\lambda},$$

or

$$\mu(y) = \exp\left\{\frac{y}{\lambda+o(1)}\right\}.$$

But $\alpha(\mu(y)) \sim \lambda/\mu(y)$, so

$$\alpha(\mu(y)) \sim \lambda \exp - \left\{\frac{y}{\lambda+o(1)}\right\}$$

and hence, for $z \geq 0$,

$$(2.10) \quad \frac{\alpha(\mu(y))}{\alpha(\mu(y+z))} \to e^{z/\lambda}, \quad y \to \infty.$$

In particular, (2.10) implies that $e^{-y/\lambda}\alpha(\mu(y))$ is a fmg and so, as before, for any small $\varepsilon > 0$,

$$\frac{\alpha(\mu(y))}{\alpha(\mu(y+z))} \leq e^{z(\varepsilon + \frac{1}{\lambda})},$$

for all large $y$. Since we can choose $\varepsilon$ small enough to ensure $\varepsilon + 1/\lambda < 1$, this inequality and (2.10) allows us to deduce from

$$\frac{\Theta_I(x)}{\chi(x)} = \int_0^\infty e^{-y} \frac{\alpha(\Lambda(x))}{\alpha(\mu(y+\Lambda(x)))}\, dy$$

that

$$\frac{\Theta_I(x)}{\chi(x)} \to \int_0^\infty e^{-y(1 - 1/\lambda)}\, dy$$

$$= \frac{\lambda}{\lambda-1}, \quad \text{as} \quad x \to \infty,$$

by dominated convergence. This completes the second part of the theorem.

To complete the theorem we need the following:

*Lemma 2.6. If* $u\alpha(u) \to \infty$, *then* $\log \alpha(x) = o(\Lambda(x))$, *as* $x \to \infty$.

*Proof of Lemma:* Take $T$ very large, then

$$\Lambda(x) \geq \int_0^T \alpha(u)du + T\alpha(T)\log(x/T).$$

Hence

$$\liminf_{x \to \infty} \frac{\Lambda(x)}{\log x} \geq T\alpha(T).$$

But $T\alpha(T)$ can be made arbitrarily large. Thus $\log x = o(\Lambda(x))$, as $x \to \infty$. The lemma follows from the fact that, since $x\alpha(x)$ is non-decreasing, $\alpha(x) \geq 1/x$ for all large $x$, implying that $|\log \alpha(x)| \leq \log x$.

Now, when $x\alpha(x)$ is unbounded, the first part of the theorem shows

$$\begin{aligned} \log T_I(x) &= -\Lambda(x) - \log \alpha(x) + o(1) \\ &= -\Lambda(x)\{1 \_ o(1)\}, \end{aligned}$$

by the lemma. Thus, if $T(x)$ is SE so that

$$\limsup_{x \to \infty} \frac{\Lambda(2x)}{\Lambda(x)} < 2,$$

it follows that

$$\limsup_{x \to \infty} \frac{\log T_I(2x)}{\log T_I(x)} < 2,$$

also. This establishes that $T_I(x)$ is SE.

Finally, when $u\alpha(u) \to \lambda < \infty$, we can see that, as $x \to \infty$,

$$\frac{T(x)}{T(2x)} \sim e^{\int_x^{2x} \alpha(u)du} \to e^{\lambda \log 2},$$

i.e.

$$T(2x) \sim 2^{-\lambda}\, T(x).$$

Thus

$$\begin{aligned} T_I(2x) &= \int_{2x}^{\infty} T(u)du \\ &= 2\int_x^{\infty} T(2u)du \end{aligned}$$

$$\sim \; 2^{1-\lambda}\, T_I(x).$$

Therefore $\log T_I(2x) \sim \log T_I(x)$, and $T_I(x)$ is again proved to be SE.

When $\tau(x)$ is a non-negative non-decreasing function such that $\tau(x)/x$ is non-increasing, we call $\tau(x)$ a sub-linear function. Such functions have proved useful in a previous investigation of the author (Smith (1969)). It is an easy exercise to show that a sub-linear function is also sub-additive:

$$\tau(x+y) \le \tau(x) + \tau(y), \quad x > 0, \quad y > 0.$$

Also: $x/\tau(x)$ is sub-linear if $\tau(x)$ is sub-linear.

A useful class of functions, in discussing tail probabilities, are of the form $\tau(x)/x^{\gamma}$, for $\gamma > 1$. There is nothing to be gained by considering functions of the form $1/\tau(x)x^{\gamma}$; such functions are included in the first class. In connection with that class we have the following.

*Theorem 2.7. Let* $T(x) \sim \tau(x)/x^{\gamma}$, *as* $x \to \infty$, *where* $\tau(x)$ *is sub-linear and* $\gamma > 0$. *Then* $T(x)$ *is a fmg. If* $\gamma > 1$ *and* $T_I(x) = \int_x^{\infty} T(u)du$, *then* $T_I(x)$ *is a sub-exponential tail function of the form* $\tau_1(x)/x^{\gamma-1}$, *where* $\tau_1(x)$ *is another sub-linear function.*

*Proof:* If we set

$$\chi(x) = \frac{\tau(x)}{x^{\gamma}}$$

then, since $\tau(x)$ and $x/\tau(x)$ are non-increasing,

$$\frac{x^{\gamma-1}}{(x+1)^{\gamma-1}} \ge \frac{\chi(x+1)}{\chi(x)} \ge \frac{x^{\gamma}}{(x+1)^{\gamma}}.$$

From this we see that

$$\frac{\chi(x+1)}{\chi(x)} \to 1, \quad \text{as} \quad x \to \infty,$$

and consequently $T(x)$ is a fmg. By Theorem 2.4 (i), it follows that $T_I(x)$ is also a fmg.

Let us set, when $\gamma > 1$,

$$\chi_I(x) = \int_x^\infty \frac{\tau(u)}{u^\gamma} du.$$

Then

$$\frac{\chi_I'(x)}{\chi_I(x)} = -\frac{\{\tau(x)/x\}}{x^{\gamma-1} \int_x^\infty \tau(u)u^{-\gamma} du} .$$

Since $\tau(u)/u$ is non-increasing we shall establish that $\chi_I'(x)/\chi_I(x)$ is non-decreasing if we can show that

$$x^{\gamma-1} \int_x^\infty \tau(u)u^{-\gamma} du = \rho(x), \text{ say,}$$

is non-decreasing. But

$$\begin{aligned} \rho'(x) &= (\gamma-1)x^{\gamma-2} \int_x^\infty \tau(u)u^{-\gamma} du - x^{-1} \tau(x) \\ &\geq (\gamma-1)x^{\gamma-2} \tau(x) \int_x^\infty u^{-\gamma} du - x^{-1} \tau(x) \\ &\geq x^{-1} \tau(x) - x^{-1} \tau(x). \end{aligned}$$

Thus $\rho(x)$ is, indeed, non-decreasing and so

$$\frac{d}{dx} \log \chi_I(x)$$

is non-decreasing. Consequently, for any integer $n$

$$\log \chi_I(n+1) - \log \chi_I(n) \geq \log \chi_I(n) - \log \chi_I(n-1)$$

i.e.

$$(2.11) \quad \frac{\chi_I(n+1)}{\chi_I(n)} \geq \frac{\chi_I(n)}{\chi_I(n-1)} .$$

If we hark back to our preliminary results on moderate growth, we see that we can set

$$\frac{\chi_I(n)}{\chi_I(n-1)} = e^{b_n}, \text{ say,}$$

where $b_n \to 0$; then (2.11) shows that $b_{n+1} \le b_n$, and we are allowed to conclude that $\chi_I(x)$, and hence $T_I(x) \sim \chi_I(x)$, is a tail function.

Let us define (still for $\gamma > 1$)

$$\frac{\tau_1(x)}{x^{\gamma-1}} = \int_x^\infty \frac{\tau(u)}{u^\gamma}\, du.$$

Then the argument just above shows that $\tau_1(x)$ is non-decreasing. Also,

$$\frac{d}{dx}\frac{\tau_1(x)}{x} = (\gamma-2)x^{\gamma-3}\int_x^\infty \frac{\tau(u)}{u^\gamma}\, du - x^{-2}\,\tau(x)$$

and the right-hand side of this equation is non-positive if $\gamma \le 2$. Suppose $\gamma > 2$; then, by the monotonicity of $\tau(u)/u$,

$$\begin{aligned}\frac{d}{dx}\left(\frac{\tau_1(x)}{x}\right) &\le (\gamma-2)x^{\gamma-4}\,\tau(x)\int_x^\infty \frac{du}{u^{\gamma-1}} - x^{-2}\,\tau(x)\\ &\le x^{-2}\,\tau(x) - x^{-2}\,\tau(x).\end{aligned}$$

Thus $\tau_1(x)/x$ is non-increasing. We have shown that, when $\gamma > 1$,

$$\chi_1(x) = \frac{\tau_1(x)}{x^{\gamma-1}}$$

where $\tau_1(x)$ is sub-linear. A particular consequence is that

$$\frac{\chi_1(2x)}{\chi_1(x)} \ge \frac{1}{2^{\gamma-1}}$$

and hence

$$\liminf_{x\to\infty} \frac{T_I(2x)}{T_I(x)} \ge \frac{1}{2^{\gamma-1}}.$$

From this it is a routine deduction that

$$\log T_I(2x) \sim \log T_I(x)$$

and, hence, that $T_I(x)$ is SE.

Of course, in matters of the present sort, the "functions of regular variation" introduced by Karamata (1930a), (1930b), and exploited, in particular, by Feller (1971), are of great importance. Before we close this section we shall show how they fit in with the present scheme. If $T(x) \sim x^{-\alpha} S(x)$, where $\alpha > 0$ and $S(x)$ is a function of slow growth, we say $T(x)$ is a function of regular variation (frv) of index $-\alpha$. Since, as is now well-known from the work of Karamata, it is true, that

$$\frac{S(x)}{\alpha x^{\alpha}} \sim \int_x^{\infty} \frac{S(u)}{u^{(1+\alpha)}}\, du, \quad \text{as} \quad x \to \infty,$$

*we may always conveniently suppose such a frv to be strictly decreasing* (a similar argument shows that when the index is strictly positive a frv can be supposed strictly increasing). The example

$$S(x) = \frac{3}{2} + \sin\sqrt{(\log x)}$$

shows, however, that a function of slow growth (which this $S(x)$ is) need not be asymptotically equal to a monotone function; thus our convenient supposition does not extend to the case of zero index.

Suppose now that, for $\alpha > 0$,

$$T(x) \sim \frac{S(x)}{x^{\alpha}}, \quad x \to \infty.$$

Let us set

$$\chi(x) = \alpha \int_x^{\infty} \frac{S(u)}{u^{1+\alpha}}\, du,$$

so that $\chi(x) \sim T(x)$, as $x \to \infty$. Then

$$\chi'(x) = -\frac{\alpha S(x)}{x^{1+\alpha}}$$

and so

$$x \left\{\frac{\chi'(x)}{\chi(x)}\right\} \to -\alpha, \quad \text{as} \quad x \to \infty.$$

Consequently, $\log \chi(x)$ is ultimately non-increasing and the conclusion that $\chi(x)$ is a tail function follows as for the case involving sub-linear functions, above. Indeed, the proof that $\chi(x)$ is sub-exponential can be constructed similarly, if we use that $S(2x) \sim S(x)$. Thus we have

*Theorem 2.8. If $T(x)$ is a function of regular variation with strictly negative index, then $T_I(x)$ is a sub-exponential tail function.*

## 3. RELATIONSHIP BETWEEN $B^c(x)$ AND $L^c(x)$

Let us write $F(x)$ for the d.f. of $X = V-U$, where $V$ is a typical service-time and $U$ a typical (independent) inter-arrival interval. Then we have

*Theorem 3.1. If either $B^c(x)$ or $F^c(x)$ is a fmg, then*

$$(3.1) \quad B^c(x) \sim F^c(x) \quad \text{as} \quad x \to \infty.$$

*Conversely, if (3.1) holds then both $B^c(x)$ and $F^c(x)$ are fmg.*

*Proof:* We base our argument on the equation

$$(3.2) \quad F^c(x) = \int_0^\infty B^c(x+u)A(du).$$

Since $B^c(x)$ is non-increasing, this immediately yields

$$(3.3) \quad F^c(x) \le B^c(x).$$

On the other hand, if $\Delta$ is any large positive constant, (3.1) gives

$$(3.4) \quad F^c(x) \ge B^c(x+\Delta)\ A(\Delta).$$

*a) If $B^c$ is given to be of moderate growth,* we have from (3.4) that

$$\liminf_{x \to \infty} \frac{F^c(x)}{B^c(x)} \ge A(\Delta),$$

since $B^c(x+\Delta) \sim B^c(x)$. But $A(\Delta)$ can be made arbitrarily near unity, by choosing $\Delta$ large enough. This remark, with (3.3) proves (3.1).

*b) If $F^c$ is given to be of moderate growth,* we have, from (3.4), that

$$F^c(x-\Delta) \;\geq\; B^c(x)\,A(\Delta)$$

for all large x. Since $F^c(x-\Delta) \sim F^c(x)$ it should be clear that a proof such as in (a) again applies, to yield (3.1).

*c) If $F^c(x) \sim B^c(x)$, as $x \to \infty$.* Let $\Delta$ be any constant such that $A(\Delta) < 1$. Then (3.2) gives

$$F^c(x) \;\leq\; B^c(x)A(\Delta) + B^c(x+\Delta)A^c(\Delta)$$

and so

$$1 \;\leq\; A(\Delta)\left\{\frac{B^c(x)}{F^c(x)}\right\} + A^c(\Delta)\left\{\frac{B^c(x+\Delta)F^c(x+\Delta)}{F^c(x+\Delta)F^c(x)}\right\}.$$

Hence, since $B^c(x) \sim F^c(x)$,

$$1 \;\leq\; A(\Delta) + A^c(\Delta)\liminf_{x\to\infty}\frac{F^c(x+\Delta)}{F^c(x)}.$$

This shows that

$$\liminf_{x\to\infty}\frac{F^c(x+\Delta)}{F^c(x)} \;\geq\; 1.$$

But $F^c(x)$ is non-increasing. Hence $F^c(x+\Delta) \sim F^c(x)$. If $\Delta$ can be chosen arbitrarily large the theorem is proved. If, however, there is a least finite $\Delta_0$ such that $A(\Delta_0) = 1$ our choice of $\Delta$ is restricted to the range $0 < \Delta < \Delta_0$. However, if k is any fixed integer it is obviously true that $F^c(x+k\Delta) \sim F^c(x)$. Thus the proof is clear.

*Corollary 3.1.1. If either $\int_0^\infty B^c(u)du$ or $\int_0^\infty F^c(u)du$ is finite then both are finite. If either $\int_x^\infty B^c(u)du$ or $\int_x^\infty F^c(u)du$ is a fmg, then they both are, and they are asymptotically equal. Conversely, if they are asymptotically equal then they are fmg.*

*Proof:* The Corollary follows from (3.2) which gives

$$(3.5)\quad \int_x^\infty F^c(v)dv = \int_0^\infty \left\{\int_{x+u}^\infty B^c(v)dv\right\} A(du).$$

The arguments applied to (3.2) in the proof of the theorem can now be applied to (3.5).

Note that $\int_0^\infty B^c(u)du$ is the mean service-time.

Before we proceed further, let us explain a notational convention we shall adopt. If $H(\cdot)$, say, is a measure function, we shall write $H(I)$, whenever $I$ is an interval, to denote $\int_I H(dx)$. If $I \equiv (-\infty,x]$, however, we shall find it convenient to write $H(x)$ in place of $H(I)$, when it is finite. In terms of the random walk $\{S_n\}$ described in §1, we shall now introduce the usual renewal measure; if $I$ is any finite interval of the real axis,

$$H(I) = \sum_{n=1}^\infty P\{S_n \in I\}.$$

This series is convergent because the walk $\{S_n\}$ is transient (Chung & Fuchs (1951)).

Let us define another measure, involving the improper random variable $N_1$, the suffix of the first ladder variable:

$$G(I) = \sum_{n=1}^\infty P\{S_n \in I \ \&\ N_1 > n\}.$$

Then $G(I) \le H(I)$ and so is also finite for any bounded interval.

Notice that $L(x)$, the d.f. of the first ladder variable, is given by $L(x) = P\{S_{N_1} \le x \mid N_1 < \infty\}$. Thus if $I \subset (0,\infty)$ is a finite interval,

$$(3.6)\quad \pi L(I) = P\{S_{N_1} \in I \ \&\ N_1 < \infty\}$$

$$= \sum_{n=1}^\infty P\{S_n \in I \ \&\ N_1 > (n-1)\}$$

$$= F(I) + \sum_{n=1}^{\infty} P\{S_{n+1} \in I \ \& \ N_1 > n\}$$

$$= F(I) + \int_{-\infty}^{0} F(I-z)\ G(dz).$$

Also, for *any* bounded interval $I$,

$$\text{(3.7)} \quad H(I) = \sum_{n=1}^{\infty} P\{S_n \in I \ \& \ N_1 > n\} + \sum_{n=1}^{\infty} P\{S_n \in I \ \& \ N_1 \le n\}$$

$$= G(I) + \sum_{n=1}^{\infty} \sum_{r=1}^{n} P\{S_n \in I \ \& \ N_1 = r\}$$

$$= G(I) + \sum_{t=0}^{\infty} P\{S_{N_1+t} \in I \ \& \ N_1 < \infty\}$$

$$= G(I) + \pi L(I) + \pi \int_{0}^{\infty} H(I-z)\ L(dz).$$

If we replace the interval $I$ by $I-t$ in (3.7), where $t$ is large and positive, we obtain

$$\text{(3.8)} \quad H(I-t) = G(I-t) + \pi L(I-t) + \pi \int_{0}^{\infty} H(I-t-z)\ L(dz).$$

It is possible for the random variables $\{X_n\}$ to be lattice, that is: almost surely multiples of some fixed $\tilde{\omega} > 0$; in this case we may suppose, with no loss of generality, that $\tilde{\omega} = 1$. Thus it follows from standard renewal theory that if $I \equiv (0,1)$ then $H(I-t)$ is a bounded function of $t$ and, in both the lattice and non-lattice case,

$$H(I-t) \to \frac{1}{\mu}, \quad \text{as} \quad t \to \infty,$$

where $\mu = |EX_n|$. It is then routine to show that

$$\int_{0}^{\infty} H(I-t-z)\ L(dz) \to \frac{1}{\mu}, \quad \text{as} \quad t \to \infty,$$

and thence to deduce from (3.8) that

$$\text{(3.9)} \quad G(I-t) \to \frac{(1-\pi)}{\mu}, \quad \text{as} \quad t \to \infty.$$

For any $\varepsilon > 0$ we can therefore find an integer $\Delta(\varepsilon)$ such that

$$(3.10)\quad G(I-t) \geq \frac{(1-\pi)}{\mu} - \varepsilon$$

for all $t \geq \Delta(\varepsilon)$. With the present choice of $I$ we can then obtain from (3.6) that, for any integer $\nu > 0$,

$$\pi \sum_{n=\nu}^{\infty} L(I+n) \geq \int_{-\infty}^{0} \sum_{n=\nu}^{\infty} F(I+n-z)\, G(dz)$$

which is the same thing as

$$(3.11)\quad \pi L^c(\nu) \geq \int_{-\infty}^{0} F^c(\nu-z)\, G(dz),$$

$$\geq \sum_{r=\Delta}^{\infty} \int_{-(r+1)}^{-r} F^c(\nu-z)\, G(dz)$$

$$\geq \left\{\frac{(1-\pi)}{\mu} - \varepsilon\right\} \int_{\nu+\Delta}^{\infty} F^c(u)\, du.$$

Suppose that $\int_x^\infty F^c(u)du$ is a fmg.

Then (3.11) yields, since $\varepsilon$ is arbitrary, and $L^c(x)$ is monotone,

$$(3.12)\quad \liminf_{x \to \infty} \frac{L^c(x)}{\int_x^\infty F^c(u)du} \geq \frac{(1-\pi)}{\pi\mu}.$$

Conversely, if we suppose that $L^c(x)$, rather than $\int_x^\infty F^c(u)du$ is a fmg, we can still deduce (3.12) from (3.11) by using that $L^c(\nu-\Delta-1) \sim L^c(\nu)$ as $\nu \to \infty$.

If, in (3.6), we let $I \equiv (x,\infty)$ we obtain

$$(3.13)\quad \pi L^c(x) = F^c(x) + \int_{-\infty}^{0} F^c(x-z)\, G(dz).$$

Hence, if we use the fact that the renewal measure $H$ dominates $G$ and suppose that, when $I \equiv [0,1)$,

$$G(I-t) \leq \frac{(1-\pi)}{\mu} + \varepsilon$$

for all $t \geq \Delta(\varepsilon)$, we have

$$(3.14)\quad \pi L^c(x) \le F^c(x) + \int_{-\Delta}^{0} F^c(x-z)\, H(dz) + \left\{\frac{1-\pi}{\mu} + \varepsilon\right\} \int_{\Delta-1}^{\infty} F^c(x+u)\, du.$$

Let us, temporarily, write

$$C = \int_{-\Delta}^{0} H(dz),$$

necessarily finite. Then (3.14) gives

$$(3.15)\quad \pi L^c(x) \le (1+C)\, F^c(x) + \left\{\frac{(1-\pi)}{\mu} + \varepsilon\right\} \int_{\Delta-1}^{\infty} F^c(x+u)\, du.$$

At this point, we need:

*Lemma 3.2. If* $\int_x^\infty F^c(u)du$ *is a fmg, then*

$$\frac{F^c(x)}{\int_x^\infty F^c(u)\, du} \to 0, \quad \text{as} \quad x \to \infty.$$

*Proof:* If $T$ is a large positive constant,

$$\int_x^\infty F^c(u)\, du \ge TF^c(x+T),$$

since $F^c(x)$ is non-increasing. Thus, since

$$\int_{x-T}^\infty F^c(u)du \sim \int_x^\infty F^c(u)du,$$

$$\limsup_{x\to\infty} \frac{F^c(x)}{\int_x^\infty F^c(u)\, du} \le \frac{1}{T}.$$

But $T$ can be arbitrarily large, so the lemma is proved.

This lemma enables us to deduce from (3.15) that if $\int_x^\infty F_c(u)du$ is fmg.

$$(3.16)\quad \limsup_{x\to\infty} \frac{L^c(x)}{\int_x^\infty F^c(u)\, du} \le \frac{(1-\pi)}{\pi\mu}.$$

However, we also wish to show that (3.16) follows from the alternative premiss *that* $L^c(x)$ *is a fmg.* In this case, we have, from (3.12) that

$$L^c(x) \geq \left\{\frac{(1-\pi)}{\pi\mu} - \varepsilon\right\} \int_x^\infty F^c(u)du$$

for all sufficiently large $x$. Thus, if $T > 0$ is fixed,

$$L^c(x) \geq \left\{\frac{(1-\pi)}{\pi\mu} - \varepsilon\right\} T\, F^c(x+T),$$

and so, since $L^c(x+T) \sim L^c(x)$

$$\begin{aligned}\limsup_{x\to\infty} \frac{F^c(x)}{L^c(x)} &= \limsup_{x\to\infty} \frac{F^c(x+T)}{L^c(x+T)} \\ &= \limsup_{x\to\infty} \frac{F^c(x+T)}{L^c(x)} \\ &\leq \frac{1}{T}\left\{\frac{(1-\pi)}{\pi\mu} - \varepsilon\right\}^{-1}.\end{aligned}$$

Thus

$$\frac{F^c(x)}{L^c(x)} \to 0, \quad \text{as} \quad x \to \infty,$$

and we can infer from (3.14) that

$$\liminf_{x\to\infty} \frac{1}{L^c(x)} \int_{x+\Delta-1}^\infty F^c(u)du \geq \frac{\pi\mu}{(1-\pi)}.$$

If we reason along what should now be familiar lines we see that (3.15) necessarily follows. Thus, when we couple (3.12) and (3.16) and use Corollary 3.1.1 we have part of the following.

*Theorem 3.3. If $L^c(x)$ or $\int_x^\infty B^c(u)du$ is of moderate growth as $x \to \infty$, then*

$$(3.17) \qquad L^c(x) \sim \frac{(1-\pi)}{\pi\mu} \int_x^\infty B^c(u)du,$$

*as $x \to \infty$. Conversely, if (3.16) holds then both sides of (3.17) are functions of moderate growth. In particular (3.17) holds if $B^c(x)$ is a fmg.*

To complete the proof of this theorem, we must show that (3.17) implies that both sides of this asymptotic equation are fmg. Now the derivation of

(3.15) has not depended on assumptions about growth-rate. From it we have, *a fortiori* (since $F^c(x) \le B^c(x)$)

$$(3.18) \qquad \pi L^c(x) \le (1+C)\, F^c(x) + \left\{\frac{(1-\pi)+\varepsilon\mu}{\mu}\right\} \int_{x+\Delta-1}^{\infty} B^c(u)du.$$

This inequality, with (3.17) and Corollary 3.1.1, yields

$$1 \le (1+C) \limsup_{x\to\infty} \frac{F^c(x)}{L^c(x)} + \frac{(1-\pi)+\varepsilon\mu}{(1-\pi)} .$$

Since $\varepsilon$ is arbitrary, we can thus infer that

$$\frac{F^c(x)}{L^c(x)} \to 0, \quad x \to \infty.$$

This allows us then to deduce from (3.17) and (3.18) that

$$1 \le \frac{(1-\pi)+\varepsilon\mu}{(1-\pi)} \liminf_{x\to\infty} \frac{L^c(x+\Delta-1)}{L^c(x)} .$$

Thus, for any $\theta > 0$, by choosing $\Delta >> \theta$,

$$\limsup_{x\to\infty} \frac{L^c(x)}{L^c(x+\theta)} \le 1.$$

But $L^c(x)$ is non-increasing, so we are forced to the conclusion that $L^c(x) \sim L^c(x+\theta)$, i.e. that $L^c(x)$ is a fmg. The theorem is thus proved.

## 4. RELATIONSHIP BETWEEN $L^c(x)$ AND $Q^c(x)$

If $A(\cdot)$ is a measure function on $[0,\infty)$ we shall denote its Laplace-Stieltjes transform thus:

$$A^*(s) = \int_{0-}^{\infty} e^{-sx} A(dx),$$

and take $s$ to be real and positive. From (1.7) we then have

$$(4.1) \qquad Q^*(s) = \frac{1-\pi}{1-\pi L^*(s)} ,$$

in a familiar way. But

$$(4.2)\quad \int_0^\infty e^{-sx}\, Q^c(x)dx = \frac{1 - Q^*(s)}{s}$$

and a similar formula holds for $L^c(x)$. From (4.1) we obtain

$$\left\{\frac{1-Q^*(s)}{s}\right\} = \left\{\frac{1-L^*(s)}{s}\right\}\left\{\frac{\pi Q^*(s)}{1-\pi}\right\},$$

and this, in view of (4.2), gives the useful relation:

$$(4.3)\quad Q^c(x) = \left(\frac{\pi}{1-\pi}\right)\int_0^x L^c(x-z)\, Q(dz).$$

Alternatively, (4.3) can be obtained from (1.3) by direct calculation with no reference to transforms.

An immediate and obvious consequence of (4.3) is the following.

*Lemma 4.1. For all* $x \geq 0$,

$$Q^c(x) \geq \left(\frac{\pi}{1-\pi}\right) L^c(x)\, Q(x)$$

*and consequently,*

$$\liminf_{x\to\infty} \frac{Q^c(x)}{L^c(x)} \geq \frac{\pi}{1-\pi}$$

One can also obtain from (4.1) the equation

$$\left\{\frac{1-Q^*(s)}{s}\right\} = \pi\left\{\frac{1-L^*(s)}{s}\right\} + \pi\left\{\frac{1-Q^*(s)}{s}\right\} L^*(s),$$

which yields the second useful relation:

$$(4.4)\quad Q^c(x) = \pi L^c(x) + \pi\int_0^x Q^c(x-z)\, L(dz).$$

(This can also be obtained, but tediously, by direct calculation.)

Let us note, in passing, the rather obvious fact that

$$(4.5)\quad Q^c(x) \leq (1-\pi), \quad x \geq 0.$$

Let us now set

$$\chi(x) = \frac{Q^c(x)}{L^c(x)}$$

and

$$\hat{\chi}(x) = \sup_{0\le y\le x} \chi(x).$$

Since we define distribution functions to be continuous to the right, it follows that $\chi(x)$ is also continuous to the right. It is then easy to show that $\hat{\chi}(x)$ is continuous to the right. Suppose $\chi(x)$ is unbounded as $x \to \infty$. Then we can find a sequence $\{x_j\}$ such that $x_j \to \infty$, $\chi(x_j) = \hat{\chi}(x_j) \to \infty$. From (4.4) and (4.5) we then have the inequality

$$(4.6) \quad Q^c(x) \le \pi L^c(x) + \pi(1-\pi)\{L^c(x-\Delta) - L^c(x)\} + \pi\int_\Delta^{x-\Delta} Q^c(x-z)\,L(dz) + \pi Q^c(x-\Delta),$$

from which, on setting $x = x_j$ and dividing by $L^c(x_j)$, we have

$$(4.7) \quad \chi(x_j) \le \pi^2 + \pi(1-\pi)\left\{\frac{L^c(x_j-\Delta)}{L^c(x_j)}\right\} + \pi\chi(x_j)\int_\Delta^{x_j-\Delta}\left\{\frac{L^c(x_j-z)}{L^c(x_j)}\right\}L(dz) + \pi\chi(x_j)\left\{\frac{L^c(x_j-\Delta)}{L^c(x_j)}\right\}.$$

*Let us now assume* $L^c(x)$ *to be a SETF.* We can then appeal to Lemma 2.2 to deal with the convolution on the right of (4.7) and obtain (since $L^c(x) \sim L^c(x-\Delta)$),

$$(1-\varepsilon\pi-\pi)\lim_{j\to\infty}\chi(x_j) \le \pi^2 + \pi(1-\pi)$$

which, since $\varepsilon$ is arbitrary, shows

$$\lim_{j\to\infty}\chi(x_j) \le \frac{\pi}{1-\pi}.$$

This contradicts the hypothesis that $\chi(x)$ is unbounded. If we let $K$ be a finite upper bound for $\chi(x)$ we can deduce from (4.6) that

$$(4.8)\quad \chi(x) \le \pi^2 + \pi(1-\pi)\left\{\frac{L^c(x-\Delta)}{L^c(x)}\right\} + \pi K \int_\Delta^{x-\Delta} \left\{\frac{L^c(x-z)}{L^c(x)}\right\} L(dz) + \pi\chi(x-\Delta)\left\{\frac{L^c(x-\Delta)}{L^c(x)}\right\}.$$

If we let $\theta = \lim\sup_{x\to\infty} \chi(x)$ then $\chi(x) \le \theta+\varepsilon$ for all sufficiently large $x$. If we let $x$ in (4.8) run through an unbounded sequence $\{x_j\}$ such that $\chi(x_j) \to \theta$, we can infer (with another appeal to Lemma 2.2) that

$$\theta \le \pi^2 + \pi(1-\pi) + \pi K\varepsilon + \pi(\theta+\varepsilon).$$

The arbitrariness of $\varepsilon$ then allows the deduction

$$\lim_{x\to\infty}\sup \frac{Q^c(x)}{L^c(x)} \le \frac{\pi}{1-\pi}.$$

This result, coupled with Lemma 4.1 shows that, when $L^c(x)$ is a SETF,

$$(1-\pi)\, Q^c(x) \sim \pi L^c(x).$$

*Suppose next that we start instead with the premiss that $Q^c(x)$ is a SETF.* From Lemma 4.1, given any small $\delta > 0$ we can find $\Delta(\delta)$ such that

$$\frac{L^c(x)}{Q^c(x)} \le \left(\frac{1-\pi}{\pi}\right)(1+\delta)$$

for all $x \ge \Delta$. Then, from (4.3), we can infer that

$$Q^c(x) \le \left(\frac{\pi}{1-\pi}\right)\{Q^c(x-\Delta) - Q^c(x)\} + (1+\delta)\int_\Delta^{x-\Delta} Q^c(x-z)\, Q(dz) + \left(\frac{\pi}{1-\pi}\right) L^c(x-\Delta)\, Q(\Delta).$$

If we now divide by $Q^c(x)$, use the fact that $Q^c(x) \sim Q^c(x-\Delta)$ and Lemma 2.2, we find

$$1 \le (1+\delta)\ \varepsilon(\Delta) + \left(\frac{\pi Q(\Delta)}{1-\pi}\right) \liminf_{x \to \infty} \frac{L^c(x)}{Q^c(x)}$$

where $\varepsilon(\Delta) \to 0$ as $\Delta \to \infty$. Thus

$$\liminf_{x \to \infty} \frac{L^c(x)}{Q^c(x)} \ge \frac{1-\pi}{\pi}$$

(since $Q(\Delta) \to 1$ as $\Delta \to \infty$). This result, coupled with Lemma 4.1, proves that $(1-\pi)\ Q^c(x) \sim \pi L^c(x)$.

Lastly, suppose we start with the premiss that

(4.9) $$(1-\pi)Q^c(x) \sim \pi L^c(x), \quad x \to \infty.$$

From (4.4) we see that, for any fixed $\theta > 0$,

$$Q^c(x) \ge \pi L^c(x) + \pi Q^c(x)\ L(\theta) + \pi Q^c(x-\theta)\ L^c(\theta).$$

Hence, from (4.7),

$$1 \ge (1-\pi) + \pi L(\theta) + \pi L^c(\theta) \limsup_{x \to \infty} \frac{Q^c(x-\theta)}{Q^c(x)}.$$

We may suppose $L^c(\theta) \ne 0$, and deduce

$$\limsup_{x \to \infty} \frac{Q^c(x-\theta)}{Q^c(x)} \le 1.$$

But $Q^c(x)$ is non-increasing. Hence $Q^c(x-\theta) \sim Q^c(x)$, as $x \to \infty$. This is enough to establish that $Q^c(x)$ is a fmg and so, by (4.7), $L^c(x)$ must be likeliwise. We have thus proved:

*Theorem 4.2. If either* $L^c(x)$ *or* $Q^c(x)$ *is a sub-exponential tail function then*

$$(1-\pi)\ Q^c(x) \sim \pi L^c(x), \quad x \to \infty.$$

*Conversely, if this asymptotic equation holds, then both* $Q^c(x)$ *and* $L^c(x)$ *are of moderate growth.*

# 5. ON MOMENTS AND THEIR EXISTENCE

In this section, we consider the relationship between the distribution functions $B(x)$, $F(x)$, $L(x)$ and $Q(x)$ with reference to the moments that may exist; we consider moments of a quite general sort. Our main result will be in terms of the *moment functions* (M.F.) introduced in §1. However, some of our subsidiary results can be expressed in terms of a more general class of functions: we shall say $S(x)$ is a *scale function* if it is non-negative, non-decreasing, and satisfies the ordeq condition $S(x) = O(S(x-1))$ for large positive $x$. It is easy to see that a moment function is a scale function (but not *vice versa*: $e^x$ is a scale function). Also, if $S(x)$ is a scale function then so is $S_J(x) =_{\text{def}} \int_0^x S(u)du$.

*Theorem 5.1. If $S(x)$ is a scale function then $\int_0^\infty S(x)B(dx) < \infty$ if and only if $\int_0^\infty S(x)F(dx) < \infty$.*

*Proof:* Let $\hat{U}$ be a median of the inter-arrival distribution. Then there must be a finite $K > 0$ such that $S(x) \leq K\{1+S(x-\hat{U})\}$. Now

$$\begin{aligned}\int_0^\infty S(x)F(dx) &= ES((V-U)^+)\\ &\geq \tfrac{1}{2}ES((V-\tilde{U})),\end{aligned}$$

and so

$$\begin{aligned}\int_0^\infty S(x)B(dx) &= ES(V)\\ &\leq K\{1 + 2\int_0^\infty S(x)F(dx)\}.\end{aligned}$$

On the other hand, $S(V) \geq S(V-U)$, so

$$\int_0^\infty S(x)B(dx) \geq \int_0^\infty S(x)F(dx).$$

The theorem follows from these last two inequalities.

*Theorem 5.2. If $S(x)$ is a scale function,*

$$(5.1)\quad \int_0^\infty S(x)L(dx) < \infty$$

*if and only if*

$$(5.2) \quad \int_0^\infty S_J(x)F(dx) < \infty,$$

*where* $S_J(x) = \int_0^x S(u)du$.

*Proof:* From (3.13) we have

$$(5.3) \quad \pi L^c(x) = F^c(x) + \sum_{n=0}^{\infty} \int_{-(n+1)}^{-n} F^c(x-z)G(dz)$$

$$\leq F^c(x) + \sum_{n=0}^{\infty} F^c(x+n) \int_{-(n+1)}^{-n} G(dz)$$

$$\leq F^c(x) + C \sum_{n=0}^{\infty} F^c(x+n)$$

$$\leq F^c(x) + C \int_{x-1}^{\infty} F^c(u)du,$$

where $C$ is the bound introduced just prior to (3.15). Thus (5.1) holds if

$$\int_0^\infty S(x) \int_{x-1}^{\infty} F^c(u)dudx < \infty.$$

Since $S(x) = O(S(x-1))$ this holds if

$$\int_0^\infty S(x) \int_{x}^{\infty} F^c(u)dudx < \infty$$

and an integration by parts shows the latter is implied by (5.2).

Next we observe, again from (3.13), that

$$\pi L^c(x) \geq \sum_{n=0}^{\infty} F^c(x+n+1) \int_{-(n+1)}^{-n} G(dz)$$

$$\geq c \sum_{n=\Delta}^{\infty} F^c(x+n+1),$$

for some suitably large $\Delta > 0$, where $c$ can be any real satisfying $0 < c < (1-\pi)/\mu$ and we have appealed to (3.9). Thus

$$\pi L^c(x) \geq c \int_{\Delta+x+1}^{\infty} F^c(u)du.$$

If (5.1) holds we can thus infer that

$$\int_0^\infty S(x) \int_{\Delta+x+1}^\infty F^c(u)du\ dx < \infty.$$

A further appeal to the defining property of $S(x)$ and an integration by parts then shows (5.2) must hold, and the theorem is complete.

To complete our discussion of moments we must discuss the "moment relationship" between $L(x)$ and $Q(x)$. This is not so simple a matter as those we have studied earlier in this section.

*Theorem 5.3. Let* $S(x)$ *be a scale function of moderate growth satisfying the condition*

$$(5.4) \quad S(x_1+x_2) \le S(x_1)\, S(x_2),$$

*for all* $x_1 \ge 0$, $x_2 \ge 0$. *Then* $\int_0^\infty S(x)L(dx)$ *and* $\int_0^\infty S(x)Q(dx)$ *converge or diverge together.*

*Note:* $S(x)$ automatically satisfies these conditions if it is a MF.

*Proof:* Let us first suppose

$$ES(Z) = \int_0^\infty S(x)\ L(dx) < \infty,$$

where $Z$ is a typical ladder variable. Then, in view of (1.7), our first task is to prove the convergence of

$$(5.5) \quad \sum_{r=1}^\infty \pi^r\, ES(Z_1+\ldots+Z_r),$$

where, as before, $\{Z_j\}$ are iid ladder variables. We observe that

$$\begin{aligned} ES(Z_1+\ldots+Z_r) &\le ES(r+Z_1+\ldots+Z_r) \\ &= ES(1+Z_1)R_2R_3\ldots R_r, \end{aligned}$$

where

$$R_j = \frac{S(j+Z_1+\ldots+Z_j)}{S(j-1+Z_1+\ldots+Z_{j-1})}.$$

Thus

$$ES(Z_1+\ldots+Z_r) \le E\{S(1+Z_1)R_2R_3\ldots R_{r-1}S_r\}$$

where

$$S_r = E\{R_r|Z_1,\ldots,Z_{r-1}\}.$$

At this point, we need the following

*Lemma 5.4. If $S(x)$ is as in Theorem 5.2,*

$$\frac{ES(Z+x)}{S(x)} \to 1, \quad as \quad x \to \infty.$$

*Proof of Lemma:* Since $S(x)$ is of moderate growth $S(y+x)/S(x) \to 1$, as $x \to \infty$, for every fixed $y \ge 0$. But, by (5.4)

$$\frac{S(y+x)}{S(x)} \le S(y).$$

Thus dominated convergence proves the lemma. We can now observe that

$$S_r = \int_0^\infty \frac{S(y+r+Z_1+\ldots+Z_{r-1})}{S(r-1+Z_1+\ldots+Z_{r-1})} L(dy)$$

tends to unity, as $r \to \infty$, uniformly with respect to $Z_1+\ldots+Z_{r-1}$ (> 0,, necessarily, since the Z's are ladder variables). Thus, given $\varepsilon > 0$ arbitrarily small

$$S_r \le (1+\varepsilon)$$

for all $r > R_0(\varepsilon)$.

Thus, for $r$ large,

$$\begin{aligned} ES(Z_1+\ldots+Z_r) &\le (1+\varepsilon)ES(1+Z_1)R_2\ldots R_{r-1} \\ &= (1+\varepsilon)ES(1+Z_1)R_2\ldots R_{r-2}S_{r-1} \\ &\le (1+\varepsilon)^2ES(1+Z_1)R_2\ldots R_{r-2}, \end{aligned}$$

and so on. It should now be clear that

$$(5.6) \quad \limsup_{r \to \infty} \{ES(Z_1+\ldots+Z_r)\}^{1/r} = 1.$$

(Note that (5.4) ensures $ES(x) \geq 1$ for all $x \geq 0$.) Plainly (5.6) implies the convergence of (5.5), whatever the value of $\pi$ $(0 < \pi < 1)$.

The completion of the theorem is trivial, for (1.7) shows that

$$Q^c(x) \geq (1-\pi)\pi L^c(x).$$

Clearly this implies that any moment finite for $Q$ is necessarily finite for $L$.

We conclude with two notes: (i) The functions satisfying Theorem 5.3 are discussed in Smith (1969) where it is shown, in particular, that if $N(x) = x^\gamma \tau(x)$ for $\gamma \geq 0$ and $\tau(x)$ sub-linear, then $N(x)$ is asymptotically equivalent to such a scale function. (ii) If $S(x)$ is a scale function then so in $S_J(x)$; this observation is relevant when combining Theorems 5.1, 5.2 and 5.3 to draw the conclusion (under suitable conditions) that $\int_0^\infty S(x)Q(dx) < \infty$ if and only if $\int_0^\infty S_J(x)\, B(dx) < \infty$.

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